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It is OK to post the homework in your web-site

1.

Set $x = u+v$, $(u+v)^3 - 9(u+v) - 28 = 0$,

$$u^3 + 3u^2v + 3uv^2 + v^3 - 9(u+v) - 28 = 0,$$
$$u^3 + v^3 + [3uv(u+v) - 9(u+v)] - 28 = 0,$$
$$u^3 + v^3 + (u+v)(3uv - 9) - 28 = 0$$

Since we want to get rid of the middle term, so we demand it is 0.

$$3uv - 9 = 0, \quad uv = 3,$$

then $u^3 + v^3 = 28$, so u^3 and v^3 are two numbers whose sum is 9, and product is 27. Hence they are both solutions of the quadratic equation

$$X^2 - 28X + 27 = 0, \text{ then } X = 1, \ X = 27$$

Hence $u^3 = 1$, $v^3 = 27$, we get $u = 1$, $v = 3$.

So one solution is $x = u + v = 4$,

The other two roots are $\frac{-1+\sqrt{3}i}{2} \cdot 1 + \frac{-1-\sqrt{3}i}{2} \cdot 3 = \frac{-1+\sqrt{3}i-3-3\sqrt{3}i}{2} = -2-\sqrt{3}i$

$$\frac{-1-\sqrt{3}i}{2} \cdot 1 + \frac{-1+\sqrt{3}i}{2} \cdot 3 = \frac{-1-\sqrt{3}i-3+3\sqrt{3}i}{2} = -2+\sqrt{3}i$$

2.

Set $x = u+v$, $(u+v)^3 - 30(u+v) - 133 = 0$,

$$u^3 + 3u^2v + 3uv^2 + v^3 - 30(u+v) - 133 = 0,$$
$$u^3 + v^3 + [3uv(u+v) - 30(u+v)] - 133 = 0,$$
$$u^3 + v^3 + (u+v)(3uv - 30) - 133 = 0$$

Since we want to get rid of the middle term, so we demand it is 0.

$$3uv - 30 = 0, \quad uv = 10,$$

then $u^3 + v^3 = 133$, so u^3 and v^3 are two numbers whose sum is 9, and product is 1000. Hence they are both solutions of the quadratic equation

$X^2 - 133X + 1000 = 0$, then $X = 8$, $X = 125$

Hence $u^3 = 8$, $v^3 = 125$, we get $u = 2$, $v = 5$.

So one solution is $x = u + v = 7$,

The other two roots are $\frac{-1+\sqrt{3}i}{2} \cdot 2 + \frac{-1-\sqrt{3}i}{2} \cdot 5 = \frac{-2+2\sqrt{3}i-5-5\sqrt{3}i}{2} = -\frac{7}{2} - \frac{3\sqrt{3}}{2}i$

$$\frac{-1-\sqrt{3}i}{2} \cdot 2 + \frac{-1+\sqrt{3}i}{2} \cdot 5 = \frac{-2-2\sqrt{3}i-5+5\sqrt{3}i}{2} = -\frac{7}{2} + \frac{3\sqrt{3}}{2}i$$

3.

Set $x = u + v$, $(u+v)^3 + p(u+v) + q = 0$,

$$u^3 + 3u^2v + 3uv^2 + v^3 + p(u+v) + q = 0 ,$$
$$u^3 + v^3 + [3uv(u+v) + p(u+v)] + q = 0 ,$$
$$u^3 + v^3 + (u+v)(3uv + p) + q = 0$$

Since we want to get rid of the middle term, so we demand it is 0.

$$3uv + p = 0 , \quad uv = -\frac{p}{3} ,$$

then $u^3 + v^3 = -q$, so u^3 and v^3 are two numbers whose sum is $-q$, and product is $(-\frac{p}{3})^3$. Hence they are both solutions of the quadratic equatio

$$X^2 - (-q)X + (-\frac{p}{3})^3 = 0 ,$$

then $X = \frac{-q \pm \sqrt{(-q)^2 - 4(-\frac{p}{3})^3}}{2} = \frac{-q \pm \sqrt{q^2 - 4(-\frac{p}{3})^3}}{2}$

Hence $u = \sqrt[3]{\frac{-q + \sqrt{q^2 - 4(-\frac{p}{3})^3}}{2}}$, $v = \sqrt[3]{\frac{-q - \sqrt{q^2 - 4(-\frac{p}{3})^3}}{2}}$

So one solution is $x = u + v = \sqrt[3]{\frac{-q + \sqrt{q^2 - 4(-\frac{p}{3})^3}}{2}} + \sqrt[3]{\frac{-q - \sqrt{q^2 - 4(-\frac{p}{3})^3}}{2}}$

The other two roots are $\frac{-1+\sqrt{3}i}{2}\cdot\sqrt[3]{\frac{-q+\sqrt{q^2-4(-\frac{p}{3})^3}}{2}}+\frac{-1-\sqrt{3}i}{2}\cdot\sqrt[3]{\frac{-q-\sqrt{q^2-4(-\frac{p}{3})^3}}{2}}$

and $\frac{-1-\sqrt{3}i}{2}\cdot\sqrt[3]{\frac{-q+\sqrt{q^2-4(-\frac{p}{3})^3}}{2}}+\frac{-1+\sqrt{3}i}{2}\cdot\sqrt[3]{\frac{-q-\sqrt{q^2-4(-\frac{p}{3})^3}}{2}}$.

4. $y=x+\frac{a}{3}$, then $y=x+1$, $x=y-1$,

Plug in, $(y-1)^3+3(y-1)^2+5(y-1)-100=0$

$y^3-3y^2+3y-1+3y^2-6y+3+5y-105=0$

$y^3+2y-103=0$, so we find a reduced cubic.