Homework 12

① $f(x) = \sin(x+x^2) \to f(0) = 0$

$f'(x) = \cos(x+x^2)\cdot(1+2x) \to f'(0) = 1$

$f''(x) = -\sin(x+x^2)(1+2x)^2 + 2\cos(x+x^2) \to f''(0) = 2$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$\underline{0 + x + x^2}$

② $\sin z = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \frac{1}{7!} z^7 + \ldots$

with $z = x+x^2$ we have

$0 + (x+x^2) - \frac{1}{3!}(x+x^2)^3 + \frac{1}{5!}(x+x^2)^5 - \frac{1}{7!}(x+x^2)^7$

$= (x+x^2) - \frac{1}{6}(x+x^2)^3 + \frac{1}{120}(x+x^2)^5 - \frac{1}{5040}(x+x^2)^7$

$= 0 + (x+x^2) - \frac{1}{6}x^3(1+x)^3 + \frac{1}{120}x^5(1+x)^5 - \frac{1}{5040}x^7(1+x)^7$

$= 0 + (x+x^2) - \frac{1}{6}x^3(1+3x+3x^2+x^3)$

$+ \frac{1}{120}x^5(1+5x+10x^2+10x^3+5x^4+x^5)$

$- \frac{1}{5040}x^7(1+7x+21x^2+35x^3+35x^4+21x^5+7x^6+x^7)$

Simplified:

$x+x^2 - \frac{1}{6}(x^3+3x^4+3x^5+x^6)$

$+ \frac{1}{120}(x^5+5x^6+10x^7+10x^8+5x^9+x^{10})$

$- \frac{1}{5040}(x^7+7x^8+21x^9+35x^{10}+35x^{11}+21x^{12}+7x^{13}+x^{14})$

$= x+x^2 - \frac{x^3}{6} - \frac{x^4}{2} - \frac{x^5}{2} - \frac{x^6}{6} + \frac{x^5}{120} + \frac{x^6}{24} + \frac{x^7}{12} + \frac{x^8}{12}$
$+ \frac{x^9}{24} + \frac{x^{10}}{120} - \frac{x^7}{5040} - \frac{x^8}{720} - \frac{x^9}{240} - \frac{x^{10}}{144} - \frac{x^{11}}{144} - \frac{x^{12}}{240} - \frac{x^{13}}{720}$ 5040

$= x+x^2 - \left(\frac{x^3+x^6}{6}\right) - \left(\frac{x^4+x^5}{2}\right) + \left(\frac{x^5+x^{10}}{120}\right)$
$+ \left(\frac{x^4+x^9}{24}\right) + \left(\frac{x^7+x^8}{12}\right) - \left(\frac{x^7+x^{14}}{5040}\right)$
$- \left(\frac{x^8+x^{13}}{720}\right) - \left(\frac{x^9+x^{12}}{240}\right) - \left(\frac{x^{10}+x^{11}}{144}\right)$

(3) $\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$, $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

If $z = t^2$, $\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (t^2)^n$

$\arctan x = \int_0^x \sum_{n=0}^{\infty} (t^2)^n \, dt$

$= \sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n+1}(-1)^n \Big]_0^x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

(4) $\tan(a+b) = \frac{\tan a + \tan B}{1 - (\tan a)(\tan b)}$ $\rightarrow$ $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$

let $\arctan x = a$ and $\arctan y = b$

$\rightarrow$ $x = \tan a$, $y = \tan b$

Since $\tan(a+b) = \frac{\tan a + \tan b}{1-(\tan a)(\tan b)}$, $\tan(a+b) = \frac{x+y}{1-xy}$

$a+b = \arctan\left(\frac{x+y}{1-xy}\right)$

$\downarrow$

$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$

(5) using the above derivation,

$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan\left(\frac{1/2 + 1/3}{1 - (1/2)(1/3)}\right)$

$= \arctan\left(\frac{5/6}{1 - 1/6}\right)$

$= \arctan(1) = \frac{\pi}{4}$