History of Math | Homework 12 | Koron Amin

1) $f(x) = \sin(x+x^2)$

$f'(x) = \cos(x+x^2)(1+2x)$

$f''(x) = -\sin(x+x^2)(1+2x)^2 + 2\cos(x+x^2)$

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 2$

therefore $f(x) \approx \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2$

$\approx 0 + x + x^2$

2) $\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots$

$\sin(x+x^2) \approx (x+x^2) - \frac{1}{3!}(x+x^2)^3 + \frac{1}{5!}(x+x^2)^5 - \frac{1}{7!}(x+x^2)^7$

3) ~~Show~~ $\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$ and $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

Let $z = -t^2$, then $\frac{1}{1-z} = \frac{1}{1+t^2} = \sum_{n=0}^{\infty}(-t^2)^n = \sum_{n=0}^{\infty}(-1)^n t^{2n}$

Then $\arctan x = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty}\int (-1)^n x^{2n}\,dx$

$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} + C$ Since $\arctan(0) = 0 \rightarrow C = 0$

Therefore $\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$

4) Let $\alpha = \arctan x$ and $\beta = \arctan y$

since $\tan(\alpha+\beta) = \dfrac{\tan(\alpha) + \tan(\beta)}{1 - \tan\alpha \tan\beta}$

then

$$\tan(\arctan(x) + \arctan(y)) = \frac{\tan(\arctan(x)) + \tan(\arctan(y))}{1 - \tan(\arctan(x))\tan(\arctan(y))}$$

$x + y = \dfrac{x+y}{1-xy}$ implies that

$$\arctan(x+y) = \arctan\left(\frac{x+y}{1-xy}\right)$$

5) Show $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$

Let $\alpha = \arctan\frac{1}{2}$ and $\beta = \arctan\frac{1}{3}$

$$\tan(\arctan 1/2 + \arctan 1/3) = \frac{\tan(\arctan 1/2) + \tan(\arctan 1/3)}{1 - \tan(\arctan 1/2)\tan(\arctan 1/3)}$$

$$= \frac{1/2 + 1/3}{1 - \frac{1}{6}} = 1$$

$\arctan(\tan(\arctan 1/2 + \arctan 1/3)) = \arctan(1)$

$\arctan 1/2 + \arctan 1/3 = \frac{\pi}{4}$