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1. Chinese Remainder Theorem, Special Case:
   If n_1 and n_2 are relatively prime ($\gcd(n_1, n_2) = 1$)
   then mapping:
   $$f : [0, n_1 n_2 - 1] \to [0, n_1 - 1] \times [0, n_2 - 1]$$
   defined by
   $$f(x) := (x \pmod{n_1}, \ x \pmod{n_2})$$
   is one-to-one and onto (i.e. a bijection).

<u>Proof:</u>
Take some $x \in [0, n_1 n_2 - 1]$ such that $f(x) = (0, 0)$.
Then $x \pmod{n_1} = 0$ and $x \pmod{n_2} = 0$, so x is divisible by n_1 and n_2. By the restriction on the domain $0 \le x \le n_1 n_2 - 1$ and to be divisible by n_1 and n_2, x has to be a multiple of $n_1 n_2$.
Therefore $x = 0$.

Now consider $x_1, x_2 \in [0, n_1 n_2 - 1]$ such that $f(x_1) = f(x_2)$
Then:
$$f(x_1) - f(x_2) = 0$$
$$(x_1 \pmod{n_1}, \ x_1 \pmod{n_2}) - (x_2 \pmod{n_1}, \ x_2 \pmod{n_2}) = 0$$
$$((x_1 - x_2) \pmod{n_1}, \ (x_1 - x_2) \pmod{n_2}) = 0$$
$$f(x_1 - x_2) = 0.$$
From above, we know that $f(x) = 0$ only if $x = 0$. Therefore, since $f(x_1 - x_2) = 0$, then $x_1 - x_2 = 0$ and so $x_1 = x_2$. This shows that

f is one-to-one

Moreover the domain, $[0, n_1 n_2 - 1]$ and the range $[0, n_1 - 1] \times [0, n_2 - 1]$ has the same number of elements and since f is one-to-one, f must also be onto. So $f(x)$ is a bijection.

2. $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{15}{30} + \frac{10}{30} + \frac{6}{30} = \frac{31}{30}$

smallest integer larger than $\frac{30}{31}$ is 1 so.

$$\frac{31}{30} = 1 + \left(\frac{31}{30} - 1\right)$$
$$= 1 + \frac{1}{30}$$

$\rightarrow \frac{31}{30} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5}$ Method 1

Divide 15 pizzas into $1/2$, distribute between 30 people.
Divide 10 pizzas into $1/3$, distribute between 30 people
Divide 6 pizzas into $1/5$ distribute between 30 people

$\rightarrow \frac{31}{30} = 1 + \frac{1}{30}$ Method 2

Give one pizza to every person.
Divide one pizza into 30 parts and give to 30 people.

Technically, it is easier to divide a pizza into $1/2$, $1/3$ and $1/5$ than $1/30$. But method 1 requires to change all pizzas, while method 2 requires only one pizza to be cut.

3. Take $f(x) = (x \pmod 3,\ x \pmod 7)$ for $0 \leq x \leq 20$. Then:

$f(0) = (0, 0)$
$f(1) = (1, 1)$
$f(2) = (2, 2)$
$f(3) = (0, 3)$
$f(4) = (1, 4)$
$f(5) = (2, 5)$
$f(6) = (0, 6)$
$f(7) = (1, 0)$

$f(8) = (2, 1)$
$f(9) = (0, 2)$
$f(10) = (1, 3)$
$f(11) = (2, 4)$
$f(12) = (0, 5)$
$f(13) = (1, 6)$
$f(14) = (2, 0)$
$f(15) = (0, 1)$

$f(16) = (1, 2)$
$f(17) = (2, 3)$
$f(18) = (0, 4)$
$f(19) = (1, 5)$
$f(20) = (2, 6)$

a) $f(x_1) = (2, 6)$ then $\underline{x_1 = 20}$

b) $f(x_2) = (1, 4)$ then $\underline{x_2 = 4}$

c) $f(x_3) = (0, 2)$ then $\underline{x_3 = 9}$