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Graph Theory Homework #6

6.1)

(i) Eulerian

(ii) Semi-Eulerian

(iii) Neither Eulerian nor semi-Eulerian

(iv) Eulerian

(v) Neither Eulerian nor semi-Eulerian

6.2)
Eulerian: 1, 4, 8, 18, 21, 25, 31
Semi-Eulerian: 2, 3, 6, 7, 9, 13, 14, 16, 17, 19, 22, 23, 26, 28, 30

6.3)

(i) When n is odd.

(ii) $K_{r,s}$, where r and s are both even.

(iii) The octahedron.

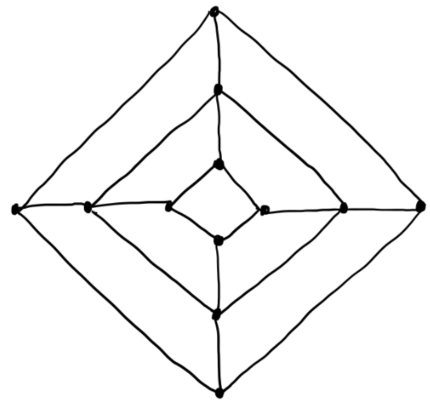
(iv) None.

(v) When k is even.

6.4)

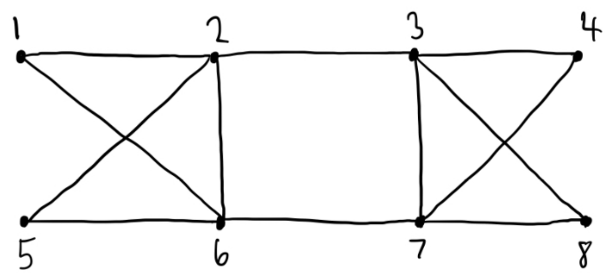
(i) Partition G into t trails which share no edges. A closed trail has no endpoints, while an open trail contributes two endpoints. Only endpoints of open trails can have odd degree. In the worst case, all trails are open and all endpoints of these open trails are distinct, meaning that $k = 2t$. In all other cases, some endpoints have even degree or some trails are closed, making the number of vertices of odd degree smaller. So, the equation that describes all cases is $k \leq 2t$, which is equivalent to $\frac{k}{2} \leq t$. This means that the minimum number of trails that include every edge of G and have no edges in common is $\frac{k}{2}$, which is exactly what we wanted to prove.

(ii) An equivalent graph for the figure is:



In this graph, there are 8 vertices of odd degree, so the minimum number of trails that together include ever edge of the graph and have no edges in common is $\frac{8}{2} = 4$. So, it will take four continuous pen strokes to draw this diagram without repeating any lines.

6.5)



Eulerian trail: $1 \to 6 \to 5 \to 2 \to 6 \to 7 \to 3 \to 8 \to 7 \to 4 \to 3 \to 2 \to 1$

6.6)

(i) Let e_{ij} be an arbitrary vertex in the line graph, representing an edge in the original graph connecting vertices v_i and v_j.

The degree of e_{ij} is $(\deg(v_i) - 1) + (\deg(v_j) - 1)$.

Since the original graph is Eulerian, $\deg(v_i)$ and $\deg(v_j)$ are even, meaning $\deg(v_i) = 2a$ and $\deg(v_j) = 2b$ for some integer a, b.

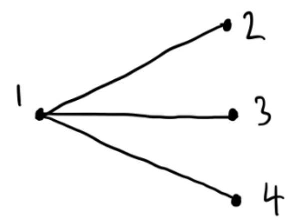
So, $\deg(e_{ij}) = (2a - 1) + (2b - 1) = 2a + 2b - 2 = 2c$, where $c = a + b - 1$.

Since $\deg(e_{ij})$ can be written as $2c$ with c being an integer, $\deg(e_{ij})$ is even.

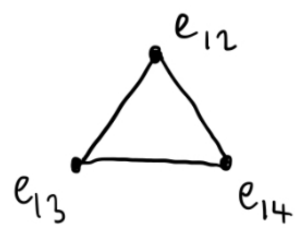
e_{ij} was chosen arbitrarily, so this is true for all vertices of the line graph.

So, since the degree of all vertices of the line graph is even, the line graph is Eulerian.

(ii) No. Consider the the following graph G:



The line graph corresponding to G is:



The degrees of all vertices of this line graph are even, so the line graph is Eulerian, but the degrees of all vertices of G are odd, so G is not Eulerian.