Enumerating Lattice Walks

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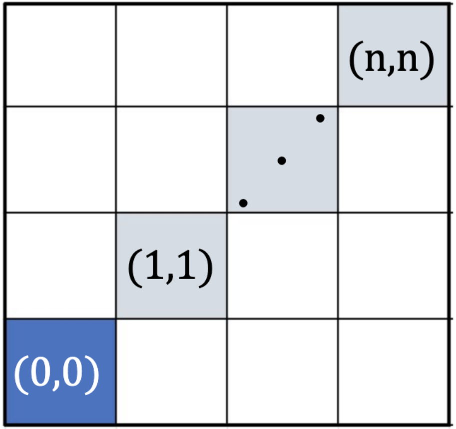
What is a Lattice Walk?

- Every position on the lattice of any dimension can be identified by its own unique coordinate
- Discrete steps of any size or direction can be taken from any position on the lattice
- Our focus is on the main diagonal of the lattice
  - Any point that has coordinates $(i, i, ..., i)$)



What Kind of Lattice Walk

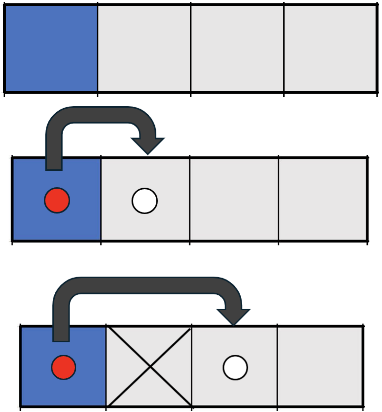
Using symmetric step sets, we want to count two types of lattice walks:

Part I: walks from the origin to a point on the main diagonal $[n, n, \ldots, n]$.

Part II: walks from the origin back to the origin in n steps.

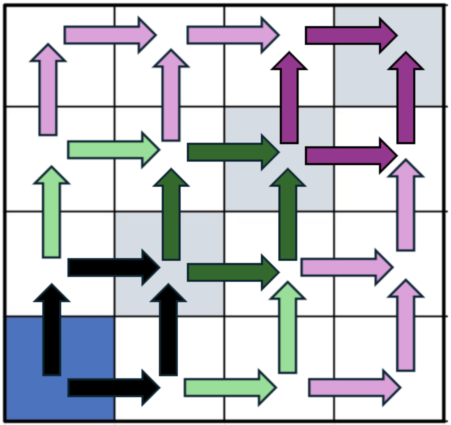
Easy 1 Dimension Walk

- Step sequence $:= \{[1]\}$
  - The is only one possible way to reach any location on a 1 dimensional lattice in the fewest number of steps
  - This yields a sequence of $[1, 1, 1, ...]$
- Step sequence $:= \{[2]\}$
  - In this case, it is impossible to ever reach the odd positions of the lattice.
  - This yields a sequence of $[0, 1, 0, 1, ...]$
- Step sequence $:= \{[i]\}$
  - In fact, $\forall\ i$ the resulting sequence is $[0\$(i-1), 1, 0\$(i-1), 1, ...]$



More Complicated 2 Dimensional Walk

- With only the set of steps being $\{[1,0],[0,1]\}$, it becomes increasingly more complicated to enumerate the possibilities of just reaching a point on the diagonal
  - There are paths that stay close to the diagonal as shown by the darker colors.
  - But one can also take multiple steps in a single direction before turning to go to the diagonal
- The resulting sequence for just getting to the point on the diagonal without returning is as follows
  - $[2, 6, 20, 70, 252, 924, 3432, 12870, 48620, 184756, \ldots]$



Approaching the Problem

- By using Dr. Z's maple packages, we can easily enumerate sequences for a given set of symmetric steps.
- Listed on the right are the sets of symmetric steps
- We can generate these sequences using NuWalksDiaSeq and NuWalksBackSeq

```
A:= {[1,0],[0,1]}:
B:= {[1,0],[0,1],[1,1]}:
C:= {[2,0],[0,2],[1,1]}:
D:= {[1,0],[0,1],[2,1],[1,2]}:
E:= {[2,1],[1,2]}:
F:= {[1,0],[0,1],[2,1],[1,2],[2,2]}:
G:= {[1,0,0],[0,1,0],[0,0,1]}:
H:= {[1,1,0],[1,0,1],[0,1,1]}:
I:= {[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[0,1,1]}:
J:= {[1,0,0],[0,1,0],[0,0,1],[1,1,1]}:
K:= {[1,1,0],[1,0,1],[0,1,1],[1,1,1]}:
L:= {[1,0,0],[0,1,0],[0,0,1],[2,0,0],[0,2,0],[0,0,2]}:
M:= {[1,0,0],[0,1,0],[0,0,1],[2,1,0],[2,0,1],[1,2,0],[1,0,2],[0,2,1],[0,1,2]}:
N:= {[1,0,0],[0,1,0],[0,0,1],[2,1,1],[1,2,1],[1,1,2]}:
O:= {[1,1,0],[1,0,1],[0,1,1],[2,1,0],[2,0,1],[1,2,0],[1,0,2],[0,2,1],[0,1,2]}:
P:= {[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[0,1,1],[1,1,1]}:
Q:= {[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[0,1,1],[2,0,0],[0,2,0],[0,0,2]}:
R:= {[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[0,1,1],[2,1,0],[2,0,1],[1,2,0],[1,0,2],[0,2,1],[0,1,2]}:
```

Two Main Counting Problems

- **Part I: Diagonal walks**
  - Given a symmetric set of positive steps, count the number of walks from the origin to $[n, n, \ldots, n]$
  - In Maple, we generated these sequences using:

```
NuWalksDiaSeq(Steps,L)
```

- **Part II: Return-to-origin walks**
  - Given a symmetric set of steps, also include the reverse of each step.
  - Count the number of walks that start and end at the origin after n steps.
  - In Maple, we generated these sequences using:

```
NuWalksBackSeq(Steps,K)
```

OEIS Example: Simplest 2D Diagonal Walks

- Step set:

$$S = \{[1,0],[0,1]\}$$

- We count walks from the origin to $[n,n]$
- Sequence:

$$2, 6, 20, 70, 252, 924, 3432, \ldots$$

- OEIS: A000984

OEIS Example: 2D Diagonal Walks

- Step set:

$$S = \{[1,0],[0,1],[2,1],[1,2],[2,2]\}$$

- We count walks from the origin to $[n, n]$
- Sequence:

$$2, 11, 52, 269, 1414, 7575, 41064, 224665, \ldots$$

- OEIS: A026933

OEIS Example: 3D Diagonal Walks

- Step set:

$$S = \{[1,1,0],[1,0,1],[0,1,1],[1,1,1]\}$$

- We count walks from the origin to $[n,n,n]$
- Sequence:

$$1, 7, 25, 151, 751, 4411, 24697, 146455, 862351, \ldots$$

- OEIS: A208425

OEIS Example: 3D Diagonal Walks

- Step set:

$$S = \{[1,0,0],[0,1,0],[0,0,1],[2,1,1],[1,2,1],[1,1,2]\}$$

- We count walks from the origin to [n,n,n] using NuWalksDiaSeq
- Sequence:

$$6, 108, 2238, 51126, 1234836, 30933846, 795124008, \ldots$$

- OEIS: A361728

OEIS Example: 2D Walk from Origin and Back

- Step set:
$$S = \{[1,0],[0,1],[1,1]\}$$
- Sequence:
$$0, 6, 12, 90, 360, 2040, 10080, 54810, 290640, 1588356, \ldots$$
- OEIS: A002898
- Interestingly, the sequence refers to the number of n-step closed paths on hexagonal lattice.

OEIS Example: 3D Walk from Origin and Back

- Step set:

$$S = \{[1,0,0],[0,1,0],[0,0,1],[1,1,1]\}$$

- Sequence:

$$0, 8, 0, 216, 0, 8000, 0, 343000, 0, 16003008, 0, 788889024, 0, 40424237568, 0, 2131746903000,$$

- OEIS: A002897 (for the nonzero terms)

OEIS Example: 3D Walk from Origin and Back

- Step set:

$$S = \{[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[0,1,1],[1,1,1]\}$$

- Sequence:

$$0, 14, 72, 882, 8400, 95180, 1060080, 12389650, \ldots$$

- OEIS: A328735

Non-OEIS Example: 2D Walk from Origin and Back

- Step set:

$$S = \{[2,0],[0,2],[1,1]\}$$

- Sequence:

$$0, 6, 0, 114, 0, 2820, 0, 77490, 0, 2256156, 0, 68181036, \ldots$$

- Asymptotic Expression

$$\frac{3^{(n+1)}}{n} * \left(\frac{1}{n}\right)^{\frac{1}{2}} * \left(-\frac{16}{3} * n + 1 - \frac{\frac{1}{96}}{n} - \frac{\frac{45}{512}}{n^2} - \frac{\frac{2233}{32768}}{n^3} + \frac{\frac{8799}{524288}}{n^4}\right)$$

Non-OEIS Example: 3D Walks to $[n, n, n]$

- Step set:

$$S = \{[1,0,0],[0,1,0],[0,0,1],[2,0,0],[0,2,0],[0,0,2]\}$$

- We count walks from the origin to $[n, n, n]$
- Sequence:

$$6, 222, 8280, 347850, 15381828, 705379416, \ldots$$

- This sequence did not appear in the OEIS.
- This was too complex for Maple to find a recurrence and thus we could not find an asymptotic expression for this.

Comparing the two Stepping Problems

- Returning back to the origin always allows for more possibilities for a given set of symmetric steps
- The clearest example uses
$$S = \{[1,0],[0,1]\}$$
- To point on diagonal:
$$2, 6, 20, 70, 252, 924, \ldots$$
- Returning to origin (without 0s)
$$4, 36, 400, 4900, 63504, 853776, \ldots$$
- Although this relationship does not hold in all cases, in this most simple 2D case, the relationship between the two sequences is clear. Returning to the origin has n^2 options.

Thank you!