History of Math | Homework 1 | Karan Amin

1 Chinese Remainder Theorem

Theorem: If n_1 and n_2 are relatively prime, then the function $f: [0, n_1 n_2 - 1] \rightarrow ([0, n_1 - 1] \times [0, n_2 - 1])$ defined by $f(x) ::= (x \bmod n_1, x \bmod n_2)$ where $[a, b] ::= \{n \in \mathbb{Z} : a \leq n \leq b\}$ is a bijection.

If $f(x) = (0,0)$ then we know that $x \equiv 0 \bmod n_1$ and $x \equiv 0 \bmod n_2$ which means both n_1 and n_2 divide x. We also know that n_1 and n_2 are relatively prime which means their greatest common divisor is 1. Hence we can show that x must be a multiple of $n_1 n_2$. Since we know that x must be between 0 and $n_1 n_2 - 1$ because of the domain, the only logical integer it can be is 0. Therefore $f(0) = (0,0)$

To show that f is one-to-one, we have to show that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. If $f(x_1) = f(x_2)$ then x_1 has the same remainder as x_2 when divided by n_1 and n_2. Hence when we divide $x_2 - x_1$ (Assuming $x_2 \geq x_1$) by n_1 and n_2, we get remainder 0, therefore $f(x_2 - x_1) = (0,0)$, hence why $x_2 - x_1 = 0$. therefore $x_1 = x_2$, so f must be one-to-one. To show that it's onto, observe that $[0, n_1 n_2 - 1]$ has the same number of elements as $[0, n_1 - 1] \times [0, n_2 - 1]$, and since it's one-to-one, it must also be onto. If it wasn't, then it would violate the pigeonhole principle

2 Convert $\frac{1}{2} + \frac{1}{3} + \frac{1}{5}$ into a usual fraction

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{15+10+6}{30} = \boxed{\frac{31}{30}}$$

Smallest integer larger than $\frac{30}{31}$ is 1, so we write

$$\frac{31}{30} = \frac{1}{1} + \left(\frac{31}{30} - \frac{30}{30}\right) = \boxed{\frac{1}{1} + \frac{1}{30}}$$

Using the first Egyptian fraction [$1/2 + 1/3 + 1/5$], take 15 pizzas and divide each into equal halves, then take 10 pizzas and divide each into three equal parts, and finally take the last 6 pizzas and divide each into 5 equal parts, and each person one of each different size pieces.

Using the second Egyptian fraction [$\frac{1}{1} + \frac{1}{30}$], take 30 pizzas and give person one whole pizza, then take the last pizza, divide it into 30 equal pieces and give person a piece.

In my opinion, I think slicing one pizza into 30 parts is easier than dividing 15 pizzas into halves, 10 pizzas into thirds, and 6 pizzas into fifths. There is less slicing involved in the second Egyptian fraction, therefore less work.

3

a) Compute the map $f(x) ::= (x \bmod 3, x \bmod 7)$ $3 \cdot 7 = 21$

$f(1) = (1,1)$
$f(2) = (2,3)$
$f(3) = (0,3)$
$f(4) = (1,4)$
$f(5) = (2,5)$
$f(6) = (0,6)$
$f(7) = (1,0)$

$f(8) = (2,1)$
$f(9) = (0,2)$
$f(10) = (1,3)$
$f(11) = (2,4)$
$f(12) = (0,5)$
$f(13) = (1,6)$
$f(14) = (2,0)$
$f(15) = (0,1)$

$f(16) = (1,2)$
$f(17) = (2,3)$
$f(18) = (0,4)$
$f(19) = (1,5)$
$f(20) = (2,6)$

$20 + 21 = 41$

The two smallest integers are $\boxed{20 \text{ and } 41}$ that leaves remainder 2 when divided by 3 and remainder 6 when divided by 7

b) Compute the map $f(x) ::= (x \bmod 3, x \bmod 7)$

Looking at the table above, we can see that $f(4) = (1,4)$. Hence $\boxed{4 \text{ and } 25}$ are the two smallest integers that leaves remainder 1 when divided by 3 and remainder 4 when divided by 7.

$4 + 21 = 25$

c) Similarily from looking at the table above, we can see that $f(9) = (0,2)$. Hence $\boxed{9 \text{ and } 30}$ are the two smallest integers that leaves remainder 0 when divided by 3 and remainder 2 when divided by 7.

$9 + 21 = 30$