HOMEWORK 12 : NINA CHALGERI

①

$$f(x) = \sin(x + x^2) \Rightarrow f(0) = 0$$

$$f'(x) = \cos(x + x^2) \cdot (1 + 2x) \Rightarrow f'(0) = 1$$

$$f''(x) = -\sin(x + x^2)(1 + 2x)^2 + 2\cos(x + x^2) \Rightarrow f''(0) = 2$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + x + x^2$$

②

$$\sin z = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \frac{1}{7!} z^7 + \dots$$

$$\Rightarrow \sin(x + x^2) = (x + x^2) - \frac{1}{3!}(x + x^2)^3 + \frac{1}{5!}(x + x^2)^5 - \frac{1}{7!}(x + x^2)^7 - \dots$$

$$= 0 + (x + x^2) - \frac{1}{6} x^3 (1+x)^3 + \frac{1}{120} x^5 (1+x)^5 - \frac{1}{5040} x^7 (1+x)^7$$

$$= 0 + (x + x^2) - \frac{1}{6} x^3 (1 + 3x + 3x^2 + x^2) + \frac{1}{120} x^5 (1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5)$$ (continued)

$$- \frac{1}{5040} x^7 (1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7)$$

$$\Rightarrow x + x^2 - \frac{1}{6}(x^3 + 3x^4 + 3x^5 + x^6) + \frac{1}{120}(x^5 + 5x^6 + 10x^7 + 10x^8 + 5x^9 + x^{20})$$

Continued

$$- \frac{1}{5040}(x^7 + 7x^8 + 21x^9 + 35x^{10} + 35x^{20} + 21x^{12} + 7x^{13} + x^{14})$$

$$= x + x^2 - \frac{x^3}{6} - \frac{x^4}{2} - \frac{x^5}{2} - \frac{x^6}{6} + \frac{x^5}{120} + \frac{x^6}{24} + \frac{x^7}{12} + \frac{x^8}{12} + \frac{x^9}{24} + \frac{x^{10}}{120} - \frac{x^7}{5040} - \frac{x^8}{720}$$

(con't)

$$- \frac{x^9}{240} - \frac{x^{10}}{144} - \frac{x^{11}}{144} - \frac{x^{12}}{240} - \frac{x^{13}}{720} - \frac{x^{14}}{5040}$$

$$= x - x^2 - \left(\frac{x^3 + x^6}{6}\right) - \left(\frac{x^4 + x^5}{2}\right) + \left(\frac{x^5 + x^{10}}{120}\right) + \left(\frac{x^4 + x^9}{24}\right) + \left(\frac{x^7 + x^8}{12}\right)$$

$$- \left(\frac{x^7 + x^{14}}{5040}\right) - \left(\frac{x^8 + x^{13}}{720}\right) - \left(\frac{x^9 + x^{12}}{240}\right) - \left(\frac{x^{10} + x^{11}}{144}\right)$$

③

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt \qquad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

$$z = t^2, \quad \frac{1}{1+t^2} = \sum_{n=0}^{\infty} (t^2)^n \Rightarrow \arctan x = \int_0^x \sum_{n=0}^{\infty} (t^2)^n \, dt$$

$$= \sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n+1}(-1)^n \Big]_0^x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

④

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-(\tan\alpha)(\tan\beta)} \Rightarrow \arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

$$\alpha+\beta = \arctan\left(\frac{x+y}{1-xy}\right) \Rightarrow \arctan x + \arctan y \cancel{\left(\frac{x+y}{1-xy}\right)}$$

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

⑤

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan\left(\frac{1/2 + 1/3}{1-(1/2)(1/3)}\right)$$

$$= \arctan\left(\frac{5/6}{1-1/6}\right)$$

$$= \arctan(1) = \frac{\pi}{4}$$