Anna Zhebrun

OK to post

10/17/2021

Homework 11

1. $x^3 - 9x - 28 = 0$

Let $x = u + v$ then:

$(u+v)^3 - 9(u+v) - 28 = 0$

$u^3 + 3u^2v + 3uv^2 + v^3 - 9(u+v) - 28 = 0$

$u^3 + v^3 + 3uv(u+v) - 9(u+v) - 28 = 0$

$u^3 + v^3 + (3uv - 9)(u+v) - 28 = 0$

We need $3uv - 9 = 0$ so $uv = 3$.

Then:

$u^3 + v^3 = 28$ an $u^3v^3 = 3^3 = 27$.

Place them in the quadratic equation:

$x^2 - 28x + 27 = 0$

$(x-27)(x-1) = 0$

So $x = 27$ or $x = 1$.

Take $u^3 = 1$ and $v = 27$

$u = 1$ and $v = 3$.

So one solution is $u + v = 1 + 3 = 4$.

The other two solutions are:

$$\frac{-1+\sqrt{3}i}{2} \cdot 1 + \frac{-1-\sqrt{3}i}{2} \cdot 3 = \frac{-4-2\sqrt{3}i}{2} = -2-\sqrt{3}i$$

$$\frac{-1-\sqrt{3}i}{2} \cdot 1 + \frac{-1+\sqrt{3}i}{2} \cdot 3 = \frac{-4+2\sqrt{3}i}{2} = -2+\sqrt{3}i$$

So the solutions to $x^3 - 9x - 28 = 0$ are:

4, $-2-\sqrt{3}i$ and $-2+\sqrt{3}i$

3. $x^3 - 30x - 133 = 0$

Set $x = u+v$, then:

$(u+v)^3 - 30(u+v) - 133 = 0$

$u^3 + v^3 + 3uv(u+v) - 30(u+v) - 133 = 0$

$u^3 + v^3 + (3uv - 30)(u+v) - 133 = 0$

We need $(3uv-30) = 0$, so $uv = 10$, and

$u^3 + v^3 = 133$ and $u^3v^3 = 10^3 = 1000$

Use the above values for the quadratic equation:

$x^2 - 133x + 1000 = 0$

$(x-8)(x-125) = 0$

$x = 8$ $x = 125$, so

$u^3 = 8$ and $v^3 = 125$

$u = 2$ and $v = 5$

Therefore, one solution to the cubic will be

$x = u + v = 2 + 5 = 7.$

The other two solutions are:

$$\frac{-1+\sqrt{3}i}{2} \cdot 2 + \frac{-1-\sqrt{3}i}{2} \cdot 5 = \frac{-7-3\sqrt{3}i}{2}$$

$$\frac{-1-\sqrt{3}i}{2} \cdot 2 + \frac{-1+\sqrt{3}i}{2} \cdot 5 = \frac{-7+3\sqrt{3}i}{2}$$

So, the solutions to the cubic equation are:

7, $\frac{-7-3\sqrt{3}i}{2}$ and $\frac{-7+3\sqrt{3}i}{2}$

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3. $x^3 + px + q = 0.$

Set $x = u+v$, then:

$$(u+v)^3 + p(u+v) + q = 0$$
$$u^3 + v^3 + 3uv(u+v) + p(u+v) + q = 0$$
$$u^3 + v^3 + (3uv + p)(u+v) + q = 0$$

We need $3uv + p = 0$, so $uv = \frac{-p}{3}$ to simplify above equation to:

$$u^3 + v^3 + q = 0 \rightarrow u^3 + v^3 = -q.$$

And calculate $u^3v^3 = (uv)^3 = \left(\frac{-p}{3}\right)^3 = \frac{-p^3}{27}.$

Since the sum of u^3 and v^3 is $-q$ and their product is $-p^3/27$, u^3 and v^3 are solutions to the following quadratic equation:

$$x^2 + qx - p^3/27 = 0$$

$$v^3/u^3 = x = \frac{-q \pm \sqrt{q^2 + 4\frac{p^3}{27}}}{2} = \frac{-q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

We set:

$$u = \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \text{and} \quad v = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

The solutions to the cubic equation are then:

1. $u+v = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$

2. $\frac{-1+\sqrt{3}i}{2}u + \frac{-1-\sqrt{3}i}{2}v$

3. $\frac{-1-\sqrt{3}i}{2}u + \frac{-1+\sqrt{3}i}{2}v.$

(where v and u are as defined above.

4. $x^3+3x^2+5x-100=0$

Set $y=x+1$. Then $x=y-1$, so:

$(y-1)^3+3(y-1)^2+5(y-1)-100=0$

$y^3-3y^2+3y-1+3y^2-6y+3+5y-5-100=0$

$y^3+2y-106=0.$

Therefore, to find the roots of $x^3+3x^2+5x-100=0$, we can use the reduced form: $y^3+2y-106=0$.