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Homework for Lecture 11 - OK to post

① Let $x = u+v$. Plugging in for x, we obtain

$$(u+v)^3 - 9(u+v) - 28 = 0$$

Expanding,

$$u^3 + 3u^2v + 3uv^2 + v^3 - 9(u+v) - 28 = 0$$

We rewrite this as

$$u^3 + v^3 + (3u^2v + 3uv^2 - 9(u+v)) - 28 = 0$$

Factoring, we obtain

$$u^3 + v^3 + 3uv(u+v) - 9(u+v) - 28 = 0$$

$$u^3 + v^3 + (u+v)(3uv - 9) - 28 = 0$$

We make it so that $3uv - 9 = 0$. This means that

$$uv = 3$$

Thus $u^3v^3 = 27$.

Returning to the original equation, we see that $u^3 + v^3 = 28$

Thus u^3 and v^3 have a sum of 28 and a product of 27. We use the formula

$$x^2-(\text{sum})(x)+\text{product}$$

to get that

$$x^2-28x+27$$

Setting equal to 0 and factoring, we get

$$x=27,\ x=1$$

So $u^3=27$ and $v^3=1$, therefore $u=3$ and $v=1$.

One solution is $x=u+v=3+1=4$.

We find the other two roots by using $\omega=\frac{-1+\sqrt{3}i}{2}$ and $\omega^2=\frac{-1-\sqrt{3}i}{2}$

$$\omega u+\omega^2 v=\left(\frac{-1+\sqrt{3}i}{2}\right)(3)+\left(\frac{-1-\sqrt{3}i}{2}\right)(1)=-2+\sqrt{3}i$$

$$\omega^2 u+\omega v=\left(\frac{-1-\sqrt{3}i}{2}\right)(3)+\left(\frac{-1+\sqrt{3}i}{2}\right)(1)=-2-\sqrt{3}i$$

So the solutions are $4, -2\pm\sqrt{3}i$

2 Let $x=u+v$. Plugging in for x, we obtain

$$(u+v)^3-30(u+v)-133=0$$

Expanding,

$$u^3+3u^2v+3uv^2+v^3-30(u+v)-133=0$$

We rewrite this as

$$u^3+v^3+\left(3u^2v+3uv^2-30(u+v)\right)-133=0$$

Factoring, we obtain

$$u^3+v^3+3uv(u+v)-30(u+v)-133=0$$

$$u^3+v^3+(u+v)(3uv-30)-133=0$$

We make it so that $3uv-30=0$. This means that

$$uv=10$$

Thus $u^3v^3=1000$.

Returning to the original equation, we see that $u^3+v^3=133$

Therefore u^3 and v^3 have a sum of 133 and a product of 1000. We use the formula

$$x^2-(\text{sum})(x)+\text{product}$$

to get that

$$x^2-133x+1000$$

Setting equal to 0 and using the quadratic formula, we get

$$\frac{133\pm\sqrt{133^2-4(1)(1000)}}{2}=\frac{133\pm117}{2}=125 \text{ and } 8$$

So $u^3 = 125$ and $v^3 = 8$, thus $u = 5$ and $v = 2$.

One solution is $x = u + v = 5 + 2 = 7$.

We find the other two solutions by using $\omega = \frac{-1+\sqrt{3}i}{2}$ and $\omega^2 = \frac{-1-\sqrt{3}i}{2}$

$$\omega u + \omega^2 v = \left(\frac{-1+\sqrt{3}i}{2}\right)(5) + \left(\frac{-1-\sqrt{3}i}{2}\right)(2) = -\frac{7}{2} + \frac{3}{2}\sqrt{3}i$$

$$\omega^2 u + \omega v = \left(\frac{-1-\sqrt{3}i}{2}\right)(5) + \left(\frac{-1+\sqrt{3}i}{2}\right)(2) = -\frac{7}{2} - \frac{3}{2}\sqrt{3}i$$

So the solutions are $7, -\frac{7}{2} \pm \frac{3}{2}\sqrt{3}i$

3 Let $x = u + v$. Plugging in for x, we obtain

$$(u+v)^3 + p(u+v) + q = 0$$

Expanding,

$$u^3 + 3u^2v + 3uv^2 + v^3 + p(u+v) + q = 0$$

We rewrite this as

$$u^3 + v^3 + (3u^2v + 3uv^2 + p(u+v)) + q = 0$$

Factoring, we obtain

$$u^3 + v^3 + 3uv(u+v) + p(u+v) + q = 0$$

$$u^3 + v^3 + (u+v)(3uv + p) + q = 0$$

We make it so that $3uv + p = 0$. This means that

$$uv = \frac{-p}{3}$$

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Thus $u^3v^3 = \frac{-p^3}{27}$

Returning to the original equation, we see that $u^3+v^3=-q$

Therefore u^3 and v^3 have a sum of $-q$ and a product of $\frac{-p^3}{27}$. We use the formula

$$x^2-(\text{sum})(x)+\text{product}$$

to get that

$$x^2+qx-\frac{p^3}{27}$$

Setting equal to 0 and using the quadratic formula, we obtain

$$\frac{-q\pm\sqrt{q^2-4(1)(-\frac{p^3}{27})}}{2} = \frac{-q\pm\sqrt{q^2+\frac{4}{27}p^3}}{2}$$

So a formula for one of the roots is

$$\left(\frac{-q+\sqrt{q^2+\frac{4}{27}p^3}}{2}\right)^{1/3} + \left(\frac{-q-\sqrt{q^2+\frac{4}{27}p^3}}{2}\right)^{1/3}$$

(4) We start by using the formula $y=x+\frac{a}{3}$. In this case, $a=3$, so we have $y=x+1$. Rearranging, we get $x=y-1$. Plugging this in, we get

$$(y-1)^3+3(y-1)^2+5(y-1)-100=0$$

Expanding, we get

$$y^3-3y^2+3y-1+3y^2-6y+3+5y-5-100=0$$

So the reduced cubic is $y^3+2y-103=0$.