Homework 11

(1) $x^3 - 9x - 28 = 0$

set $x = u + v \rightarrow (u+v)^3 - 9(u+v) - 28 = 0$

$$u^3 + 3u^2v + 3uv^3 + v^3 - 9(u+v) - 28 = 0$$

$$= u^3 + v^3 + (3u^2v + 3uv^2 - 9(u+v)) - 28 = 0$$

Since $3u^2 + 3uv^2 = 3uv(u+v)$ we get

$$u^3 + v^3 + [3uv(u+v) - 9(u+v)] - 28 = 0$$

$$= u^3 + v^3 + (u+v)(3uv - 9) - 28 = 0$$

$\downarrow$

We want to get $3uv - 9 = 0$ so $uv = 3$

$$(u+v)(3uv-9) = 0$$

$$u^3 + v^3 = 28$$

If $uv = 3$ then $u^3v^3 = 27$

Hence, $u^3 + v^3 = 28$, $u^3v^3 = 27$

so they are both solutions to

$$X^2 - 28X + 27 = 0$$

$(X - 1)(X - 27)$ or $X = 1$ and $X = 27$

$u^3 = 1 \qquad v^3 = 27$

$u = 1 \qquad v = 3$

$x = u + v = 1 + 3 = 4$ (one solution)

$$\frac{-1 + \sqrt{3}i}{2} \cdot 1 + \frac{-1 - \sqrt{3}i}{2} \cdot 3 = \frac{-4 - 2\sqrt{3}i}{2} \quad \text{(2nd solution)}$$

$$\frac{-1 - \sqrt{3}i}{2} \cdot 1 + \frac{-1 + \sqrt{3}i}{2} \cdot 3 = \frac{-4 + 2\sqrt{3}i}{2} \quad \text{(3rd solution)}$$

2. $x^3 - 30x - 133$

Set $x = u + v \rightarrow (u+v)^3 - 30(u+v) - 133 = 0$

$$u^3 + 3u^2v + 3uv^2 + v^3 - 30(u+v) - 133 = 0$$

$$= u^3 + v^3 + (3u^2 + 3uv^2 - 30(u+v)) - 133 = 0$$

Since $3u^2v + 3uv^2 = 3uv(u+v)$ we get

$$u^3 + v^3 + [3uv(u+v) - 30(u+v)] - 133 = 0$$

$$= u^3 + v^3 + (u+v)(3uv - 30) - 133 = 0$$

$\downarrow$

We want to get $3uv - 30 = 0$, so $uv = 10$

$(u+v)(3uv - 30) = 0$

$u^3 + v^3 = 133$

If $uv = 10$, $u^3v^3 = 1000$

Hence, $u^3 + v^3 = 133$, $u^3v^3 = 1000$

$$X^2 - 133X + 1000 = 0$$

$(X-8)(X-125)$ or $X = 8, 125$

$u^3 = 8$ $\quad$ $v^3 = 125$

$u = 2$ $\quad$ $v = 5$

$x = u + v = 7$ (one solution)

$$\frac{-1+\sqrt{3}i}{2} \cdot 2 + \frac{-1-\sqrt{3}i}{2} \cdot 5 = \frac{-7-3\sqrt{3}i}{2} \quad (2^{nd} \text{ solution})$$

$$\frac{-1-\sqrt{3}i}{2} \cdot 2 + \frac{-1+\sqrt{3}i}{2} \cdot 5 = \frac{-7+3\sqrt{3}i}{2} \quad (3^{rd} \text{ solution})$$

③ $x^3 + px + q = 0$

$x = u + v \rightarrow (u+v)^3 + p(u+v) + q = 0$

$u^3 + v^3 + (3u^2v + 3uv^2 + p(u+v)) + q$

$= u^3 + v^3 + [3uv(u+v) + p(u+v)] + q = 0$

$= u^3 + v^3 + (u+v)(3uv + p) + q = 0$

$\downarrow$

$3uv + p = 0$

$uv = \frac{-p}{3}$

$u^3 + v^3 = -q$

$u^3 v^3 = \left(\frac{-p}{3}\right)^3 = \frac{-p}{27}$

$X^2 + qX - \frac{p}{27} = 0$

quadratic formula: $X = \dfrac{-q \pm \sqrt{q^2 + \frac{4p}{27}}}{2}$

$u^3 = \dfrac{-q + \sqrt{q^2 + \frac{4p}{27}}}{2} \rightarrow u = \sqrt[3]{\dfrac{-q + \sqrt{q^2 + \frac{4p}{27}}}{2}}$

$v^3 = \dfrac{-q - \sqrt{q^2 + \frac{4p}{27}}}{2} \rightarrow v = \sqrt[3]{\dfrac{-q - \sqrt{q^2 + \frac{4p}{27}}}{2}}$

$x_1 = u + v$

$x_2 = \dfrac{-1 - \sqrt{3}i}{2} \cdot u + \dfrac{-1 + \sqrt{3}i}{2} \cdot v$

$x_3 = \dfrac{-1 + \sqrt{3}i}{2} \cdot u + \dfrac{-1 - \sqrt{3}i}{2} \cdot v$

(4) Reduced cubic of $x^3 + 3x^2 + 5x - 100$

$$y = x + \frac{3}{3} = x + 1 \rightarrow y - 1 = x$$

$$(y-1)^3 + 3(y-1)^2 + 5(y-1) - 100 = 0$$

$$y^3 - 3y^2 + 3y - 1 + 3(y-1) + 5(y-1) - 100 = 0$$

$$y^3 - 3y^2 + 3y - 1 + 3y + 3(y^2 - 2y + 1) + 5(y-1) - 100 = 0$$

$$y^3 - 3y^2 + 3y - 1 + 3y + 3y^2 + 6y - 3 + 5y - 5 - 100 = 0$$

$$= y^3 - 3y^2 + 3y^2 + 3y + 3y + 6y + 5y - 1 - 3 - 5 - 100 = 0$$

$$y^3 + 17y - 109 = 0$$