Holen Yee
Graph Theory Homework #7

7.1)

(i) Hamiltonian

(ii) Semi-Hamiltonian

(iii) Hamiltonian

(iv) Hamiltonian

(v) Hamiltonian

7.2)
Hamiltonian: 1, 4, 8, 9, 10, 18, 22, 26, 27, 28, 29, 30, 31
Semi-Hamiltonian: 2, 3, 6, 7, 13, 15, 16, 17, 19, 20, 21, 23, 24, 25

7.3)

(i) When $n \geq 3$.

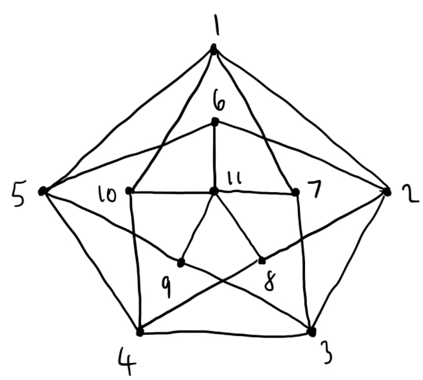
(ii) $K_{r,s}$ where $r = s$ and $r, s \geq 2$.

(iii) All of them.

(iv) When $n \geq 4$.

(v) When $k \geq 2$.

7.4)



Hamiltonian path: $1 \to 2 \to 8 \to 4 \to 10 \to 11 \to 6 \to 5 \to 9 \to 3 \to 7 \to 1$

7.5)

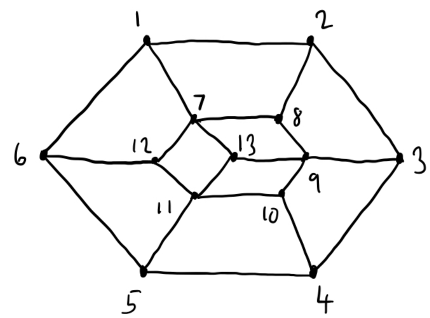
(i) Suppose for contradiction that G is a bipartite graph with an odd number of vertices and is Hamiltonian.

Let $v_1 \to \ldots \to v_n \to$ be the Hamiltonian cycle of G.

The length of this cycle is n, which is odd.

However, all cycles in bipartite graphs are of even length.

This is a contradiction, so bipartite graphs with an odd number of vertices can not be Hamiltonian.



(ii) Let $A = \{1, 3, 5, 8, 10, 12, 13\}$ and $B = \{2, 4, 6, 7, 9, 11\}$. There are no vertices x, y such that an edge exists between x and y and $x, y \in A$ or $x, y \in B$. So, the graph is bipartite. Alos, the graph has an odd number of vertices. Since the graph is bipartite and has an odd number of vertices, the graph can not be Hamiltonian.

(iii) We can construct a graph G to represent this problem. Let each vertex of G represent a square of the board and let an edge exist between two vertices if they are accessible to each other by a single knight's move. Finding a Hamiltonian path in this graph would be equivalent to finding a sequence of knight's moves which visits all squares exactly once and returns to its starting point.

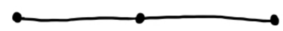
There are n^2 vertices. Since n is odd, n^2 is odd, meaning that G has an odd number of vertices.

Notice that a knight's move always connects squares of different colors. So, if we partition the set of vertices into sets A and B, defined as $A = \{v \in V : v \text{ corresponds to a white square}\}$ and $B = \{v \in V : v \text{ corresponds to a black square}\}$, we can see that the graph is bipartite.

Since G is bipartite with an odd number of vertices, G can not be Hamiltonian.

So, it is impossible to find a sequence of knight's moves which visits all squares exactly once and returns to its starting point.

7.6)
Consider the following graph:



The degree sequence of the graph is $1, 1, 2$. Both 1 and 2 are greater than or equal to $\frac{3-1}{2} = \frac{2}{2} = 1$, but the graph is not Hamiltonian.