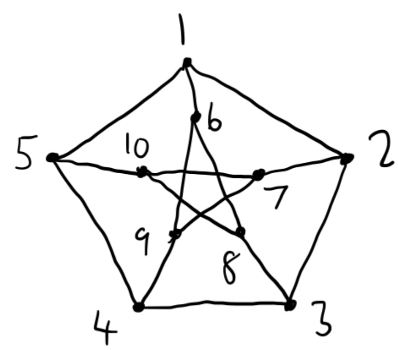
Holen Yee
Graph Theory Homework #5

5.1)



(i) $1 \to 2 \to 3 \to 4 \to 5 \to 10$

(ii) $1 \to 2 \to 3 \to 4 \to 5 \to 10 \to 7 \to 9 \to 6 \to 8$

(iii) $1 \to 2 \to 3 \to 4 \to 5 \to 1$
$1 \to 2 \to 3 \to 4 \to 9 \to 6 \to 1$
$1 \to 2 \to 3 \to 4 \to 5 \to 10 \to 8 \to 6 \to 1$
$1 \to 6 \to 9 \to 7 \to 10 \to 8 \to 3 \to 4 \to 5 \to 1$

(iv) $\{\{1,2\},\{1,5\},\{1,6\}\}$
$\{\{1,2\},\{1,5\},\{6,8\},\{6,9\}\}$
$\{\{1,6\},\{2,3\},\{2,7\},\{4,5\},\{5,10\}\}$

5.2)

(i) 3

(ii) 4

(iii) 8

(iv) 3

(v) 4

(vi) 5

(vii) 5

5.3)
Lemma: If there is an odd length closed walk in a graph, then there is an odd length closed cycle.

Proof:
We will perform induction on the number of edges k of the odd length closed walk.
The base case $k = 1$, when the closed walk is a loop, holds trivially.
Assume that, for some positive integer $r > 1$, the lemma is true for all odd numbers $k \leq 2r - 1$.
Let $W = w_1 \to \cdots \to w_{2r+1} \to w_1$ be a closed walk of $2r + 1$ edges.
If there exists two identical vertices $w_i = w_j$ for $1 < i < j \leq 2r + 1$, then W can be written as $w_1 \to \cdots \to w_i \to \cdots \to w_j \to \cdots \to w_1$.
Thus, we now have two closed walks $W_1 = w_i \to w_{i+1} \to \cdots \to w_j$ and $W_2 = w_j \to w_{j+1} \to \cdots \to w_i$.
The summation of the length of W_1 and W_2 is equal to the length of W. Since W is of odd length, one of W_1 or W_2 must be of odd length $\leq 2r - 1$.
By our assumption, there must be an odd cycle in W_1 or W_2, and thus in W.

Proof of converse of Theorem 5.1:
Let G be an arbitrary graph whose cycles are all of even length.
Assume without loss of generality that G is connected. (If G was disconnected, then we would just perform the procedure described below for each connected component of G)
Choose an arbitrary vertex of G, which we will call v_0.
Partition the set of vertices of G, V, into two sets A and B defined by the following:
$A = \{v \in V : \text{the shortest path from } v_0 \text{ to } v \text{ is even in length}\}$
$B = \{v \in V : \text{the shortest path from } v_0 \text{ to } v \text{ is odd in length}\}$
We must show that there are no edges between any two vertices in A or B.
Suppose for contradiction that there exists an edge $\{x, y\} \in E$ such that $x, y \in A$ or $x, y \in B$.
Then, we can construct the closed walk $v_0 \to \cdots \to x \to y \to \cdots \to v_0$, which is of odd length.
By the lemma, G contains an odd cycle, which is a contradiction.
Therefore, G is a bipartite graph between A and B.

5.5)

(i) $\kappa(G) = 2, \lambda(G) = 2$

(ii) $\kappa(G) = 3, \lambda(G) = 3$

(iii) $\kappa(G) = 4, \lambda(G) = 4$

(iv) $\kappa(G) = 4, \lambda(G) = 4$

5.7)

(i) $\implies$ direction:

We will prove this using induction on the length of the shortest walk between two arbitrary vertices u, v in the graph.

Base case:

Let u, v be adjacent vertices.

Let z be any vertex in G distinct from u and v.

Since G is 2-connected, the removal of u or v does not disconnect G.

So, there exists a path from v to z that does not use v, which we will call P_1, and a path from z to u that does not use u, which we will call P_2.

So, the cycle containing u, v consists of $u \to v \to P_1 \to P_2$.

Inductive step:

Assume that the proposition is true for all pairs of vertices with distance less than or equal to k, where $k \geq 1$.

Let the distance between u and v, $d(u, v)$ be $k + 1$.

Let w be the vertex adjacent to v on the shortest path from u to v.

Since $d(u, w) = k$, there is a cycle containing u and w, which consists of two paths, one from u to w, which we will call P_1, and a path from w to u, which we will call P_2.

Since G is 2-connected, there is a path from u to v that does not contain w, which we will call P_3.

If P_3 shares vertices with P_1 or P_2 (assume without loss of generality that it is P_1) let z be the shared vertex closest to v. Then, the cycle containing u and v consists of: the portion of P_1 that goes from u to $z \to$ the portion of P_3 that goes from z to $v \to w \to P_2$.

If P_3 does not share vertices with P_1 and does not share vertices with P_2, then the cycle containing u and v consists of: $P_3 \to w \to P_2$.

In both cases, we were able to construct a cycle containing u and v.

$\Longleftarrow$ direction:

If every pair of vertices u, v is in a cycle $u \to \cdots \to w \to \cdots \to v \to \cdots \to u$, with w being some other vertex, if w is removed, the path $v \to \cdots \to u$ still exists, since the cycle is a path which does not repeat vertices.

So, the removal of any one vertex does not disconnect G.

So, G is 2-connected.

(ii) A graph is 2-edge-connected if and only if each pair of edges is contained in a closed trail.

5.8)

(i) $A = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix}$

$A^2_{ij} = A_{i1}A_{1j} + \cdots + A_{in}A_{nj}$

The kth term of this summation is equal to the number of walks of length 2 from v_i to v_j via v_k. So, A^2_{ij} is equal to the number of 2-length walks from v_i to v_j.

(ii) A^2_{ii} = the number of 2-length walks from v_i to v_i. In other words, this is counting the number of edges incident to v_i or the degree of v_i. So, $\sum_{i=1}^{n} A^2_{ii}$ = the sum of the degrees of the vertices= $2m$

(iii) $A^3_{ij} = A^2_{i1}A_{1j} + \cdots + A^2_{in}A_{nj}$

The kth term of this summation is equal to the number of 3-length walks from v_i to v_j, via a 2-length walk from v_i to v_k then crossing the edge between v_k and v_j.

So, the entire sum is the number of 3-length walks from v_i to v_j.

$A^3_{ii} = 2 \cdot \#$ of 3-length walks from v_i to v_i (i.e. triangles with v_i) in them. There is a factor of 2 because both directions of the triangle are counted. For example, $v_i \to v_j \to v_k \to v_i$ and $v_i \to v_k \to v_j \to v_i$ are both counted, for some i, j, k.

$\sum_{i=1}^{n} A^3_{ii} = 6t$. The factor of 6 is there because each triangle is counted 3 times for each vertex in the triangle, and for each of those times, it is counted twice for both directions.

5.9)

(i) Let the shortest path from v to w be $v \to \cdots \to z \to \cdots \to w$, where z is some vertex distinct from v and w.

Call the fragment of the path we just described from v to z P and the fragment from z to w Q.

The sum of the lengths of P and Q is $d(v, w)$.

We claim that P and Q are the shortest paths from v to z and z to w, respectively.

Suppose for contradiction that P is not the shortest path from v to z. Call the shortest path from v to z R. Then, the path $R \to Q$ is a shorter path from v to w. This is a contradiction, so P must be the shortest path from v to z.

We can use a similar proof to show the same for Q.

So, $d(v, z) + d(z, w) = d(v, w)$.

(ii) Choose an arbitrary vertex x. Since the Petersen graph is regular of degree 3, x has 3 neighbors, call them u, v, w. Since the Petersen-graph is 3-regular, u, v, w each have exactly 2 other neighbors besides x. Since the Petersen graph has no triangles, none of the vertices in $\{u, v, w\}$ are adjacent to each other. So, the six other neighbors are distinct from x, u, v, w.

$|\{x\}| \cup |\{u, v, w\}| \cup |\{\text{six other neighbors}|\} = 1 + 3 + 6 = 10$.

The Petersen graph has 10 vertices, so all vertices are exhausted.

Every vertex distinct from x is either one of its neighbors (distance 1) or one of the six vertices at distance 2 from x.

So, for every pair of vertices in the Petersen graph v, w, $d(v, w) = 1$ or 2