CHAPTER IV. THE CHARACTERS OF GL_n OVER A FINITE FIELD

CHAPTER V. THE HECKE RING OF GL_n OVER A LOCAL FIELD

FROM "Symmetric Functions and Hall polynomials" by I. G. Macdonald

I

SYMMETRIC FUNCTIONS

1. Partitions

Many of the objects we shall consider in this book will turn out to be parametrized by partitions. The purpose of this section is to lay down some notation and terminology which will be used throughout, and to collect together some elementary results on orderings of partitions which will be used later.

Partitions

A *partition* is any (finite or infinite) sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \tag{1.1}$$

of non-negative integers in decreasing order;

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r \geqslant \ldots$$

and containing only finitely many non-zero terms. We shall find it convenient not to distinguish between two such sequences which differ only by a string of zeros at the end. Thus, for example, we regard (2, 1), (2, 1, 0), (2, 1, 0, 0,...) as the same partition.

The non-zero λ_i in (1.1) are called the *parts* of λ. The number of parts is the *length* of λ, denoted by $l(\lambda)$; and the sum of the parts is the *weight* of λ, denoted by $|\lambda|$:

$$|\lambda| = \lambda_1 + \lambda_2 + \ldots .$$

If $|\lambda| = n$ we say that λ is a *partition of n*. The set of all partitions of n is denoted by $\mathcal{P}_n$, and the set of all partitions by $\mathcal{P}$. In particular, $\mathcal{P}_0$ consists of a single element, the unique partition of zero, which we denote by 0.

Sometimes it is convenient to use a notation which indicates the number of times each integer occurs as a part:

$$\lambda = (1^{m_1} 2^{m_2} \ldots r^{m_r} \ldots)$$

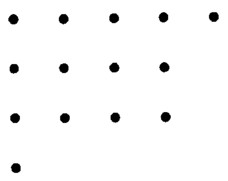
means that exactly m_i of the parts of λ are equal to i. The number

$$m_i = m_i(\lambda) = \mathrm{Card}\{j : \lambda_j = i\} \tag{1.2}$$

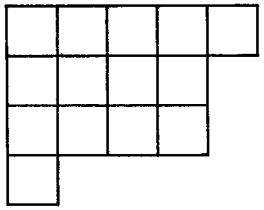
is called the *multiplicity* of i in λ.

Diagrams

The *diagram* of a partition λ may be formally defined as the set of points $(i, j) \in \mathbf{Z}^2$ such that $1 \leqslant j \leqslant \lambda_i$. In drawing such diagrams we shall adopt the convention, as with matrices, that the first coordinate i (the row index) increases as one goes downwards, and the second cordinate j (the column index) increases as one goes from left to right.† For example, the diagram of the partition (5441) is



consisting of 5 points of nodes in the top row, 4 in the second row, 4 in the third row, and 1 in the fourth row. More often it is convenient to replace the nodes by squares, in which case the diagram is



We shall usually denote the diagram of a partition λ by the same symbol λ.

The *conjugate* of a partition λ is the partition λ′ whose diagram is the transpose of the diagram λ, i.e. the diagram obtained by reflection in the main diagonal. Hence λ'_i is the number of nodes in the ith column of λ, or equivalently

$$(1.3) \qquad \lambda'_i = \mathrm{Card}\{j : \lambda_j \geqslant i\}.$$

In particular, $\lambda'_1 = l(\lambda)$ and $\lambda_1 = l(\lambda')$. Obviously $\lambda'' = \lambda$.

For example, the conjugate of (5441) is (43331).

From (1.2) and (1.3) we have

$$(1.4) \qquad m_i(\lambda) = \lambda'_i - \lambda'_{i+1}.$$

For each partition λ we define

$$(1.5) \qquad n(\lambda) = \sum_{i \geqslant 1} (i-1)\lambda_i,$$

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† Some authors (especially Francophones) prefer the convention of coordinate geometry (in which the first coordinate increases from left to right and the second coordinate from bottom to top) and define the diagram of λ to be the set of $(i, j) \in \mathbf{Z}^2$ such that $1 \leqslant i \leqslant \lambda_j$. Readers who prefer this convention should read this book upside down in a mirror.

so that $n(\lambda)$ is the sum of the numbers obtained by attaching a zero to each node in the top row of the diagram of λ, a 1 to each node in the second row, and so on. Adding up the numbers in each column, we see that

$$(1.6) \qquad n(\lambda) = \sum_{i \geqslant 1} \binom{\lambda'_i}{2}.$$

Another notation for partitions which is occasionally useful is the following, due to Frobenius. Suppose that the main diagonal of the diagram of λ consists of r nodes (i, i) $(1 \leqslant i \leqslant r)$. Let $\alpha_i = \lambda_i - i$ be the number of nodes in the ith row of λ to the right of (i, i), for $1 \leqslant i \leqslant r$, and let $\beta_i = \lambda'_i - i$ be the number of nodes in the ith column of λ below (i, i), for $1 \leqslant i \leqslant r$. We have $\alpha_1 > \alpha_2 > \ldots > \alpha_r \geqslant 0$ and $\beta_1 > \beta_2 > \ldots > \beta_r \geqslant 0$, and we denote the partition λ by

$$\lambda = (\alpha_1, \ldots, \alpha_r \mid \beta_1, \ldots, \beta_r) = (\alpha \mid \beta).$$

Clearly the conjugate of $(\alpha \mid \beta)$ is $(\beta \mid \alpha)$.

For example, if $\lambda = (5441)$ we have $\alpha = (421)$ and $\beta = (310)$.

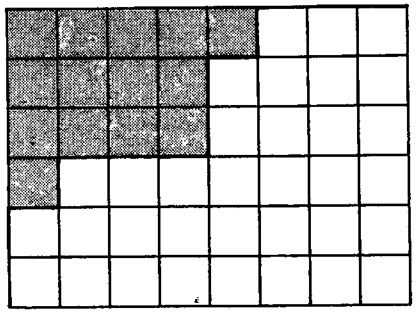
(1.7) *Let λ be a partition and let $m \geqslant \lambda_1$, $n \geqslant \lambda'_1$. Then the $m+n$ numbers*

$$\lambda_i + n - i \quad (1 \leqslant i \leqslant n), \qquad n - 1 + j - \lambda'_j \quad (1 \leqslant j \leqslant m)$$

are a permutation of $\{0, 1, 2, \ldots, m+n-1\}$.

Proof. The diagram of λ is contained in the diagram of (m^n), which is an $n \times m$ rectangle. Number the successive segments of the boundary line between λ and its complement in (m^n) (marked thickly in the picture) with the numbers $0, 1, \ldots, m+n-1$, starting at the bottom. The numbers attached to the vertical segments are $\lambda_i + n - i$ $(1 \leqslant i \leqslant n)$, and by transposition those attached to the horizontal segments are

$$(m+n-1) - (\lambda'_j + m - j) = n - 1 + j - \lambda'_j \qquad (1 \leqslant j \leqslant m). \quad |$$



$\lambda = (54^21)$, $m = 8$, $n = 6$

Let

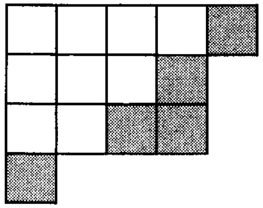
$$f_{\lambda,n}(t)=\sum_{i=1}^{n} t^{\lambda_i+n-i}.$$

Then (1.7) is equivalent to the identity

(1.7′) $$f_{\lambda,n}(t)+t^{m+n-1}f_{\lambda',m}(\overset{\vee}{t})=(1-t^{m+n})/(1-t). \qquad {}^{-1}$$

Skew diagrams and tableaux

If λ, μ are partitions, we shall write $\lambda\supset\mu$ to mean that the diagram of λ contains the diagram of μ, i.e. that $\lambda_i\geqslant\mu_i$ for all $i\geqslant 1$. The set-theoretic difference $\theta=\lambda-\mu$ is called a *skew diagram*. For example, if $\lambda=(5441)$ and $\mu=(432)$, the skew diagram $\lambda-\mu$ is the shaded region in the picture below:



A *path* in a skew diagram θ is a sequence $x_0, x_1,\ldots, x_m$ of squares in θ such that x_{i-1} and x_i have a common side, for $1\leqslant i\leqslant m$. A subset φ of θ is said to be connected if any two squares in φ can be connected by a path in φ. The maximal connected subsets of θ are themselves skew diagrams, called the *connected components* of θ. In the example above, there are three connected components.

The *conjugate* of a skew diagram $\theta=\lambda-\mu$ is $\theta'=\lambda'-\mu'$. Let $\theta_i=\lambda_i-\mu_i$, $\theta'_i=\lambda'_i-\mu'_i$, and

$$|\theta|=\sum\theta_i=|\lambda|-|\mu|.$$

A skew diagram θ is a *horizontal m-strip* (resp. a *vertical m-strip*) if $|\theta|=m$ and $\theta'_i\leqslant 1$ (*resp.* $\theta_i\leqslant 1$) for each $i\geqslant 1$. In other words, a horizontal (resp. vertical) strip has at most one square in each column (resp. row).

A *tableau T* is a sequence of partitions

$$\mu=\lambda^{(0)}\subset\lambda^{(1)}\subset\ldots\subset\lambda^{(r)}=\lambda$$

such that each skew diagram $\theta^{(i)}=\lambda^{(i)}-\lambda^{(i-1)}$ $(1\leqslant i\leqslant r)$ is a horizontal strip. Graphically, T may be be described by numbering each square of the skew diagram $\theta^{(i)}$ with the number i, for $1\leqslant i\leqslant r$, and we shall often think of a tableau as a numbered skew diagram in this way. The numbers inserted in $\lambda-\mu$ must increase strictly down each column, and weakly from left to right along each row. The skew diagram $\lambda-\mu$ is called the *shape* of the tableau T, and the sequence $(|\theta^{(1)}|,\ldots,|\theta^{(r)}|)$ is called the *weight* of T.

A *standard tableau* is a tableau T in which each number $1,2,\ldots,r$ occurs exactly once, so that its weight is $(1,1,\ldots,1)$.

Addition of partitions

Let λ, μ be partitions. We define $\lambda+\mu$ to be the sum of the sequences λ, μ:

$$(\lambda+\mu)_i=\lambda_i+\mu_i.$$

Also we define $\lambda\cup\mu$ to be the partition whose parts are those of λ and of μ, arranged in descending order. For example, if $\lambda=(321)$ and $\mu=(22)$, then $\lambda+\mu=(541)$ and $\lambda\cup\mu=(32221)$.

The operations $+$ and $\cup$ are dual to each other:

(1.8) $$(\lambda\cup\mu)'=\lambda'+\mu'.$$

Proof. The diagram of $\lambda\cup\mu$ is obtained by taking the rows of the diagrams of λ and μ and reassembling them in order of decreasing length. Hence the length of the ith column of $\lambda\cup\mu$ is the sum of the lengths of the ith columns of λ and of μ, i.e.

$$(\lambda\cup\mu)'_i=\lambda'_i+\mu'_i. \quad |$$

Orderings

Let L_n denote the *reverse lexicographic ordering* on the set $\mathscr{P}_n$ of partitions of n: that is to say, L_n is the subset of $\mathscr{P}_n\times\mathscr{P}_n$ consisting of all (λ,μ) such that either $\lambda=\mu$ or else the first non-vanishing difference $\lambda_i-\mu_i$ is *positive*. L_n is a total ordering. For example, when $n=5$, L_n arranges $\mathscr{P}_5$ in the sequence

$$(5), (41), (32), (31^2), (2^21), (21^3), (1^5).$$

Another total ordering on $\mathscr{P}_n$ is L'_n, the set of all (λ,μ) such that either $\lambda=\mu$ or else the first non-vanishing difference $\tilde{\lambda}_i-\tilde{\mu}_i$ is *negative*, where $\tilde{\lambda}_i=\lambda_{n+1-i}$. The orderings L_n, L'_n are distinct as soon as $n\geqslant 6$. For example, if $\lambda=(31^3)$ and $\mu=(2^3)$ we have $(\lambda,\mu)\in L$ and $(\mu,\lambda)\in L'$.

(1.9) *Let* $\lambda,\mu\in\mathscr{P}_n$. *Then*

$$(\lambda,\mu)\in L'_n\Leftrightarrow(\mu',\lambda')\in L_n.$$

Proof. Suppose that $(\lambda,\mu)\in L'_n$ and $\lambda\neq\mu$. Then for some integer $i\geqslant 1$ we have $\lambda_i<\mu_i$, and $\lambda_j=\mu_j$ for $j>i$. If we put $k=\lambda_i$ and consider the diagrams of λ and μ, we see immediately that $\lambda'_j=\mu'_j$ for $1\leq j\leq k$, and that $\lambda'_{k+1}<\mu'_{k+1}$, so that $(\mu',\lambda')\in L_n$. The converse is proved similarly. |

An ordering which is more important than either L_n or L'_n is the *natural* (partial) *ordering* N_n on $\mathscr{P}_n$, defined as follows:

$$(\lambda,\mu)\in N_n\Leftrightarrow\lambda_1+\dots+\lambda_i\geq\mu_1+\dots+\mu_i\quad\text{for all } i\geq 1.$$

As soon as $n\geq 6$, N_n is not a total ordering. For example, the partitions (31^3) and (2^3) are incomparable with respect to N_6.

We shall write $\lambda\geq\mu$ in place of $(\lambda,\mu)\in N_n$.

(1.10) *Let* $\lambda,\mu\in\mathscr{P}_n$. *Then*

$$\lambda\geq\mu\Rightarrow(\lambda,\mu)\in L_n\cap L'_n.$$

Proof. Suppose that $\lambda\geq\mu$. Then either $\lambda_1>\mu_1$, in which case $(\lambda,\mu)\in L_n$, or else $\lambda_1=\mu_1$. In that case either $\lambda_2>\mu_2$, in which case again $(\lambda,\mu)\in L_n$, or else $\lambda_2=\mu_2$. Continuing in this way, we see that $(\lambda,\mu)\in L_n$.

Also, for each $i\geq 1$, we have

$$\begin{aligned}\lambda_{i+1}+\lambda_{i+2}+\dots&=n-(\lambda_1+\dots+\lambda_i)\\&\leq n-(\mu_1+\dots+\mu_i)\\&=\mu_{i+1}+\mu_{i+2}+\dots.\end{aligned}$$

Hence the same reasoning as before shows that $(\lambda,\mu)\in L'_n$. |

Remark. It is not true in general that $N_n=L_n\cap L'_n$. For example, when $n=12$ and $\lambda=(63^2)$, $\mu=(5^21^2)$ we have $(\lambda,\mu)\in L_{12}\cap L'_{12}$, but $(\lambda,\mu)\notin N_{12}$.

(1.11) *Let* $\lambda,\mu\in\mathscr{P}_n$. *Then*

$$\lambda\geq\mu\Leftrightarrow\mu'\geq\lambda'.$$

Proof. Clearly it is enough to prove one implication. Suppose then that $\mu'\not\geq\lambda'$. Then for some $i\geq 1$ we have

$$\lambda'_1+\dots+\lambda'_j\leq\mu'_1+\dots+\mu'_j\qquad(1\leq j\leq i-1)$$

and

$$\lambda'_1+\dots+\lambda'_i>\mu'_1+\dots+\mu'_i\tag{1}$$

from which it follows that $\lambda'_i>\mu'_i$.

Let $l=\lambda'_i$, $m=\mu'_i$. From (1) it follows that

$$\lambda'_{i+1}+\lambda'_{i+2}+\dots<\mu'_{i+1}+\mu'_{i+2}+\dots.\tag{2}$$

Now $\lambda'_{i+1}+\lambda'_{i+2}+\dots$ is equal to the number of nodes in the diagram of λ

which lie to the right of the ith column, and therefore

$$\lambda'_{i+1}+\lambda'_{i+2}+\dots=\sum_{j=1}^{l}(\lambda_j-i).$$

Likewise

$$\mu'_{i+1}+\mu'_{i+2}+\dots=\sum_{j=1}^{m}(\mu_j-i).$$

Hence from (2) we have

$$\sum_{j=1}^{m}(\mu_j-i)>\sum_{j=1}^{l}(\lambda_j-i)\geq\sum_{j=1}^{m}(\lambda_j-i)\tag{3}$$

in which the right-hand inequality holds because $l>m$ and $\lambda_j\geq i$ for $1\leq j\leq l$. From (3) we have

$$\mu_1+\dots+\mu_m>\lambda_1+\dots+\lambda_m$$

and therefore $\lambda\not\geq\mu$. |

Raising operators

In this subsection we shall work not with partitions but with integer vectors $a=(a_1,\dots,a_n)\in\mathbf{Z}^n$. The symmetric group S_n acts on $\mathbf{Z}^n$ by permuting the coordinates, and the set

$$P_n=\{b\in\mathbf{Z}^n:b_1\geq b_2\geq\dots\geq b_n\}$$

is a fundamental domain for this action, i.e. the S_n-orbit of each $a\in\mathbf{Z}^n$ meets P_n in exactly one point, which we denote by a^+. Thus a^+ is obtained by rearranging $a_1,\dots,a_n$ in descending order of magnitude.

For $a,b\in\mathbf{Z}^n$ we define $a\geq b$ as before to mean

$$a_1+\dots+a_i\geq b_1+\dots+b_i\qquad(1\leq i\leq n).$$

(1.12) *Let* $a\in\mathbf{Z}^n$. *Then*

$$a\in P_n\Leftrightarrow a\geq wa\quad\textit{for all } w\in S_n.$$

Proof. Suppose that $a\in P_n$, i.e., $a_1\geq\dots\geq a_n$. If $wa=b$, then $(b_1,\dots,b_n)$ is a permutation of $(a_1,\dots,a_n)$, and therefore

$$a_1+\dots+a_i\geq b_1+\dots+b_i\qquad(1\leq i\leq n)$$

so that $a\geq b$.

Conversely, if $a\geq wa$ for all $w\in S_n$ we have in particular

$$(a_1,\dots,a_n)\geq(a_1,\dots,a_{i-1},a_{i+1},a_i,a_{i+2},\dots,a_n)$$

for $1 \leqslant i \leqslant n-1$, from which it follows that

$$a_1 + \dots + a_{i-1} + a_i \geqslant a_1 + \dots + a_{i-1} + a_{i+1},$$

i.e., $a_i \geqslant a_{i+1}$. Hence $a \in P_n$. |

Let $\delta = (n-1, n-2, \dots, 1, 0) \in P_n$.

(1.13) *Let* $a \in P_n$. *Then for each* $w \in S_n$ *we have*

$$(a + \delta - w\delta)^+ \geqslant a.$$

Proof. Since $\delta \in P_n$ we have $\delta \geqslant w\delta$ by (1.12), hence

$$a + \delta - w\delta \geqslant a.$$

Let $b = (a + \delta - w\delta)^+$. Then again by (1.12) we have

$$b \geqslant a + \delta - w\delta.$$

Hence $b \geqslant a$. |

For each pair of integers i, j such that $1 \leqslant i < j \leqslant n$ define R_{ij}: $\mathbf{Z}^n \to \mathbf{Z}^n$ by

$$R_{ij}(a) = (a_1, \dots, a_i + 1, \dots, a_j - 1, \dots, a_n).$$

Any product $R = \prod_{i<j} R_{ij}^{r_{ij}}$ is called a *raising operator.* The order of the terms in the product is immaterial, since they commute with each other.

(1.14) *Let* $a \in \mathbf{Z}^n$ *and let* R *be a raising operator. Then*

$$(Ra)^+ \geqslant a.$$

Proof. Let R be as above. If $b = Ra$ we have

$$b_i = a_i + \sum_{j=1}^{n} r_{ij}$$

where, if $i > j$, r_{ij} is defined to be $-r_{ji}$, and $r_{ii} = 0$. Hence if $c = b^+$ we have

$$c_1 + \dots + c_k \geqslant b_1 + \dots + b_k = a_1 + \dots + a_k + \sum_{i=1}^{k} \sum_{j=1}^{n} r_{ij}.$$

But the matrix (r_{ij}) is skew-symmetric, so that

$$\sum_{i=1}^{k} \sum_{j=1}^{k} r_{ij} = 0.$$

Hence

$$c_1 + \dots + c_k \geqslant a_1 + \dots + a_k + \sum_{i=1}^{k} \sum_{j=k+1}^{n} r_{ij} \geqslant a_1 + \dots + a_k$$

because $r_{ij} \geqslant 0$ for $i < j$. Hence $c \geqslant a$, as required. |

Conversely:

(1.15) *Let* $a, b \in \mathbf{Z}^n$ *be such that* $a \leqslant b$ *and* $a_1 + \dots + a_n = b_1 + \dots + b_n$. *Then there exists a raising operator* R *such that* $b = Ra$.

Proof. We may take

$$R = \prod_{k=1}^{n-1} R_{k,k+1}^{r_k}$$

where

$$r_k = \sum_{i=1}^{k} (b_i - a_i) \geqslant 0. \quad |$$

Examples

1. Let λ be a partition. The *hook-length* of λ at $x = (i, j) \in \lambda$ is defined to be

$$h(x) = h(i, j) = \lambda_i + \lambda'_j - i - j + 1.$$

From (1.7′), with λ and λ' interchanged, and $m = \lambda_1$, we have

$$\sum_{j=1}^{\lambda_1} t^{\lambda'_j + \lambda_1 - j} + \sum_{j=1}^{n} t^{\lambda_1 - 1 + j - \lambda_j} = \sum_{j=0}^{\lambda_1 + n - 1} t^j$$

or, putting $\mu_i = \lambda_i + n - i$ $(1 \leqslant i \leqslant n)$,

$$\sum_{j=1}^{\lambda_1} t^{h(1,j)} + \sum_{j=2}^{n} t^{\mu_1 - \mu_j} = \sum_{j=1}^{\mu_1} t^j. \tag{1}$$

By writing down this identity for the partition $(\lambda_i, \lambda_{i+1}, \dots)$ and then summing over $i = 1, 2, \dots, l(\lambda)$ we obtain

$$\sum_{x \in \lambda} t^{h(x)} + \sum_{i<j} t^{\mu_i - \mu_j} = \sum_{i \geqslant 1} \sum_{j=1}^{\mu_i} t^j. \tag{2}$$

From (2) it follows that

$$\prod_{x \in \lambda} (1 - t^{h(x)}) = \frac{\prod_{i \geqslant 1} \prod_{j=1}^{\mu_i} (1 - t^j)}{\prod_{i<j} (1 - t^{\mu_i - \mu_j})} \tag{3}$$

and in particular, by dividing both sides of (3) by $(1-t)^{|\lambda|}$ and then setting $t = 1$, that

$$\prod_{x \in \lambda} h(x) = \frac{\prod_{i \geqslant 1} \mu_i!}{\prod_{i<j} (\mu_i - \mu_j)}. \tag{4}$$

2. The sum of the hook-lengths of λ is

$$\sum_{x\in\lambda} h(x) = n(\lambda)+n(\lambda')+|\lambda|.$$

3. For each $x=(i,j)\in\lambda$, the *content* of x is defined to be $c(x)=j-i$. We have

$$\sum_{x\in\lambda} c(x) = n(\lambda')-n(\lambda).$$

If n is any integer $\geqslant l(\lambda)$, the numbers $n+c(x)$ for x in the ith row of λ are $n-i+1,\dots,n-i+\lambda_i$, and therefore

$$\prod_{x\in\lambda}(1-t^{n+c(x)}) = \prod_{i\geqslant 1}\frac{\varphi_{\lambda_i+n-i}(t)}{\varphi_{n-i}(t)}$$

where $\varphi_r(t)=(1-t)(1-t^2)\dots(1-t^r)$.

4. If $\lambda=(\lambda_1,\dots,\lambda_n)=(\alpha_1,\dots,\alpha_r\mid\beta_1,\dots,\beta_r)$ in Frobenius notation, then

$$\sum_{i=1}^{n} t^i(1-t^{-\lambda_i}) = \sum_{j=1}^{r}(t^{\beta_j+1}-t^{-\alpha_j}).$$

Notes and references

The idea of representing a partition by a diagram goes back to Ferrers and Sylvester, and the diagram of a partition is called by some authors the Ferrers (often misspelt) diagram or graph. Tableaux and raising operators were introduced by Alfred Young in his series of papers on quantitative substitutional analysis [55].

2. The ring of symmetric functions

Consider the ring $\mathbf{Z}[x_1,\dots,x_n]$ of polynomials in n independent variables $x_1,\dots,x_n$ with rational integer coefficients. The symmetric group S_n acts on this ring by permuting the variables, and a polynomial is *symmetric* if it is invariant under this action. The symmetric polynomials form a subring

$$\Lambda_n = \mathbf{Z}[x_1,\dots,x_n]^{S_n}.$$

Λ_n is a graded ring: we have

$$\Lambda_n = \bigoplus_{k\geqslant 0}\Lambda_n^k$$

where Λ_n^k consists of the homogeneous symmetric polynomials of degree k, together with the zero polynomial.

For each $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbf{N}^n$ we denote by x^α the monomial

$$x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n}.$$

Let λ be any partition of length $\leqslant n$. The polynomial

$$m_\lambda(x_1,\dots,x_n)=\sum x^\alpha \tag{2.1}$$

summed over all distinct permutations α of $\lambda=(\lambda_1,\dots,\lambda_n)$, is clearly symmetric, and the m_λ (as λ runs through all partitions of length $\leqslant n$) form a $\mathbf{Z}$-basis of Λ_n. Hence the m_λ such that $l(\lambda)\leqslant n$ and $|\lambda|=k$ form a $\mathbf{Z}$-basis of Λ_n^k; in particular, as soon as $n\geqslant k$, the m_λ such that $|\lambda|=k$ form a $\mathbf{Z}$-basis of Λ_n^k.

In the theory of symmetric functions, the number of variables is usually irrelevant, provided only that it is large enough, and it is often more convenient to work with symmetric functions in infinitely many variables. To make this idea precise, let $m\geqslant n$ and consider the homomorphism

$$\mathbf{Z}[x_1,\dots,x_m]\to\mathbf{Z}[x_1,\dots,x_n]$$

which sends each of $x_{m+1},\dots,x_n$ to zero and the other x_i to themselves. On restriction to Λ_m this gives a homomorphism

$$\rho_{m,n}:\Lambda_m\to\Lambda_n$$

whose effect on the basis (m_λ) is easily described; it sends $m_\lambda(x_1,\dots,x_m)$ to $m_\lambda(x_1,\dots,x_n)$ if $l(\lambda)\leqslant n$, and to 0 if $l(\lambda)>n$. It follows that $\rho_{m,n}$ is surjective. On restriction to Λ_m^k we have homomorphisms

$$\rho_{m,n}^k:\Lambda_m^k\to\Lambda_n^k$$

for all $k\geqslant 0$ and $m\geqslant n$, which are always surjective, and are bijective for $m\geqslant n\geqslant k$.

We now form the inverse limit

$$\Lambda^k = \varprojlim_n \Lambda_n^k$$

of the $\mathbf{Z}$-modules Λ_n^k relative to the homomorphisms $\rho_{m,n}^k$: an element of Λ^k is by definition a sequence $f=(f_n)_{n\geqslant 0}$, where each $f_n=f_n(x_1,\dots,x_n)$ is a homogeneous symmetric polynomial of degree k in $x_1,\dots,x_n$, and $f_m(x_1,\dots,x_n,0,\dots,0)=f_n(x_1,\dots,x_n)$ whenever $m\geqslant n$. Since $\rho_{m,n}^k$ is an isomorphism for $m\geqslant n\geqslant k$, it follows that the projection

$$\rho_n^k:\Lambda^k\to\Lambda_n^k,$$

which sends f to f_n, is an isomorphism for all $n\geqslant k$, and hence that Λ^k has a $\mathbf{Z}$-basis consisting of the *monomial symmetric functions* m_λ (for all partitions λ of k) defined by

$$\rho_n^k(m_\lambda)=m_\lambda(x_1,\dots,x_n)$$

for all $n \geqslant k$. Hence Λ^k is a free $\mathbf{Z}$-module of rank $p(k)$, the number of partitions of k.

Now let

$$\Lambda = \bigoplus_{k \geqslant 0} \Lambda^k,$$

so that Λ is the free $\mathbf{Z}$-module generated by the m_λ for *all* partitions λ. We have surjective homomorphisms

$$\rho_n = \bigoplus_{k \geqslant 0} \rho_n^k : \Lambda \to \Lambda_n$$

for each $n \geqslant 0$, and ρ_n is an isomorphism in degrees $k \leqslant n$.

It is clear that Λ has a structure of a graded ring such that the ρ_n are ring homomorphisms. The graded ring Λ thus defined is called the *ring of symmetric functions*† in countably many independent variables $x_1, x_2, \dots$.

Remarks. 1. Λ is *not* the inverse limit (in the category of rings) of the rings Λ_n relative to the homomorphisms $\rho_{m,n}$. This inverse unit, $\hat{\Lambda}$ say, contains for example' the infinite product $\prod_{i=1}^{\infty} (1+x_i)$, which does not belong to Λ, since the elements of Λ are by definition finite sums of monomial symmetric functions m_λ. However, Λ is the inverse limit of the Λ_n in the category of *graded* rings.

2. We could use any commutative ring A in place of $\mathbf{Z}$ as coefficient ring; in place of Λ we should obtain $\Lambda_A \cong \Lambda \otimes_{\mathbf{Z}} A$.

Elementary symmetric functions

For each integer $r \geqslant 0$ the rth *elementary symmetric function* e_r is the sum of all products of r distinct variables x_i, so that $e_0 = 1$ and

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \dots x_{i_r} = m_{(1^r)}$$

for $r \geqslant 1$. The generating function for the e_r is

$$E(t) = \sum_{r \geqslant 0} e_r t^r = \prod_{i \geqslant 1} (1 + x_i t) \tag{2.2}$$

(t being another variable), as one sees by multiplying out the product on the right. (If the number of variables is finite, say n, then e_r (i.e., $\rho_n(e_r)$) is

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† The elements of Λ (unlike those of Λ_n) are no longer polynomials: they are formal infinite sums of monomials. We have therefore reverted to the older terminology of 'symmetric functions'.

zero for all $r > n$, and (2.2) then takes the form

$$\sum_{r=0}^{n} e_r t^r = \prod_{i=1}^{n} (1 + x_i t)$$

both sides now being elements of $\Lambda_n[t]$. Similar remarks will apply to many subsequent formulas, and we shall usually leave it to the reader to make the necessary (and obvious) adjustments.)

For each partition $\lambda = (\lambda_1, \lambda_2, \dots)$ define

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots .$$

(2.3) *Let λ be a partition, λ' its conjugate. Then*

$$e_{\lambda'} = m_\lambda + \sum_{\mu} a_{\lambda\mu} m_\mu$$

where the $a_{\lambda\mu}$ are non-negative integers, and the sum is over partitions μ which are later than λ in the reverse lexicographic ordering.

Proof. We have

$$e_{\lambda'} = e_1^{m_1} e_2^{m_2} \dots$$

where $m_i = m_i(\lambda') = \lambda_i - \lambda_{i+1}$ by (1.4). If we now multiply out this product, we obtain a sum of monomials; if these are arranged in reverse lexicographic order, the leading term in the expansion is clearly

$$x_1^{m_1} (x_1 x_2)^{m_2} (x_1 x_2 x_3)^{m_3} \dots = x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} \dots = x^\lambda .$$

Since $e_{\lambda'}$ is a symmetric function, it follows that the monomial symmetric function m_λ occurs in the expansion with coefficient 1, and that all other m_μ which occur are lower than m_λ in the reverse lexicographic ordering. |

(2.4) *We have*

$$\Lambda = \mathbf{Z}[e_1, e_2, \dots]$$

and the e_r are algebraically independent over $\mathbf{Z}$.

Proof. The m_λ form a $\mathbf{Z}$-basis of Λ, and (2.3) shows that the e_λ form another $\mathbf{Z}$-basis: in other words, every element of Λ is uniquely expressible as a polynomial in the e_r. |

Remark. When there are only finitely many variables $x_1, \dots, x_n$, (2.4) states that $\Lambda_n = \mathbf{Z}[e_1, \dots, e_n]$, and that $e_1, \dots, e_n$ are algebraically independent. This is the usual statement of the 'fundamental theorem on symmetric functions'.

Complete symmetric functions

For each $r \geqslant 0$ the rth *complete symmetric function* h_r is the sum of all monomials of total degree r in the variables $x_1, x_2, \ldots$, so that

$$h_r = \sum_{|\lambda|=r} m_\lambda.$$

In particular, $h_0 = 1$ and $h_1 = e_1$. It is convenient to define h_r and e_r to be zero for $r < 0$.

The generating function for the h_r is

$$H(t) = \sum_{r\geqslant 0} h_r t^r = \prod_{i\geqslant 1} (1 - x_i t)^{-1}. \tag{2.5}$$

To see this, observe that

$$(1 - x_i t)^{-1} = \sum_{k\geqslant 0} x_i^k t^k,$$

and multiply these geometric series together.

From (2.2) and (2.5) we have

$$H(t)E(-t) = 1 \tag{2.6}$$

or, equivalently,

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0 \tag{2.6$'$}$$

for all $n \geqslant 1$.

Since the e_r are algebraically independent (2.4), we may define a homomorphism of graded rings

$$\omega : \Lambda \to \Lambda$$

by

$$\omega(e_r) = h_r$$

for all $r \geqslant 0$. The symmetry of the relations (2.6′) as between the e's and the h's shows that

(2.7) *ω is an involution, i.e. ω^2 is the identity map.* |

It follows that ω is an automorphism of Λ, and hence from (2.4) that

(2.8) *We have*

$$\Lambda = \mathbf{Z}[h_1, h_2, \ldots]$$

and the h_r are algebraically independent over $\mathbf{Z}$. |

Remark. If the number of variables is finite, say n (so that $e_r = 0$ for $r > n$) the mapping $\omega : \Lambda_n \to \Lambda_n$ is defined by $\omega(e_r) = h_r$ for $1 \leqslant r \leqslant n$, and is still an involution by reason of (2.6′); we have $\Lambda_n = \mathbf{Z}[h_1, \ldots, h_n]$ with $h_1, \ldots, h_n$ algebraically independent, but $h_{n+1}, h_{n+2}, \ldots$ are non-zero polynomials in $h_1, \ldots, h_n$ (or in $e_1, \ldots, e_n$).

As in the case of the e's, we define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots$$

for any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$. By (2.8), the h_λ form a $\mathbf{Z}$-basis of Λ. We now have three $\mathbf{Z}$-bases, all indexed by partitions: the m_λ, the e_λ and the h_λ, the last two of which correspond under the involution ω. If we define

$$f_\lambda = \omega(m_\lambda)$$

for each partition λ, the f_λ form a fourth $\mathbf{Z}$-basis of Λ. (The f_λ are the 'forgotten' symmetric functions: they have no particularly simple direct description.)

The relations (2.6′) lead to a determinant identity which we shall make use of later. Let N be a positive integer and consider the matrices of $N+1$ rows and columns

$$H = (h_{i-j})_{0\leqslant i,j\leqslant N}, \qquad E = ((-1)^{i-j} e_{i-j})_{0\leqslant i,j\leqslant N}$$

with the convention mentioned earlier that $h_r = e_r = 0$ for $r < 0$. Both H and E are lower triangular, with 1's down the diagonal, so that $\det H = \det E = 1$; moreover the relations (2.6′) show that they are inverses of each other. It follows that each minor of H is equal to the complementary cofactor of E', the transpose of E.

Let λ, μ be two partitions of length $\leqslant p$, such that λ' and μ' have length $\leqslant q$, where $p + q = N + 1$. Consider the minor of H with row indices $\lambda_i + p - i$ $(1 \leqslant i \leqslant p)$ and column indices $\mu_i + p - i$ $(1 \leqslant i \leqslant p)$. By (1.7) the complementary cofactor of E' has row indices $p - 1 + j - \lambda'_j$ $(1 \leqslant j \leqslant q)$ and column indices $p - 1 + j - \mu'_j$ $(1 \leqslant j \leqslant q)$. Hence we have

$$\det(h_{\lambda_i-\mu_j-i+j})_{1\leqslant i,j\leqslant p} = (-1)^{|\lambda|+|\mu|} \det((-1)^{\lambda'_i-\mu'_j-i+j} e_{\lambda'_i-\mu'_j-i+j})_{i\leqslant i,j\leqslant q}.$$

The minus signs cancel out, and therefore we have (Aitken [1])

$$\det(h_{\lambda_i-\mu_j-i+j})_{1\leqslant i,j\leqslant p} = \det(e_{\lambda'_i-\mu'_j-i+j})_{1\leqslant i,j\leqslant q}. \tag{2.9}$$

In particular, taking $\mu = 0$:

$$\det(h_{\lambda_i-i+j}) = \det(e_{\lambda'_i-i+j}). \tag{2.9$'$}$$

Power sums

For each $r \geqslant 1$ the rth *power sum* is

$$p_r = \sum x_i^r = m_{(r)}.$$

The generating function for the p_r is

$$\begin{aligned}P(t)&=\sum_{r\geq 1} p_r t^{r-1}=\sum_{i\geq 1}\sum_{r\geq 1} x_i^r t^{r-1}\\&=\sum_{i\geq 1}\frac{x_i}{1-x_i t}\\&=\sum_{i\geq 1}\frac{d}{dt}\log\frac{1}{1-x_i t}\end{aligned}$$

so that

(2.10) $$P(t)=\frac{d}{dt}\log\prod_{i\geq 1}(1-x_i t)^{-1}=\frac{d}{dt}\log H(t)=H'(t)/H(t).$$

Likewise we have

(2.10′) $$P(-t)=\frac{d}{dt}\log E(t)=E'(t)/E(t).$$

From (2.10) we obtain

(2.11) $$nh_n=\sum_{r=1}^{n} p_r h_{n-r}$$

for $n\geq 1$, and these equations enable us to express the h's in terms of the p's, and vice versa. From (2.11) it is clear that $h_n\in\mathbf{Q}[p_1,\ldots,p_n]$ and $p_n\in\mathbf{Z}[h_1,\ldots,h_n]$, and hence that

$$\mathbf{Q}[p_1,\ldots,p_n]=\mathbf{Q}[h_1,\ldots,h_n].$$

Since the h_r are algebraically independent over $\mathbf{Z}$, and hence also over $\mathbf{Q}$, it follows that

(2.12) *We have*

$$\Lambda_{\mathbf{Q}}=\Lambda\otimes_{\mathbf{Z}}\mathbf{Q}=\mathbf{Q}[p_1,p_2,\ldots]$$

and the p_r are algebraically independent over $\mathbf{Q}$. |

Hence, if we define

$$p_\lambda=p_{\lambda_1}p_{\lambda_2}\cdots$$

for each partition $\lambda=(\lambda_1,\lambda_2,\ldots)$, then the p_λ form a $\mathbf{Q}$-basis of $\Lambda_{\mathbf{Q}}$. But they do *not* form a $\mathbf{Z}$-basis of Λ: for example, $h_2=\frac{1}{2}(p_1^2+p_2)$ does not have integral coefficients when expressed in terms of the p_λ.

Since the involution ω interchanges $E(t)$ and $H(t)$ it follows from (2.10) and (2.10′) that

$$\omega(p_n)=(-1)^{n-1}p_n$$

for all $n\geq 1$, and hence that for any partition λ we have

(2.13) $$\omega(p_\lambda)=\varepsilon_\lambda p_\lambda$$

where $\varepsilon_\lambda=(-1)^{|\lambda|-l(\lambda)}$.

Finally, we shall express h_n and e_n as linear combinations of the p_λ. For any partition λ, define

$$z_\lambda=\prod_{i\geq 1} i^{m_i}.m_i!$$

where $m_i=m_i(\lambda)$ is the number of parts of λ equal to i. Then we have

(2.14) $$\begin{aligned}H(t)&=\sum_\lambda z_\lambda^{-1}p_\lambda t^{|\lambda|},\\E(t)&=\sum_\lambda \varepsilon_\lambda z_\lambda^{-1}p_\lambda t^{|\lambda|},\end{aligned}$$

or equivalently

(2.14′) $$\begin{aligned}h_n&=\sum_{|\lambda|=n} z_\lambda^{-1}p_\lambda,\\e_n&=\sum_{|\lambda|=n}\varepsilon_\lambda z_\lambda^{-1}p_\lambda.\end{aligned}$$

Proof. It is enough to prove the first of the identities (2.14), since the second then follows by applying the involution ω and using (2.13). From (2.10) we have

$$\begin{aligned}H(t)&=\exp\sum_{r\geq 1}p_r t^r/r\\&=\prod_{r\geq 1}\exp(p_r t^r/r)\\&=\prod_{r\geq 1}\sum_{m_r=0}^{\infty}(p_r t^r)^{m_r}/r^{m_r}.m_r!\\&=\sum_\lambda z_\lambda^{-1}p_\lambda t^{|\lambda|}.\quad|\end{aligned}$$

(2.15) *Remark.* In the language of λ-rings ([5], [23]) the ring Λ is the 'free λ-ring in one variable' (or, more precisely, is the underlying ring). Consequently *all* the formulas and identities in this chapter can be translated into this language. It is not our intention to write a text on the theory of λ-rings: we shall merely provide a brief dictionary.

If R is any λ-ring and x any element of R, there exists a unique λ-homomorphism $\Lambda\rightarrow R$ under which $e_1(=h_1=p_1)$ is mapped to x.

Under this homomorphism

e_r is mapped to $\lambda^r(x)$ (rth exterior power)
h_r $\sigma^r(x)=(-1)^r\lambda^r(-x)$ (rth symmetric power)
$E(t)$ $\lambda_t(x)$
$H(t)$ $\sigma_t(x)=\lambda_{-t}(-x)$
p_r $\psi^r(x)$ (Adams operations)

and the involution ω corresponds in R to $x\mapsto -x$. So, for example, (2.14′) becomes

$$\sigma^n(x)=\sum_{|\lambda|=n} z_\lambda^{-1}\psi^\lambda(x)$$

valid for any element x of any λ-ring (where of course $\psi^\lambda(x)=\psi^{\lambda_1}(x)\psi^{\lambda_2}(x)\dots$).

Examples

1. Let $x_1=\dots=x_n=1$, $x_{n+1}=x_{n+2}=\dots=0$. Then $E(t)=(1+t)^n$, $H(t)=(1-t)^{-n}$, so that

$$e_r=\binom{n}{r},\qquad h_r=\binom{n+r-1}{r}$$

and $p_r=n$ for all $r\geq 1$.

2. Let $x_i=1/n$ for $1\leq i\leq n$, $x_i=0$ for $i>n$, and then let $n\to\infty$. From Ex. 1 we have

$$e_r=\lim_{n\to\infty}\frac{1}{n^r}\binom{n}{r}=\frac{1}{r!}$$

and likewise $h_r=1/r!$, so that $E(t)=H(t)=e^t$. We have $p_1=1$ and $p_r=0$ for $r>1$; more generally, $m_\lambda=0$ for all partitions λ except $\lambda=(1^r)(r\geq 0)$.

3. Let $x_i=q^{i-1}$ for $1\leq i\leq n$, and $x_i=0$ for $i>n$, where q is an indeterminate. Then

$$E(t)=\prod_{i=0}^{n-1}(1+q^it)=\sum_{r=0}^{n}q^{r(r-1)/2}\begin{bmatrix}n\\r\end{bmatrix}t^r$$

where $\left[\begin{smallmatrix}n\\r\end{smallmatrix}\right]$ denotes the 'q-binomial coefficient' or Gaussian polynomial

$$\begin{bmatrix}n\\r\end{bmatrix}=\frac{(1-q^n)(1-q^{n-1})\dots(1-q^{n-r+1})}{(1-q)(1-q^2)\dots(1-q^r)},$$

and

$$H(t)=\prod_{i=0}^{n-1}(1-q^it)^{-1}=\sum_{r=0}^{\infty}\begin{bmatrix}n+r-1\\r\end{bmatrix}t^r.$$

These identities are easily proved by induction on n. It follows that

$$e_r=q^{r(r-1)/2}\begin{bmatrix}n\\r\end{bmatrix},\qquad h_r=\begin{bmatrix}n+r-1\\r\end{bmatrix}.$$

h_r is the generating function for partitions λ such that $l(\lambda)\leq r$ and $l(\lambda')\leq n-1$, and e_r is the generating function for such partitions with all parts distinct.

4. Let $n\to\infty$ in Ex. 3, i.e., let $x_i=q^{i-1}$ for all $i\geq 1$. Then

$$E(t)=\prod_{i=0}^{\infty}(1+q^it)=\sum_{r=0}^{\infty}q^{r(r-1)/2}t^r/\varphi_r(q)$$

$$H(t)=\prod_{i=0}^{\infty}(1-q^it)^{-1}=\sum_{r=0}^{\infty}t^r/\varphi_r(q)$$

where

$$\varphi_r(q)=(1-q)(1-q^2)\dots(1-q^r).$$

Hence in this case

$$e_r=q^{r(r-1)/2}/\varphi_r(q),\qquad h_r=1/\varphi_r(q)$$

and $p_r=(1-q^r)^{-1}$.

5. Since the h_r are algebraically independent we may specialize them in any way, and forget about the original variables x_i: in other words, we may take $H(t)$ (or $E(t)$) to be any power series in t with constant term 1. (We have already done this in Ex. 2 above, where $H(t)=e^t$.) Let a, b, q be variables and take

$$H(t)=\prod_{i=0}^{\infty}\frac{1-bq^it}{1-aq^it}.$$

Then we have

$$h_r=\prod_{i=1}^{r}\frac{a-bq^{i-1}}{1-q^i},\qquad e_r=\prod_{i=1}^{r}\frac{aq^{i-1}-b}{1-q^i}$$

(see e.g. Andrews [2], Ch. II.) Also $p_r=(a^r-b^r)/(1-q^r)$.

6. Take $H(t)=\prod_{n=1}^{\infty}(1-t^n)^{-1}$, so that $h_n=p(n)$, the number of partitions of n. Then $E(-t)=\prod_{n=1}^{\infty}(1-t^n)$, and so by Euler's pentagonal number theorem $e_n=0$ unless n is a pentagonal number, i.e. of the form $\frac{1}{2}m(3m+1)$ for some $m\in\mathbf{Z}$; and $e_n=(-1)^{m(m+1)/2}$ if $n=\frac{1}{2}m(3m+1)$.

From (2.10) we obtain $p_r=\sigma(r)$, the sum of the divisors of r. Hence (2.11) gives in this case

$$p(n)=\frac{1}{n}\sum_{r=1}^{n}\sigma(r)p(n-r).\tag{1}$$

7. Take $H(t)=\prod_{n=1}^{\infty}(1-t^n)^{-n}$, so that $h_n=p_2(n)$, the number of plane partitions of n (§5, Ex. 13). From (2.10) we obtain $p_r=\sigma_2(r)$, the sum of the squares of the

divisors of r. Hence by (2.11)

$$p_2(n)=\frac{1}{n}\sum_{r=1}^{n}\sigma_2(r)p_2(n-r). \tag{2}$$

It is perhaps only fair to warn the reader that the obvious generalization of (1) and (2) to m-dimensional partitions ($m>2$) is false.

8. By solving the equations (2.6′) for e_n we obtain

$$e_n=\det(h_{1-i+j})_{1\leqslant i,j\leqslant n}$$

and dually

$$h_n=\det(e_{1-i+j})_{1\leqslant i,j\leqslant n}.$$

Likewise from (2.11) we obtain the determinant formulas

$$p_n=\begin{vmatrix} e_1 & 1 & 0 & \dots 0\\ 2e_2 & e_1 & 1 & \dots 0\\ \vdots & \vdots & \vdots & \vdots\\ ne_n & e_{n-1} & e_{n-2} & \dots e_1 \end{vmatrix}$$

$$n!e_n=\begin{vmatrix} p_1 & 1 & 0 & \dots & 0\\ p_2 & p_1 & 2 & \dots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ p_{n-1} & p_{n-2} & \cdot & \dots & n-1\\ p_n & p_{n-1} & \cdot & \dots & p_1 \end{vmatrix}$$

and dually

$$(-1)^{n-1}p_n=\begin{vmatrix} h_1 & 1 & 0 & \dots & 0\\ 2h_2 & h_1 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ nh_n & h_{n-1} & h_{n-2} & \dots & h_1 \end{vmatrix}$$

$$n!h_n=\begin{vmatrix} p_1 & -1 & 0 & \dots & 0\\ p_2 & p_1 & -2 & \dots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ p_{n-1} & p_{n-2} & \cdot & \dots & -n+1\\ p_n & p_{n-1} & \cdot & \dots & p_1 \end{vmatrix}.$$

9. Let G be any subgroup of the symmetric group S_n. The *cycle indicator* of G is the symmetric function

$$c(G)=\frac{1}{|G|}\sum_{\rho}n_G(\rho)p_\rho$$

where $n_G(\rho)$ is the number of elements in G of cycle-type ρ, and the sum is over all partitions ρ of n. In particular,

$$c(S_n)=\sum_{|\rho|=n}z_\rho^{-1}p_\rho=h_n$$

by (2.14′), and for the alternating group A_n we have

$$c(A_n)=h_n+e_n.$$

If G is a subgroup of S_n and H a subgroup of S_m, then $G\times H$ is a subgroup of $S_n\times S_m\subset S_{n+m}$, and we have

$$c(G\times H)=c(G).c(H).$$

10. From Exx. 8 and 9 it follows that the number of elements of cycle-type ρ in S_n is equal to the coefficient of p_ρ in the determinant

$$d_n=n!h_n=\begin{vmatrix} p_1 & -1 & 0 & \dots & 0\\ p_2 & p_1 & -2 & \dots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ p_{n-1} & p_{n-2} & \cdot & \dots & -n+1\\ p_n & p_{n-1} & \cdot & \dots & p_1 \end{vmatrix}$$

Let l be a prime number. We may use this formula to count the number of conjugacy classes in S_n in which the number of elements is prime to l, by reducing the determinant d_n modulo l. Suppose that $n=a_0+n_1l$, where $0\leqslant a_0\leqslant l-1$. Then since the multiples of l above the diagonal in d_n become zero on reduction, it follows that

$$d_n\equiv d_l^{n_1}d_{a_0}\qquad(\text{mod. } l)$$

Now it is clear from the original definition of $d_n=n!c(S_n)$ that

$$d_l\equiv p_1^l-p_l\qquad(\text{mod. } l)$$

and therefore we have

$$d_n\equiv(p_1^l-p_l)^{n_1}.d_{a_0}\qquad(\text{mod. } l). \tag{1}$$

Hence if $n=a_0+a_1l+a_2l^2+\dots$, with $0\leqslant a_i\leqslant l-1$ for all $i\geqslant 0$, it follows from (1) that

$$d_n\equiv d_{a_0}\prod_{i\geqslant 1}(p_1^{l^i}-p_l^{l^{i-1}})^{a_i}\qquad(\text{mod. } l).$$

Consequently, if $\mu_l(S_n)$ denotes the number of conjugacy classes in S_n of order prime to l, we have

$$\begin{aligned}\mu_l(S_n)&=\mu_l(S_{a_0})\prod_{i\geqslant 1}(a_i+1)\\ &=p(a_0)\prod_{i\geqslant 1}(a_i+1)\end{aligned}$$

where $p(a_0)$ is the number of partitions of a_0. In particular, if $l=2$, we see that $\mu_2(S_n)$ is always a power of 2, because each a_i is then either 0 or 1: namely $\mu_2(S_n)=2^r$ if n is a sum of r distinct powers of 2.

11. Let

$$f(t)=\sum_{n=0}^{\infty}\frac{f_nt^n}{n!},\qquad g(t)=\sum_{n=0}^{\infty}\frac{g_nt^n}{n!}$$

be formal power series (with coefficients in a commutative **Q**-algebra) such that $g(0)=0$. We may substitute $g(t)$ for t in $f(t)$, and obtain say

$$H(t)=f(g(t))=\sum_0^\infty \frac{H_n t^n}{n!}.$$

Clearly each coefficient H_n is of the form

$$H_n=\sum_{k=1}^n f_k B_{n,k}(g)$$

where the $B_{n,k}$ are polynomials in the coefficients of g, called the *partial Bell polynomials.* Since each H_n is linear in the coefficients of *f*, in order to compute the polynomials $B_{n,k}$ we may take $f_k=a^k$, so that $f(t)=e^{at}$. Writing

$$H(t)=\sum_{n=0}^\infty h_n t^n$$

as usual, we have $H(t)=\exp(ag(t))$ and therefore by (2.10)

$$P(t)=\frac{d}{dt}\log H(t)=ag'(t)=\sum_1^\infty \frac{ag_n t^{n-1}}{(n-1)!},$$

so that $p_n=ag_n/(n-1)!$ for all $n\geqslant 1$. Hence by (2.14′)

$$H_n=n!h_n=\sum_{|\lambda|=n}\frac{n!}{z_\lambda}p_\lambda$$

and consequently

$$B_{n,k}=\sum_\lambda \frac{n!}{z_\lambda}p_\lambda=\sum_\lambda c_\lambda g_\lambda$$

where the sum is over partitions λ of n such that $l(\lambda)=k$, and

$$g_\lambda=g_{\lambda_1}g_{\lambda_2}\ldots,\qquad c_\lambda=n!\Big/\prod_{i\geqslant 1} r_i!(i!)^{r_i}$$

if $\lambda=(1^{r_1}2^{r_2}\ldots)$. These coefficients c_λ are *integers*, because c_λ is the number of decompositions of a set of n elements into disjoint subsets containing $\lambda_1,\lambda_2,\ldots$ elements. Hence each $B_{n,k}$ is a polynomial in the g_n's with integer coefficients.

Particular cases:

(a) if $g(t)=\log(1+t)$, then $B_{n,k}=s(n,k)$ are the *Stirling numbers of the first kind*; $(-1)^{n-k}s(n,k)$ is the number of elements of S_n which are products of k disjoint cycles. We have

$$\sum_{n,k\geqslant 0} s(n,k)\frac{t^n}{n!}a^k=(1+t)^a=\sum_{n\geqslant 0}\binom{a}{n}t^n,$$

from which it follows that

$$\sum_{k=0}^n s(n,k)a^n=a(a-1)\ldots(a-n+1)$$

and hence that $s(n,k)$ is the $(n-k)$th elementary symmetric function of $-1,-2,\ldots,-n+1$.

(b) if $g(t)=e^t-1$, so that $g_n=1$ for all $n\geqslant 1$, then $B_{n,k}=S(n,k)$ are the *Stirling numbers of the second kind*; $S(n,k)$ is the number of decompositions of a set of n elements into k disjoint subsets, and is also the $(n-k)$th complete symmetric function of $1,2,\ldots,k$.

$h_{n-k}(1,..\,k)$

12. Deduce from Ex. 11 that if f and g are n times differentiable functions of a real variable, and if f_k, g_k, $(f\circ g)_k$ denote the kth derivatives of f, g and $f\circ g$, then

$$(f\circ g)_n=\sum_{k=1}^n B_{n,k}(g_1,g_2,\ldots)(f_k\circ g).$$

13. If $H(t)=(1-t^r)/(1-t)^r$, we have

$$h_n=\binom{n+r-1}{r-1}-\binom{n-1}{r-1}$$

and by (2.10) we find that $p_n=r$ if $n\not\equiv 0 \pmod r$, whereas $p_n=0$ if $n\equiv 0 \pmod r$. Hence from (2.14′)

$$\sum_\lambda z_\lambda^{-1}r^{l(\lambda)}=\binom{n+r-1}{r-1}-\binom{n-1}{r-1}\qquad (r\geqslant 2)$$

where the sum on the left is over partitions λ of n none of whose parts is divisible by r.

In particular $(r=2)$

$$\sum_\lambda z_\lambda^{-1}2^{l(\lambda)}=2$$

summed over all partitions of n into *odd* parts.

Notes and references

Ex. 10. The calculation of $\mu_p(S_n)$, and in particular the fact that $\mu_2(S_n)$ is a power of 2, is new, as far as I know. More generally, if G is any finite Coxeter group, $\mu_2(G)$ is a power of 2 [33].

Ex. 11. For more information on Bell polynomials, Stirling numbers etc. see for example L. Comtet, *Analyse Combinatoire* (2 vols.), Presses Universitaires de France, Paris (1970).

Ex. 13. This result is due to A. O. Morris [38].

3. *S*-functions

Suppose to begin with that the number of variables is finite, say $x_1,\ldots,x_n$. Let $x^\alpha=x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ be a monomial, and consider the polynomial a_α obtained by antisymmetrizing x^α: that is to say,

$$a_\alpha=a_\alpha(x_1,\ldots,x_n)=\sum_{w\in S_n}\varepsilon(w).w(x^\alpha)$$

where $\varepsilon(w)$ is the *sign* (± 1) of the permutation w. This polynomial a_α is skew-symmetric, i.e. we have

$$w(a_\alpha)=\varepsilon(w)a_\alpha$$

for any $w \in S_n$; in particular, therefore, a_α vanishes unless $\alpha_1, \dots, \alpha_n$ are all distinct. Hence we may as well assume that $\alpha_1 > \alpha_2 > \dots > \alpha_n \geq 0$, and therefore we may write $\alpha = \lambda + \delta$, where λ is a partition of length $\leq n$, and $\delta = (n-1, n-2, \dots, 1, 0)$. Then

$$a_\alpha = a_{\lambda+\delta} = \sum_w \varepsilon(w) . w(x^{\lambda+\delta})$$

which can be written as a determinant:

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j+n-j})_{1\leq i,j\leq n}.$$

This determinant is divisible in $\mathbf{Z}[x_1, \dots, x_n]$ by each of the differences $x_i - x_j$ $(1 \leq i < j \leq n)$, and hence by their product, which is the *Vandermonde determinant*

$$\prod_{1\leq i<j\leq n} (x_i - x_j) = \det(x_i^{n-j}) = a_\delta.$$

So $a_{\lambda+\delta}$ is divisible by a_δ in $\mathbf{Z}[x_1, \dots, x_n]$, and the quotient

(3.1) $$s_\lambda = s_\lambda(x_1, \dots, x_n) = a_{\lambda+\delta}/a_\delta$$

is *symmetric*, i.e. is in Λ_n. It is called the *S-function* in the variables $x_1, \dots, x_n$, corresponding to the partition λ (where $l(\lambda) \leq n$), and is homogeneous of degree $|\lambda|$.

Notice that the definition (3.1) makes sense for any integer vector $\lambda \in \mathbf{Z}^n$ such that $\lambda + \delta$ has no negative parts. If the numbers $\lambda_i + n - i$ $(1 \leq i \leq n)$ are not all distinct, then $s_\lambda = 0$. If they are all distinct, then we have $\lambda + \delta = w(\mu + \delta)$ for some $w \in S_n$ and some partition μ, and $s_\lambda = \varepsilon(w) s_\mu$.

The polynomials $a_{\lambda+\delta}$, where λ runs through all partitions of length $\leq n$, form a basis of the $\mathbf{Z}$-module A_n of skew-symmetric polynomials in $x_1, \dots, x_n$. Multiplication by a_δ is an isomorphism of Λ_n onto A_n (i.e., A_n is the free Λ_n-module generated by a_δ), and therefore

(3.2) *The S-functions* $s_\lambda(x_1, \dots, x_n)$, *where* $l(\lambda) \leq n$, *form a* $\mathbf{Z}$-*basis of* Λ_n. |

Now let us consider the effect of increasing the number of variables. If $m \geq n$, it is clear that $a_\alpha(x_1, \dots, x_n, 0, \dots, 0) = a_\alpha(x_1, \dots, x_n)$. Hence

$$\rho_{m,n}(s_\lambda(x_1, \dots, x_m)) = s_\lambda(x_1, \dots, x_n)$$

in the notation of §2. It follows that for each partition λ the polynomials $s_\lambda(x_1, \dots, x_n)$, as $n \to \infty$, define a unique element $s_\lambda \in \Lambda$, homogeneous of degree $|\lambda|$. From (3.2) we have immediately:

(3.3) *The* s_λ *form a* $\mathbf{Z}$-*basis of* Λ, *and for each* $k \geq 0$ *the* s_λ *such that* $|\lambda| = k$ *form a* $\mathbf{Z}$-*basis of* Λ^k. |

From (2.4) and (2.8) it follows that each S-function s_λ can be expressed as a polynomial in the elementary symmetric functions e_r and as a polynomial in the complete symmetric functions h_r. The formulas are:

(3.4) $$s_\lambda = \det(h_{\lambda_i-i+j})_{1\leq i,j\leq n}$$

where $n \geq l(\lambda)$, and

(3.5) $$s_\lambda = \det(e_{\lambda'_i-i+j})_{1\leq i,j\leq m}$$

where $m \geq l(\lambda')$.

By (2.9′), it is enough to prove one of these formulas, say (3.4). We shall work with n variables $x_1, \dots, x_n$. For $1 \leq k \leq n$ let $e_r^{(k)}$ denote the elementary symmetric functions of $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n$ (omitting x_k), and let M denote the $n \times n$ matrix

$$M = ((-1)^{n-i} e_{n-i}^{(k)})_{1\leq i,k\leq n}.$$

The formula (3.4) will be a consequence of

(3.6) *For any* $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$, *let*

$$A_\alpha = (x_j^{\alpha_i}), \qquad H_\alpha = (h_{\alpha_i-n+j})$$

($n \times n$ *matrices*). *Then* $A_\alpha = H_\alpha M$.

Proof. Let

$$E^{(k)}(t) = \sum_{r=0}^{n-1} e_r^{(k)} t^r = \prod_{i\neq k} (1 + x_i t).$$

Then

$$H(t) E^{(k)}(-t) = (1 - x_k t)^{-1}.$$

By picking out the coefficient of t^{α_i} on either side, we obtain

$$\sum_{j=1}^{n} h_{\alpha_i-n+j} . (-1)^{n-j} e_{n-j}^{(k)} = x_k^{\alpha_i}$$

and hence $H_\alpha M = A_\alpha$. |

Now take determinants in (3.6): we obtain

$$a_\alpha = \det(A_\alpha) = \det(H_\alpha) \det(M)$$

for any $\alpha \in \mathbf{N}^n$, and in particular $\det M = a_\delta$, since $\det(H_\delta) = 1$. Hence

(3.7) $$a_\alpha = a_\delta \det(H_\alpha)$$

or equivalently

(3.7′) $$a_\alpha = a_\delta \sum_{w\in S_n} \varepsilon(w) h_{\alpha-w\delta}$$

for any $a \in \mathbf{N}^n$. Taking $\alpha = \lambda + \delta$ in (3.7), we obtain (3.4), or equivalently from (3.7′)

(3.4′)
$$s_\lambda = \sum_{w \in S_n} \varepsilon(w) h_{\lambda+\delta-w\delta}.$$

From (3.4) and (3.5) it follows that

(3.8)
$$\omega(s_\lambda) = s_{\lambda'}$$

for all partitions λ.

Also from (3.4) and (3.5) we obtain, in particular,

(3.9)
$$s_{(n)} = h_n, \qquad s_{(1^n)} = e_n.$$

Finally, the formula (3.4) or (3.4′) which expresses s_λ as a polynomial in the h's can also be expressed in terms of raising operators (§1):

(3.4″)
$$s_\lambda = \prod_{i<j} (1 - R_{ij}) h_\lambda$$

where, for any raising operator R, Rh_λ means $h_{R\lambda}$.

Proof. In the ring $\mathbf{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ we have

$$\begin{aligned}\sum_{w \in S_n} \varepsilon(w) x^{\lambda+\delta-w\delta} = x^{\lambda+\delta} a_{-\delta} &= x^{\lambda+\delta} \prod_{i<j} (x_i^{-1} - x_j^{-1}) \\ &= \prod_{i<j} (1 - x_i x_j^{-1}) . x^\lambda \\ &= \prod_{i<j} (1 - R_{ij}) x^\lambda \end{aligned}$$

where $R(x^\lambda) = x^{R\lambda}$ for any raising operator R. If we now apply the $\mathbf{Z}$-linear map $\varphi : \mathbf{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \to \Lambda_n$ defined by $\varphi(x^\alpha) = h_\alpha$ for all $\alpha \in \mathbf{Z}^n$, we see that

$$\sum_{w \in S_n} \varepsilon(w) h_{\lambda+\delta-w\delta} = \prod_{i<j} (1 - R_{ij}) h_\lambda$$

and therefore (3.4″) follows from (3.4′). |

(3.10) *Remark.* In view of (2.15) we may use (3.4) or (3.5) to define 'S-operations' in any λ-ring R. If μ is any partition and x is any element of R, we define

$$\begin{aligned} S^\mu(x) &= \det(\sigma^{\mu_i - i + j}(x))_{1 \leqslant i,j \leqslant n} \\ &= \det(\lambda^{\mu'_i - i + j}(x))_{1 \leqslant i,j \leqslant m} \end{aligned}$$

where $n \geqslant l(\mu)$ and $m \geqslant l(\mu')$. We have

$$S^\mu(-x) = (-1)^{|\mu|} S^{\mu'}(x)$$

and in particular

$$S^{(n)}(x) = \sigma^n(x), \qquad S^{(1^n)}(x) = \lambda^n(x).$$

For example, the results of Exx. 1–3 below evaluate $S^\lambda(1 + q + \dots + q^{n-1})$, $S^\lambda((1-q)^{-1})$ and $S^\lambda((a-b)/(1-q))$, where a, b, q are elements of rank 1 in a λ-ring R such that $1-q$ is a unit in R.

Since each $f \in \Lambda$ is an integral linear combination of the s_μ, say

$$f = \sum a_\mu s_\mu,$$

it follows that f determines a 'natural operation'

$$F = \sum a_\mu S^\mu$$

on the category of λ-rings. F is *natural* in the sense that it commutes with all λ-homomorphisms (because it is a polynomial in the λ^r). Conversely, any natural operation F arises in this way, from $f = F(e_1)$.

Examples

1. Take $x_i = q^{i-1}$ $(1 \leqslant i \leqslant n)$ as in §2, Ex. 3. If λ is any partition of length $\leqslant n$, we have

$$a_{\lambda+\delta} = \det(q^{(i-1)(\lambda_j + n - j)})_{1 \leqslant i,j \leqslant n}$$

which is a Vandermonde determinant in the variables $q^{\lambda_j + n - j}$ $(1 \leqslant j \leqslant n)$, so that

$$\begin{aligned} a_{\lambda+\delta} &= \prod_{i<j} (q^{\lambda_j + n - j} - q^{\lambda_i + n - i}) \\ &= q^{n(\lambda) + n(n-1)/2} \prod_{i<j} (1 - q^{\lambda_i - \lambda_j - i + j}) \end{aligned}$$

which by use of §1, Ex. 1 is equal to

$$\frac{q^{n(\lambda)+n(n-1)/2} \prod_{i \geqslant 1} \varphi_{\lambda_i + n - i}(q)}{\prod_{x \in \lambda} (1 - q^{h(x)})}$$

where $h(x)$ is the hook-length at $x \in \lambda$, and $\varphi_r(q) = (1-q) \dots (1-q^r)$. Hence (§1, Ex. 3)

$$s_\lambda = a_{\lambda+\delta}/a_\delta = q^{n(\lambda)} \prod_{x \in \lambda} \frac{1 - q^{n+c(x)}}{1 - q^{h(x)}}$$

where $c(x)$ is the content (§1, Ex. 3) of $x \in \lambda$.

For any partition λ define

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} = \prod_{x \in \lambda} \frac{1 - q^{n-c(x)}}{1 - q^{h(x)}}$$

(which when $\lambda = (r)$ agrees with the notation $\left[\begin{smallmatrix} n \\ r \end{smallmatrix}\right]$ for the q-binomial coefficients introduced in §2, Ex. 3). Then we have

$$s_\lambda(1, q, \dots, q^{n-1}) = q^{n(\lambda)} \begin{bmatrix} n \\ \lambda' \end{bmatrix}.$$

$\left[{n \atop \lambda}\right]$ is a polynomial in q, of degree

$$d = \sum_{x \in \lambda} (n - c(x) - h(x)) = \sum_{i=1}^{n} (n + 1 - 2i)\lambda_i'$$

by using §1, Ex. 2 and 3. If a_i is the coefficient of q^i in $\left[{n \atop \lambda}\right]$ for $0 \leqslant i \leqslant d$, then clearly $a_i = a_{d-i}$. We shall show in §8, Ex. 4 that $\left[{n \atop \lambda}\right]$ is *unimodal* (or 'spindle-shaped'), i.e. that $a_0 \leqslant a_1 \leqslant \dots \leqslant a_{[d/2]}$.

Finally, we can express $\left[{n \atop \lambda}\right]$ as a determinant in the q-binomial coefficients $\left[{n \atop \lambda}\right]$, by using (3.5).

2. Let $n \to \infty$ in Ex. 1, so that $H(t) = \prod_{i=0}^{\infty} (1 - q^i t)^{-1}$. From Ex. 1 we have

$$s_\lambda = q^{n(\lambda)} \prod_{x \in \lambda} (1 - q^{h(x)})^{-1} = q^{n(\lambda)} H_\lambda(q)^{-1}$$

where $H_\lambda(q)$ is the 'hook polynomial' $\prod_{x \in \lambda} (1 - q^{h(x)})$.

3. More generally, let

$$H(t) = \prod_{i=0}^{\infty} \frac{1 - bq^i t}{1 - aq^i t}$$

as in §2, Ex. 5. Then

$$(*) \qquad s_\lambda = q^{n(\lambda)} \prod_{x \in \lambda} \frac{a - bq^{c(x)}}{1 - q^{h(x)}}.$$

For if we replace t by $a^{-1}t$, the effect is to replace s_λ by $a^{-|\lambda|}s_\lambda$. Hence we may assume that $a = 1$. Both sides of $(*)$ are then polynomials in b, hence it is enough to show that they are equal for infinitely many values of b. But when $b = q^n$ and $a = 1$ we are back in the situation of Ex. 1, and $(*)$ is therefore true for $b = q^n$.

4. Suppose $x_i = 1$ $(1 \leqslant i \leqslant n)$, $x_i = 0$ for $i > n$. Then $E(t) = (1 + t)^n$, and

$$s_\lambda = \prod_{x \in \lambda} \frac{n + c(x)}{h(x)}$$

by setting $q = 1$ in Ex. 1.

More generally, if $E(t) = (1 + t)^X$, where X need not be a positive integer, then

$$s_\lambda = \prod_{x \in \lambda} \frac{X + c(x)}{h(x)}$$

for the same reason as in Ex. 3: both sides are polynomials in X which take the same values at all positive integers.

These polynomials may be regarded as generalized binomial coefficients, and

they take integer values whenever X is an integer. For any partition λ define

$$\binom{X}{\lambda} = \prod_{x \in \lambda} \frac{X - c(x)}{h(x)}$$

(which is consistent with the usual notation for binomial coefficients). Then

$$\binom{X}{\lambda} = \det\left(\binom{X}{\lambda_i - i + j}\right)$$

by (3.5). Also

$$\binom{-X}{\lambda} = (-1)^{|\lambda|} \binom{X}{\lambda'}.$$

5. As in §2, Ex. 2, take $x_i = 1/n$ for $1 \leqslant i \leqslant n$, $x_i = 0$ for $i > n$, and let $n \to \infty$. Then $E(t) = H(t) = e^t$, and from Ex. 4 we have

$$\begin{aligned} s_\lambda &= \lim_{n \to \infty} \frac{1}{n^{|\lambda|}} \prod_{x \in \lambda} \frac{n + c(x)}{h(x)} \\ &= \prod_{x \in \lambda} h(x)^{-1}. \end{aligned}$$

6. Let $p(n)$ denote the number of partitions of n. Then

$$\det(p(i - j + 1))_{1 \leqslant i, j \leqslant n}$$

is equal to ±1 or 0 according as n is or is not a pentagonal number. (Use §2, Ex. 6 together with (3.4).)

7. Let m be a positive integer. Then

$$\begin{aligned} \prod_{1 \leqslant i, j \leqslant n} \frac{x_i^m - x_j^m}{x_i - x_j} = \frac{a_{m\delta}}{a_\delta} &= s_{(m-1)\delta} \\ &= \det(h_{(m-1)(n-i)-i+j})_{1 \leqslant i, j \leqslant n} \\ &= \det(h_{mi-j})_{1 \leqslant i, j \leqslant n-1}. \end{aligned}$$

In particular,

$$\prod_{i<j} (x_i + x_j) = \det(h_{2i-j}).$$

8. Consider the ring $Q_n = \mathbf{Q}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of polynomials in $x_1, \dots, x_n$ and their inverses. For each $\alpha \in \mathbf{Z}^n$ the monomial $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ generates a symmetric function

$$\tilde{m}_\alpha = \sum_{w \in S_n} x^{w\alpha}$$

and the $\tilde{m}_\alpha$ such that $\alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant \alpha_n$ form a basis of $Q_n^{S_n}$.

Define a linear mapping $\varphi : Q_n^{S_n} \to \Lambda_n \otimes \mathbf{Q}$ by $\varphi(\tilde{m}_\alpha) = h_\alpha$ (with the usual convention that $h_\alpha = 0$ if any α_i is negative).

(a) For all $\alpha, \beta \in \mathbf{Z}^n$ we have

$$\varphi(a_\alpha a_\beta) = \det(h_{\alpha_i + \beta_j})_{1 \leqslant i, j \leqslant n}.$$

For

$$
\begin{aligned}
a_\alpha a_\beta &= \sum_{w_1, w_2 \in S_n} \varepsilon(w_1 w_2) x^{w_1\alpha + w_2\beta} \\
&= \sum_{w \in S_n} \varepsilon(w) \sum_{w_1 \in S_n} x^{w_1(\alpha + w\beta)} \\
&= \sum_{w \in S_n} \varepsilon(w) \tilde{m}_{\alpha + w\beta}
\end{aligned}
$$

so that

$$\varphi(a_\alpha a_\beta) = \sum_{w \in S_n} \varepsilon(w) h_{\alpha + w\beta} = \det(h_{\alpha_i + \beta_j}).$$

(b) In particular, if λ is any partition of n, we have

$$\varphi(s_\lambda a_\delta a_{-\delta}) = \varphi(a_{\lambda+\delta} a_{-\delta}) = \det(h_{\lambda_i - i + j}) = s_\lambda$$

by (3.4). Since the s_λ form a **Z**-basis of Λ_n, it follows that $\varphi(f a_\delta a_{-\delta}) = f$ for all $f \in \Lambda_n$.

(c) Let $\alpha, \beta \in \mathbf{N}^n$ and let $\bar{\beta} = (\beta_n, \ldots, \beta_1)$ be the reverse of β. Then $s_{\bar{\beta}} = a_{\beta-\delta}/a_{-\delta}$, and hence from (6) we have

$$s_\alpha s_{\bar{\beta}} = \varphi(a_{\alpha+\delta} a_{\beta-\delta}) = \det(h_{\alpha_i + \beta_j - i + j})_{1 \leqslant i, j \leqslant n},$$

a formula which expresses the product of two S-functions (in a finite number of variables) as a determinant in the h_r.

9. Let $a, b \geqslant 0$, then $(a \mid b)$ is the Frobenius notation (§1) for the partition $(a+1, 1^b)$. From the determinant formula (3.4) we have

$$s_{(a|b)} = h_{a+1} e_b - h_{a+2} e_{b-1} + \ldots + (-1)^b h_{a+b+1}$$

If a or b is negative, we *define* $s_{(a|b)}$ by this formula. It follows that (when a or b is negative) $s_{(a|b)} = 0$ except when $a + b = -1$, in which case $s_{(a|b)} = (-1)^b$.

Now let λ be any partition of length $\leqslant n$. By multiplying the matrix $(h_{\lambda_i - i + j})_{1 \leqslant i, j \leqslant n}$ on the right by the matrix $((-1)^{j-1} e_{n+1-j-k})_{1 \leqslant j, k \leqslant n}$, we obtain the matrix $(s_{(\lambda_i - i | n-k)})_{1 \leqslant i, k \leqslant n}$. By taking determinants and using §1, Ex. 4 we arrive at the formula

$$s_{(\alpha|\beta)} = \det(s_{(\alpha_i|\beta_j)})_{1 \leqslant i, j \leqslant r}$$

where $(\alpha \mid \beta) = (\alpha_1, \ldots, \alpha_r \mid \beta_1, \ldots, \beta_r)$.

10. $$s_\lambda(1 + x_1, 1 + x_2, \ldots, 1 + x_n) = \sum_\mu d_{\lambda\mu} s_\mu(x_1, \ldots, x_n)$$

summed over all partitions $\mu \subset \lambda$, where

$$d_{\lambda\mu} = \det\left(\binom{\lambda_i + n - i}{\mu_j + n - j}\right)_{1 \leqslant i, j \leqslant n}.$$

(Calculate $a_{\lambda+\delta}(1 + x_1, \ldots, 1 + x_n)$ and observe that $a_\delta(1 + x_1, \ldots, 1 + x_n) = a_\delta(x_1, \ldots, x_n)$.)

This formula can be used to calculate the Chern classes of the exterior square $\Lambda^2 E$ and symmetric square $\mathrm{S}^2 E$ of a vector bundle E. If $c(E) = \prod_{i=1}^{m} (1 + x_i)$ is the total Chern class of E, then

$$
\begin{aligned}
c(\Lambda^2 E) &= \prod_{i<j} (1 + x_i + x_j) \\
&= 2^{-m(m-1)/2} \prod_{i<j} (1 + 2x_i + 1 + 2x_j) \\
&= 2^{-m(m-1)/2} s_\delta(1 + 2x_1, \ldots, 1 + 2x_m)
\end{aligned}
$$

by Ex. 7, where $\delta = (m-1, m-2, \ldots, 0)$, and therefore

$$c(\Lambda^2 E) = 2^{-m(m-1)/2} \sum_{\mu \subset \delta} d_{\delta\mu} 2^{|\mu|} s_\mu(x_1, \ldots, x_m).$$

Likewise

$$
\begin{aligned}
c(\mathrm{S}^2 E) &= \prod_{i \leqslant j} (1 + x_i + x_j) \\
&= 2^{-m(m-1)/2} \sum_{\nu \subset \varepsilon} d_{\varepsilon\nu} 2^{|\nu|} s_\nu(x_1, \ldots, x_m)
\end{aligned}
$$

where $\varepsilon = (m, m-1, \ldots, 1)$.

11. Let $\mu = (\mu_1, \ldots, \mu_n)$ be a partition of length $\leqslant n$, and r a positive integer. Then, the variables being $x_1, \ldots, x_n$, we have

$$a_{\mu+\delta} p_r = \sum_{q=1}^{n} a_{\mu+\delta+r\varepsilon_q} \tag{1}$$

where ε_q is the sequence with 1 in the qth place and 0 elsewhere. We shall rearrange the sequence $\mu + \delta + r\varepsilon_q$ in descending order. If it has two terms equal, it will contribute nothing to (1). We may therefore assume that for some $p \leqslant q$ we have

$$\mu_{p-1} + n - p + 1 > \mu_q + n - q + r > \mu_p + n - p,$$

in which case $a_{\mu+\delta+r\varepsilon_q} = (-1)^{q-p} a_{\lambda+\delta}$, where λ is the partition

$$\lambda = (\mu_1, \ldots, \mu_{p-1}, \mu_q + p - q + r, \mu_p + 1, \ldots, \mu_{q-1} + 1, \mu_{q+1}, \ldots, \mu_n),$$

and therefore $\theta = \lambda - \mu$ is such that $\theta_i = 0$ if $i \notin [p, q]$,

$$\theta_p = \mu_q - \mu_p + p - q + r, \qquad \theta_i = \mu_{i-1} - \mu_i + 1 \quad \text{if} \quad p + 1 \leqslant i \leqslant q.$$

Let us call a skew diagram θ a *border strip* if θ is connected and contains no 2×2 block of squares (i.e. the successive rows (or columns) of θ overlap by exactly one square). The *length* of a border strip θ is the total number $|\theta|$ of squares it contains, and its *height* $\mathrm{ht}(\theta)$ is defined to be one less than the number of rows it occupies. With this terminology, the preceding discussion shows that

$$s_\mu p_r = \sum_\lambda (-1)^{\mathrm{ht}(\lambda-\mu)} s_\lambda \tag{2}$$

summed over all partitions $\lambda \supset \mu$ such that $\lambda - \mu$ is a border strip of length r.

From (2) it follows that, for any partitions λ, μ, ρ such that $|\lambda| = |\mu| + |\rho|$, the coefficient of s_λ in $s_\mu p_\rho$ is

$$\sum_S (-1)^{\mathrm{ht}(S)}$$

summed over all sequences of partitions $S = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(m)})$ such that $\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(m)} = \lambda$, with each $\lambda^{(i)} - \lambda^{(i-1)}$ a border strip of length ρ_i, and

$$\mathrm{ht}(S) = \sum_i \mathrm{ht}(\lambda^{(i)} - \lambda^{(i-1)}).$$

12. Let $\sigma : \mathbf{Z}[x_1, \dots, x_n] \to \Lambda_n$ be the $\mathbf{Z}$-linear mapping defined by $\sigma(x^\alpha) = s_\alpha$ for all $\alpha \in \mathbf{N}^n$. Then σ is Λ_n-linear, i.e. $\sigma(fg) = f\sigma(g)$ for $f \in \Lambda_n$ and $g \in \mathbf{Z}[x_1, \dots, x_n]$. For $\sigma(x^\alpha) = a_\delta^{-1} \boldsymbol{a}(x^{\alpha+\delta})$, where

$$\boldsymbol{a} = \sum_{w \in S_n} \varepsilon(w) w$$

is the antisymmetrization operator. By linearity it follows that $\sigma(g) = a_\delta^{-1} \boldsymbol{a}(g x^\delta)$ for all $g \in \mathbf{Z}[x_1, \dots, x_n]$, and the result follows from the fact that $\boldsymbol{a}$ is Λ_n-linear.

Notes and references

S-functions were first considered by Jacobi [20], as quotients of skew-symmetric polynomials by the polynomial a_δ, just as we have introduced them. Their relevance to the representation theory of the symmetric groups and the general linear groups, which we shall describe later, was discovered by Schur [44] much later. The name S-function (or Schur function) is due to Littlewood and Richardson [28]. The identity (3.4) which expresses s_λ in terms of the h's is due originally to Jacobi [20], and is often called the Jacobi–Trudi identity.

The results of Exx. 1–4 may be found in Littlewood [29], Chapter VII, which gives other results of the same sort. The formula in Ex. 8 for the product of two S-functions as a determinant in the h's is essentially due to Jacobi (*loc. cit.*), though rediscovered since. The result of Ex. 9 is due to Giambelli [56]. Ex. 10 is due to A. Lascoux [26].

4. Orthogonality

Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be two finite or infinite sequences of independent variables. We shall denote the symmetric functions of the x's by $s_\lambda(x)$, $p_\lambda(x)$, etc., and the symmetric functions of the y's by $s_\lambda(y)$, $p_\lambda(y)$, etc.

We shall give three series expansions for the product

$$\prod_{i,j} (1 - x_i y_j)^{-1}.$$

The first of these is

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y) \tag{4.1}$$

summed over all partitions λ.

This follows from (2.14), applied to the set of variables $x_i y_j$.

Next we have

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_\lambda m_\lambda(x) h_\lambda(y) \tag{4.2}$$

summed over all partitions λ.

Proof. We have

$$\begin{aligned}\prod_{i,j} (1 - x_i y_j)^{-1} &= \prod_j H(y_j) \\ &= \prod_j \sum_{r=0}^{\infty} h_r(x) y_j^r \\ &= \sum_\alpha h_\alpha(x) y^\alpha \\ &= \sum_\lambda h_\lambda(x) m_\lambda(y)\end{aligned}$$

where α runs through all sequences $(\alpha_1, \alpha_2, \dots)$ of non-negative integers such that $\sum \alpha_i < \infty$, and λ runs through all partitions. |

The third identity is

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_\lambda s_\lambda(x) s_\lambda(y) \tag{4.3}$$

summed over all partitions λ.

Proof. This is a consequence of (4.2) and (3.7′). Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ be two finite sets of variables, and as usual let $\delta = (n-1, n-2, \dots, 0)$. Then from (4.2)

$$a_\delta(x) a_\delta(y) \prod_{i,j=1}^{n} (1 - x_i y_j)^{-1} = a_\delta(x) \sum_{\alpha, w} h_\alpha(x) \varepsilon(w) y^{\alpha + w\delta}$$

summed over $\alpha \in \mathbf{N}^n$ and $w \in S_n$,

$$= a_\delta(x) \sum_{\beta, w} \varepsilon(w) h_{\beta - w\delta}(x) y^\beta$$

$$= \sum_\beta a_\beta(x) y^\beta$$

by (3.7′). Since $a_{w\beta} = \varepsilon(w) a_\beta$, it follows that this last sum is equal to $\sum a_\gamma(x) a_\gamma(y)$ summed over $\gamma_1 > \gamma_2 > \ldots > \gamma_n \geqslant 0$, i.e. to

$$\sum_\lambda a_{\lambda+\delta}(x) a_{\lambda+\delta}(y),$$

summed over partitions λ of length $\leqslant n$. This proves (4.3) in the case of n variables x_i and n variables y_i; now let $n \to \infty$ as usual. |

We now define a scalar product on Λ, i.e. a $\mathbf{Z}$-valued bilinear form $\langle u, v \rangle$, by requiring that the bases (h_λ) and (m_λ) should be dual to each other:

(4.5) $$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$$

for all partitions λ, μ, where $\delta_{\lambda\mu}$ is the Kronecker delta.

(4.6) *For each* $n \geqslant 0$, *let* (u_λ), (v_λ) *be* $\mathbf{Z}$*-bases of* Λ^n, *indexed by the partitions of n. Then the following conditions are equivalent:*

(a) $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$ *for all* λ, μ;

(b) $\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$.

Proof. Let

$$u_\lambda = \sum_\rho a_{\lambda\rho} h_\rho, \qquad v_\mu = \sum_\sigma b_{\mu\sigma} m_\sigma.$$

Then

$$\langle u_\lambda, v_\mu \rangle = \sum_\rho a_{\lambda\rho} b_{\mu\rho}$$

so that (a) is equivalent to

(a′) $$\sum_\rho a_{\lambda\rho} b_{\mu\rho} = \delta_{\lambda\mu}.$$

Also (b) is equivalent to the identity

$$\sum_\lambda u_\lambda(x) v_\lambda(y) = \sum_\rho h_\rho(x) m_\rho(y)$$

by (4.2), hence is equivalent to

(b′) $$\sum_\lambda a_{\lambda\rho} b_{\lambda\sigma} = \delta_{\rho\sigma}.$$

Since (a′) and (b′) are equivalent, so are (a) and (b). |

From (4.6) and (4.1) it follows that

(4.7) $$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$$

so that the p_λ form an *orthogonal* basis of Λ. Likewise from (4.6) and (4.3) we have

(4.8) $$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$$

so that the s_λ form an *orthonormal* basis of Λ, and the s_λ such that $|\lambda| = n$ form an orthonormal basis of Λ^n. Any other orthonormal basis of Λ^n must therefore be obtained from the basis (s_λ) by transformation by an orthogonal integer matrix. The only such matrices are signed permutation matrices, and therefore (4.8) characterizes the s_λ, up to order and sign.

Also from (4.7) or (4.8) we see that

(4.9) *The bilinear form* $\langle u, v \rangle$ *is symmetric and positive definite.* |

(4.10) *The involution* ω *is an isometry, i.e.* $\langle \omega u, \omega v \rangle = \langle u, v \rangle$.

Proof. From (2.13) we have $\omega(p_\lambda) = \pm p_\lambda$, hence by (4.7)

$$\langle \omega(p_\lambda), \omega(p_\mu) \rangle = \langle p_\lambda, p_\mu \rangle$$

which proves (4.10), since the p_λ form a $\mathbf{Q}$-basis of $\Lambda_{\mathbf{Q}}$ (2.12). |

Finally, from (4.10) and (4.5) we have

$$\langle e_\lambda, f_\mu \rangle = \delta_{\lambda\mu}$$

where $f_\mu = \omega(m_\mu)$, i.e. (e_λ) and (f_λ) are dual bases of Λ.

Remarks. 1. By applying the involution ω to the symmetric functions of the x's we obtain from (4.1), (4.2) and (4.3) three series expansions for the product $\prod_{i,j}(1 + x_i y_j)$, namely

(4.1′) $$\prod_{i,j} (1 + x_i y_j) = \sum_\lambda \varepsilon_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y),$$

(4.2′) $$\prod_{i,j} (1 + x_i y_j) = \sum_\lambda m_\lambda(x) e_\lambda(y) = \sum_\lambda e_\lambda(x) m_\lambda(y),$$

(4.3′) $$\prod_{i,j} (1 + x_i y_j) = \sum_\lambda s_\lambda(x) s_{\lambda'}(y),$$

the last by virtue of (3.8).

2. If x, y are elements of a λ-ring R, we have

$$\begin{aligned}\sigma_t(xy) &= \sum_\lambda z_\lambda^{-1}\psi^\lambda(x)\psi^\lambda(y)t^{|\lambda|}\\ &= \sum_\lambda S^\lambda(x)S^\lambda(y)t^{|\lambda|}\end{aligned}$$

from (4.1) and (4.3), and

$$\begin{aligned}\lambda_t(xy) &= \sum_\lambda \varepsilon_\lambda z_\lambda^{-1}\psi^\lambda(x)\psi^\lambda(y)t^{|\lambda|}\\ &= \sum_\lambda S^\lambda(x)S^{\lambda'}(y)t^{|\lambda|}\end{aligned}$$

from (4.1′) and (4.3′).

Examples

1. If we take $y_1 = \dots = y_n = t$, $y_{n+1} = y_{n+2} = \dots = 0$ in (4.3′), we obtain

$$\begin{aligned}E(t)^n &= \sum_\lambda s_\lambda(x)s_{\lambda'}(y)\\ &= \sum_\lambda \binom{n}{\lambda} s_\lambda(x)t^{|\lambda|}\end{aligned}$$

in the notation of §3, Ex. 4.

The coefficients of the powers of t on each side are polynomials in n (with coefficients in Λ) which are equal for all positive integral values of n, and hence identically equal. Consequently we have

$$E(t)^X = \sum_\lambda \binom{X}{\lambda} s_\lambda t^{|\lambda|}$$

for all X. By replacing X, t by $-X$, $-t$ we obtain

$$H(t)^X = \sum_\lambda \binom{X}{\lambda'} s_\lambda t^{|\lambda|}.$$

These identities generalize the binomial theorem.

2. Let $y_i = q^{i-1}$ for $1 \leqslant i \leqslant n$, and $y_i = 0$ for $i > n$. From (4.3′) we obtain

$$\prod_{i=1}^n E(q^{i-1}) = \sum_\lambda q^{n(\lambda')}\begin{bmatrix} n\\ \lambda\end{bmatrix} s_\lambda$$

in the notation of §3, Ex. 1. Likewise, from (4.3),

$$\prod_{i=1}^n H(q^{i-1}) = \sum_\lambda q^{n(\lambda)}\begin{bmatrix} n\\ \lambda'\end{bmatrix} s_\lambda.$$

In these formulas we may let $n \to \infty$ and obtain

$$\begin{aligned}\prod_{i,j\geqslant 1}(1 + x_j q^{i-1}) &= \sum_\lambda \frac{q^{n(\lambda')}}{H_\lambda(q)} s_\lambda(x),\\ \prod_{i,j\geqslant 1}(1 - x_j q^{i-1})^{-1} &= \sum_\lambda \frac{q^{n(\lambda)}}{H_\lambda(q)} s_\lambda(x),\end{aligned}$$

where $H_\lambda(q) = \prod_{x\in\lambda}(1 - q^{h(x)})$ is the hook-length polynomial corresponding to the partition λ.

3. Let $y_1 = \dots = y_n = t/n$, $y_{n+1} = y_{n+2} = \dots = 0$, and then let $n \to \infty$. We have

$$\prod_i \left(1 + \frac{x_i t}{n}\right)^n \to \prod_i \exp(x_i t) = \exp(e_1 t)$$

and

$$\frac{1}{n^{|\lambda|}}\binom{n}{\lambda} \to \prod_{x\in\lambda} h(x)^{-1} = h(\lambda)^{-1}$$

where $h(\lambda)$ is the product of the hook-lengths of λ. Hence from (4.3′) we obtain

$$\exp(e_1 t) = \sum_\lambda \frac{s_\lambda}{h(\lambda)} t^{|\lambda|}$$

and therefore

$$e_1^n = \sum_{|\lambda|=n} \frac{n!}{h(\lambda)} s_\lambda$$

or equivalently

$$\langle e_1^n, s_\lambda\rangle = n!/h(\lambda).$$

4. From (2.14′) and (4.7) we have

$$\langle h_n, p_\lambda\rangle = 1$$

for all partitions λ of n. Dually,

$$\langle e_n, p_\lambda\rangle = \varepsilon_\lambda.$$

5. $$\prod_{i=1}^m \prod_{j=1}^n (x_i + y_j) = \sum_\lambda s_\lambda(x) s_{\tilde\lambda'}(y)$$

summed over all partitions $\lambda = (\lambda_1, \dots, \lambda_m)$ such that $\lambda_1 \leqslant n$ (i.e. $\lambda \subset (n^m)$), where $\tilde\lambda' = (m - \lambda'_n, \dots, m - \lambda'_1)$ (Replace y_i by y_i^{-1} in (4.1′), and clear of fractions.)

Hence from §3, Ex. 10 we have

$$\prod_{i=1}^m \prod_{j=1}^n (1 + x_i + y_j) = \sum_{\lambda,\mu} d_{\lambda\mu} s_\mu(x) s_{\tilde\lambda'}(y)$$

summed over pairs of partitions λ, μ such that $\mu \subset \lambda \subset (n^m)$. (This formula gives the Chern classes of a tensor product $E \otimes F$ of vector bundles, since if $c(E) = \prod(1 + x_i)$ and $c(F) = \prod(1 + y_j)$ are the total Chern classes of E and F, we have $c(E \otimes F) = \prod(1 + x_i + y_j)$.)

6. Let $\Delta=\det((1-x_iy_j)^{-1})_{1\leqslant i,j\leqslant n}$ (Cauchy's determinant). Then

$$\Delta=a_\delta(x)a_\delta(y)\prod_{i,j=1}^{n}(1-x_iy_j)^{-1}.$$

For if we multiply each element of the ith row of the matrix $((1-x_iy_j)^{-1})$ by $\prod_{j=1}^{n}(1-x_iy_j)$, we shall obtain a matrix D whose (i,j) element is

$$\prod_{r\neq j}(1-x_iy_r)=\sum_{k=1}^{n}x_i^{n-k}.(-1)^{n-k}e_{n-k}^{(j)}(y)$$

in the notation of (3.6). This shows that $D=A_\delta(x)M(y)$, so that $\det(D)=a_\delta(x)a_\delta(y)$. On the other hand, it is clear from the definition of D that $\det(D)=\Delta.\prod(1-x_iy_j)$.

Since also

$$\Delta=\det(1+x_iy_j+x_i^2y_j^2+\ldots)$$
$$=\sum_\alpha\det(x_i^{\alpha_j}y_j^{\alpha_j})$$
$$=\sum_\alpha a_\alpha(x)y^\alpha$$

the summation being over all $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbf{N}^n$, it follows that

$$\Delta=\sum_\lambda a_{\lambda+\delta}(x)a_{\lambda+\delta}(y)$$

summed over all partitions λ of length $\leqslant n$. Hence we have another proof of (4.3).

7. Likewise the identity (4.3′) can be proved directly, without recourse to duality. Consider the Vandermonde determinant $a_\delta(x,y)$ in $2n$ variables $x_1,\ldots,x_n,y_1,\ldots,y_n$; on the one hand, this is equal to $a_\delta(x)a_\delta(y)\prod(x_i-y_j)$; on the other hand, expanding the determinant by Laplace's rule, we see that it is equal to

$$\sum_\mu(-1)^{e(\mu)}a_\mu(x)a_{\bar\mu}(y),\tag{1}$$

summed over $\mu\in\mathbf{N}^n$ such that $2n-1\geqslant\mu_1>\mu_2>\ldots>\mu_n\geqslant 0$, where $\bar\mu$ is the strictly increasing sequence consisting of the integers in $[0,2n-1]$ not equal to any of the μ_i, and $e(\mu)=\sum(2n-i-\mu_i)$. By writing $\mu=\lambda+\delta$ and using (1.7), we see that (1) is equal to

$$(y_1\ldots y_n)^{2n-1}\sum_\lambda a_{\lambda+\delta}(x)a_{\lambda'+\delta}(y^{-1})$$

summed over all partitions λ such that $l(\lambda)\leqslant n$ and $l(\lambda')\leqslant n$. If we now replace each y_i by y_i^{-1}, we obtain (4.3′).

Notes and references

The scalar product on Λ was apparently first introduced by P. Hall [17]. Ex. 5 is due to A. Lascoux [26].

5. Skew *S*-functions

Any symmetric function $f\in\Lambda$ is uniquely determined by its scalar products with the s_λ: namely

$$f=\sum_\lambda\langle f,s_\lambda\rangle s_\lambda$$

since the s_λ form an orthonormal basis of Λ (4.8).

Let λ, μ be partitions, and define a symmetric function $s_{\lambda/\mu}$ by the relations

$$\langle s_{\lambda/\mu},s_\nu\rangle=\langle s_\lambda,s_\mu s_\nu\rangle\tag{5.1}$$

for all partitions ν. The $s_{\lambda/\mu}$ are called *skew S-functions.* Equivalently, if $c_{\mu\nu}^\lambda$ are the integers defined by

$$s_\mu s_\nu=\sum_\lambda c_{\mu\nu}^\lambda s_\lambda,\tag{5.2}$$

then we have

$$s_{\lambda/\mu}=\sum_\nu c_{\mu\nu}^\lambda s_\nu.\tag{5.3}$$

In particular, it is clear that $s_{\lambda/0}=s_\lambda$, where 0 denotes the zero partition. Also $c_{\mu\nu}^\lambda=0$ unless $|\lambda|=|\mu|+|\nu|$, so that $s_{\lambda/\mu}$ is homogeneous of degree $|\lambda|-|\mu|$, and is zero if $|\lambda|<|\mu|$. (We shall see shortly that $s_{\lambda/\mu}=0$ unless $\lambda\supset\mu$.)

Now let $x=(x_1,x_2,\ldots)$ and $y=(y_1,y_2,\ldots)$ be two sets of variables. Then

$$\sum_\lambda s_{\lambda/\mu}(x)s_\lambda(y)=\sum_{\lambda,\nu}c_{\mu\nu}^\lambda s_\nu(x)s_\lambda(y)$$
$$=\sum_\nu s_\nu(x)s_\mu(y)s_\nu(y)$$

by (5.2) and (5.3), and therefore

$$\sum_\lambda s_{\lambda/\mu}(x)s_\lambda(y)=s_\mu(y)\sum_\nu h_\nu(x)m_\nu(y)$$

by (4.2) and (4.3). Now suppose that $y=(y_1,\ldots,y_n)$, so that the sums above are restricted to partitions λ and ν of length $\leqslant n$. Then the previous equation can be rewritten in the form

$$\sum_\lambda s_{\lambda/\mu}(x)a_{\lambda+\delta}(y)=\sum_\nu h_\nu(x)m_\nu(y)a_{\mu+\delta}(y)$$
$$=\sum_\alpha h_\alpha(x)\sum_{w\in S_n}\varepsilon(w)y^{\alpha+w(\mu+\delta)}$$

summed over $\alpha \in \mathbf{N}^n$. Hence $s_{\lambda/\mu}(x)$ is equal to the coefficient of $y^{\lambda+\delta}$ in this sum, i.e. we have

$$s_{\lambda/\mu} = \sum_{w \in S_n} \varepsilon(w) h_{\lambda+\delta-w(\mu+\delta)}$$

with the usual convention that $h_\alpha = 0$ if any component α_i of α is negative. This formula can also be written as a determinant

(5.4) $$s_{\lambda/\mu} = \det(h_{\lambda_i-\mu_j-i+j})_{1 \leqslant i,j \leqslant n}$$

where $n \geqslant l(\lambda)$.

When $\mu = 0$, (5.4) becomes (3.4).

From (5.4) and (2.9) we have also

(5.5) $$s_{\lambda/\mu} = \det(e_{\lambda'_i-\mu'_j-i+j})_{1 \leqslant i,j \leqslant m}$$

where $m \geqslant l(\lambda')$, and therefore

(5.6) $$\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}.$$

From (5.4) it follows that $s_{\lambda/\mu} = 0$ unless $\lambda_i \geqslant \mu_i$ for all i, i.e. unless $\lambda \supset \mu$. For if $\lambda_r < \mu_r$ for some r, we have $\lambda_i \leqslant \lambda_r < \mu_r \leqslant \mu_j$ for $1 \leqslant j \leqslant r \leqslant i \leqslant n$, and therefore $\lambda_i - \mu_j - i + j < 0$ for this range of values of (i, j). Consequently the matrix $(h_{\lambda_i-\mu_j-i+j})$ has an $(n-r+1)\times r$ block of zeros in the bottom left-hand corner, and therefore its determinant vanishes.

The same considerations show that if $\lambda \supset \mu$ and $\mu_r \geqslant \lambda_{r+1}$ for some $r < n$, the matrix $(h_{\lambda_i-\mu_j-i+j})$ is of the form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, where A has r rows and columns, and B has $n-r$ rows and columns, so that its determinant is equal to $\det(A)\det(B)$. Hence if the skew diagram $\lambda - \mu$ consists of two disjoint pieces θ, φ (each of which is a skew diagram), then we have $s_{\lambda/\mu} = s_\theta . s_\varphi$. To summarize:

(5.7) *The skew S-function $s_{\lambda/\mu}$ is zero unless $\lambda \supset \mu$, in which case it depends only on the skew diagram $\lambda - \mu$. If θ_i are the components (§1) of $\lambda - \mu$, we have $s_{\lambda/\mu} = \prod s_{\theta_i}$.* |

If the number of variables x_i is finite, we can say more:

(5.8) *We have $s_{\lambda/\mu}(x_1, \dots, x_n) = 0$ unless $0 \leqslant \lambda'_i - \mu'_i \leqslant n$ for all $i \geqslant 1$.*

Proof. Suppose that $\lambda'_r - \mu'_r > n$ for some $r \geqslant 1$. Since $e_{n+1} = e_{n+2} = \dots = 0$, it follows as above that the matrix $(e_{\lambda'_i-\mu'_j-i+j})$ has a rectangular block of zeros in the top right-hand corner, with one vertex of the rectangle on the main diagonal, hence its determinant vanishes. |

Now let $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$, $z = (z_1, z_2, \dots)$ be three sets of independent variables. Then by (5.2) we have

(a) $$\sum_{\lambda,\mu} s_{\lambda/\mu}(x)s_\lambda(z)s_\mu(y) = \sum_\mu s_\mu(y)s_\mu(z).\prod_{i,k}(1-x_iz_k)^{-1}$$

which by (4.3) is equal to

$$\prod_{i,k}(1-x_iz_k)^{-1}.\prod_{j,k}(1-y_jz_k)^{-1}$$

and therefore also equal to

(b) $$\sum_\lambda s_\lambda(x, y)s_\lambda(z)$$

where $s_\lambda(x, y)$ denotes the S-function corresponding to λ in the set of variables $(x_1, x_2, \dots, y_1, y_2, \dots)$. From the equality of (a) and (b) we conclude that

(5.9) $$\begin{aligned} s_\lambda(x, y) &= \sum_\mu s_{\lambda/\mu}(x)s_\mu(y) \\ &= \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(y)s_\nu(x). \end{aligned}$$

More generally, we have

(5.10) $$s_{\lambda/\mu}(x, y) = \sum_\nu s_{\lambda/\nu}(x)s_{\nu/\mu}(y)$$

summed over partitions ν such that $\lambda \supset \nu \supset \mu$.

Proof. From (5.9) we have

$$\begin{aligned} \sum_\mu s_{\lambda/\mu}(x, y)s_\mu(z) &= s_\lambda(x, y, z) \\ &= \sum_\nu s_{\lambda/\nu}(x)s_\nu(y, z) \\ &= \sum_{\mu,\nu} s_{\lambda/\nu}(x)s_{\nu/\mu}(y)s_\mu(z) \end{aligned}$$

by (5.9) again; now equate the coefficients of $s_\mu(z)$ at either end of this chain of equalities. |

The formula (5.10) may clearly be generalized, as follows. Let $x^{(1)}, \dots, x^{(n)}$ be n sets of variables, and let λ, μ be partitions. Then

(5.11) $$s_{\lambda/\mu}(x^{(1)}, \dots, x^{(n)}) = \sum_{(\nu)} \prod_{i=1}^n s_{\nu^{(i)}/\nu^{(i-1)}}(x^{(i)})$$

summed over all sequences $(\nu) = (\nu^{(0)}, \nu^{(1)}, \dots, \nu^{(n)})$ of partitions, such that $\nu^{(0)} = \mu$, $\nu^{(n)} = \lambda$ and $\nu^{(0)} \subset \nu^{(1)} \subset \dots \subset \nu^{(n)}$. |

We shall apply (5.11) in the case where each set of variables $x^{(i)}$ consists of a single variable x_i. For a single x, it follows from (5.8) that

$s_{\lambda/\mu}(x)=0$ unless $\lambda-\mu$ is a *horizontal strip* (§1), in which case $s_{\lambda/\mu}(x)=x^{|\lambda-\mu|}$. Hence each of the products in the sum on the right-hand side of (5.11) is a monomial $x_1^{\alpha_1}\dots x_n^{\alpha_n}$, where $\alpha_i=|\nu^{(i)}-\nu^{(i-1)}|$, and hence we have $s_{\lambda/\mu}(x_1,\dots,x_n)$ expressed as a sum of monomials x^α, one for each *tableau* (§1) T of shape $\lambda-\mu$. If the weight of T is $\alpha=(\alpha_1,\dots,\alpha_n)$, we shall write x^T for x^α. Then:

$$s_{\lambda/\mu}=\sum_T x^T \tag{5.12}$$

summed over all tableaux T of shape $\lambda-\mu$. |

For each partition ν such that $|\nu|=|\lambda-\mu|$, let $K_{\lambda-\mu,\nu}$ denote the number of tableaux of shape $\lambda-\mu$ and weight ν. From (5.12) we have

$$s_{\lambda/\mu}=\sum_\nu K_{\lambda-\mu,\nu}m_\nu \tag{5.13}$$

and therefore

$$K_{\lambda-\mu,\nu}=\langle s_{\lambda/\mu},h_\nu\rangle=\langle s_\lambda,s_\mu h_\nu\rangle \tag{5.14}$$

so that

$$s_\mu h_\nu=\sum_\nu K_{\lambda-\mu,\nu}s_\lambda. \tag{5.15}$$

In particular, suppose that $\nu=(r)$, a partition with only one non-zero part. Then $K_{\lambda-\mu,(r)}$ is 1 or 0 according as $\lambda-\mu$ is or is not a horizontal r-strip, and therefore from (5.15) we have

$$s_\mu h_r=\sum_\lambda s_\lambda \tag{5.16}$$

summed over all partitions λ such that $\lambda-\mu$ is a horizontal r-strip. |

By applying the involution ω to (5.16), we obtain

$$s_\mu e_r=\sum_\lambda s_\lambda \tag{5.17}$$

summed over all partitions λ such that $\lambda-\mu$ is a vertical r-strip. |

Remarks. 1. It is easy to give a direct proof of (5.17). Consider (for a finite set of variables $x_1,\dots,x_n$) the product

$$\begin{aligned}a_{\mu+\delta}e_r&=\sum_{w\in S_n}\varepsilon(w)x^{w(\mu+\delta)}\sum_\alpha x^\alpha\\&=\sum_\alpha a_{\mu+\alpha+\delta}\end{aligned}$$

where the sum is over all $\alpha\in\mathbf{N}^n$ such that each α_i is 0 or 1, and $\sum\alpha_i=r$. For each such α, the sequence

$$\mu+\alpha+\delta=(\mu_1+\alpha_1+n-1,\ \mu_2+\alpha_2+n-2,\dots,\ \mu_n+\alpha_n)$$

is in descending order, so that we have only to reject those α for which two consecutive terms are equal. We are then left with those α for which $\lambda=\mu+\alpha$ is a *partition*, i.e. such that $\lambda-\mu$ is a vertical r-strip. This proves (5.17), hence also (5.16) by duality. We can now play back the rest of the argument: (5.16) implies (5.15) by induction on the length of ν, hence (5.14), which in turn is merely a restatement of (5.13).

2. Proposition (5.12) is the origin of the application of S-functions to enumeration of plane partitions (see the examples at the end of this section). For this reason, combinatorialists often prefer to take (5.12) as the definition of S-functions (see e.g. Stanley [51]). This approach has the advantage of starting directly with a simple explicit definition, but it is not clear a priori why one should be led to make such a definition in the first place.

3. In any λ-ring we can define operations $S^{\lambda/\mu}$ by the formula (5.3):

$$S^{\lambda/\mu}=\sum_\nu c^\lambda_{\mu\nu}S^\nu$$

Then (5.9), for example, takes the form of an addition theorem:

$$S^\lambda(x+y)=\sum_\mu S^{\lambda/\mu}(x)S^\mu(y)$$

for any two elements x, y of a λ-ring. Similarly for the other formulas in this section.

Examples

1. Let $\lambda-\mu$ be a horizontal strip. Then $s_{\lambda/\mu}=h_\nu=h_{\nu_1}h_{\nu_2}\dots$ where the integers ν_i are the lengths of the components of the strip. (Use (5.7).) Likewise, if $\lambda-\mu$ is a vertical strip, we have $s_{\lambda/\mu}=e_{\nu_1}e_{\nu_2}\dots$ where again the ν_i are the lengths of the components of the strip.

2. Let λ be a partition of n. Then the number of standard tableaux of shape λ is

$$K_{\lambda,(1^n)}=\langle s_\lambda,h_1^n\rangle$$

by (5.14). By §4, Ex. 3 it follows that the number of standard tableaux of shape λ is equal to $n!/h(\lambda)$, where $h(\lambda)$ is the product of the hook-lengths of λ.

3. For each symmetric function $f\in\Lambda$, let $D(f):\Lambda\to\Lambda$ be the adjoint of multiplication by f, i.e.

$$\langle D(f)u,v\rangle=\langle u,fv\rangle$$

for all u, $v\in\Lambda$. Then $D:\Lambda\to\mathrm{End}\,(\Lambda)$ is a ring homomorphism.

(a) For each partition μ, let D_μ denote $D(s_\mu)$. Then since

$$\langle D_\mu s_\lambda, s_\nu\rangle = \langle s_\lambda, s_\mu s_\nu\rangle = \langle s_{\lambda/\mu}, s_\nu\rangle$$

for all λ, μ, ν, it follows that $D_\mu s_\lambda = s_{\lambda/\mu}$.

Hence from (5.9) we have

$$s_\lambda(x, y) = \sum_\mu D_\mu s_\lambda(x) . s_\mu(y)$$

and therefore, for any $f \in \Lambda$,

$$f(x, y) = \sum_\mu D_\mu f(x) . s_\mu(y).$$

(b) We have $D(h_\lambda)m_\mu = 0$ unless $\mu = \lambda \cup \nu$ for some partition ν, and in that case $D(h_\lambda)m_\mu = m_\nu$. For

$$\langle D(h_\lambda)m_\mu, h_\nu\rangle = \langle m_\mu, h_\lambda h_\nu\rangle = \langle m_\mu, h_{\lambda\cup\nu}\rangle$$

which is zero unless $\mu = \lambda \cup \nu$.

In particular, $D(h_n)m_\mu = 0$ if n is not a part of μ, and $D(h_n)m_\mu = m_\nu$ if n is a part of μ, where ν is the partition obtained by removing one part n from μ. It follows that for every $f(x_0, x_1, x_2, \dots) \in \Lambda$, $(D(h_n)f)(x_1, x_2, \dots)$ is the coefficient of x_0^n in f.

(c) Next consider $D(p_n)$. If $N \geqslant n$ we have

$$\langle D(p_n)h_N, p_\lambda\rangle = \langle h_N, p_n p_\lambda\rangle = 1 = \langle h_{N-n}, p_\lambda\rangle$$

for all partitions λ of $N-n$, by §4, Ex. 4. Hence

$$D(p_n)h_N = h_{N-n}$$

and therefore

$$D(p_n) = \sum_{r\geqslant 0} h_r\, \partial/\partial h_{n+r}$$

acting on symmetric functions expressed as polynomials in the h's.

Dually

$$D(p_n) = (-1)^{n-1}\sum_{r\geqslant 0} e_r\, \partial/\partial e_{n+r}$$

acting on symmetric functions expressed as polynomials in the e's.

Further, we have $\langle D(p_n)p_\lambda, p_\mu\rangle = \langle p_\lambda, p_n p_\mu\rangle$, which is 0 if $\lambda \neq \mu \cup (n)$, and is equal to z_λ if $\lambda = \mu \cup (n)$. It follows that $D(p_n)p_\lambda = 0$ if n is not a part of λ, and that $D(p_n)p_\lambda = z_\lambda z_\mu^{-1}$ if n is a part of λ, and μ is the partition obtained by removing one part n from λ. From the definition of z_λ it follows that $z_\lambda z_\mu^{-1} = n . m_n(\lambda)$, where $m_n(\lambda)$ is the multiplicity of n as a part of λ, and therefore

$$D(p_n) = n\partial/\partial p_n,$$

acting on symmetric functions expressed as polynomials in the p's. In particular, each $D(p_n)$ is a derivation of Λ.

Since each $f \in \Lambda$ can be expressed as a polynomial $\varphi(p_1, p_2, \dots)$ with rational coefficients, it follows that

$$D(f) = \varphi(\partial/\partial p_1, 2\partial/\partial p_2, \dots)$$

is a linear differential operator with constant coefficients.

4. We have

$$\sum_\lambda s_\lambda = \prod_i (1-x_i)^{-1} \prod_{i<j} (1-x_i x_j)^{-1},$$

where the sum on the left is over all partitions λ.

It is enough to prove this for a finite set of variables $x_1, \dots, x_n$. Let $\Phi(x_1, \dots, x_n)$ denote $\sum_\lambda s_\lambda(x_1, \dots, x_n)$, which is now a sum over partitions λ of length $\leqslant n$. By induction on n, it is enough to show that

$$\Phi(x_1, \dots, x_n, y) = \Phi(x_1, \dots, x_n)(1-y)^{-1} \prod_{i=1}^n (1-x_i y)^{-1}.$$

From (5.9) it follows that

$$\Phi(x_1, \dots, x_n, y) = \sum_{\lambda,\mu} y^{|\lambda-\mu|} s_\mu(x_1, \dots, x_n)$$

where the sum on the right is over all pairs of partitions $\lambda \supset \mu$ such that $l(\mu) \leqslant n$ and $\lambda - \mu$ is a horizontal strip. For each such pair λ, μ, define $\nu \subset \mu$ by $\mu_i - \nu_i = \lambda_{i+1} - \mu_{i+1} (i \geqslant 1)$, so that $|\lambda - \mu| = \lambda_1 - \mu_1 + |\mu - \nu|$. Then λ can be reconstructed from μ, ν and the integer $\lambda_1 - \mu_1$, and hence

$$(*) \qquad \sum_{\lambda,\mu} y^{|\lambda-\mu|} s_\mu(x_1, \dots, x_n) = \sum_{\mu,\nu} y^{|\mu-\nu|}(1-y)^{-1} s_\mu(x_1, \dots, x_n),$$

the sum on the right being over pairs of partitions $\mu \supset \nu$ such that $l(\mu) \leqslant n$ and $\mu - \nu$ is a horizontal strip. By (5.16), the right-hand side of $(*)$ is equal to

$$\sum_{\nu,r} y^r (1-y)^{-1} h_r(x_1, \dots, x_n) s_\nu(x_1, \dots, x_n)$$

summoned over all partitions ν of length $\leqslant n$, and all integers $r \geqslant 0$; and this last sum is equal to $(1-y)^{-1} \prod_{i=1}^n (1-x_i y)^{-1} \Phi(x_1, \dots, x_n)$, as required.

5. We have

$$\sum_{\mu \text{ even}} s_\mu = \prod_i (1-x_i^2)^{-1} \prod_{i<j} (1-x_i x_j)^{-1},$$

where the sum on the left is over all *even* partitions μ (i.e., with all parts μ_i even).

Each partition λ can be reduced to an even partition μ by removing a vertical strip, in exactly one way: we take $\mu_i = \lambda_i$ if λ_i is even, and $\mu_i = \lambda_i - 1$ if λ_i is odd. From this observation and (5.17) it follows that

$$\Big(\sum_{\mu \text{ even}} s_\mu\Big)\Big(\sum_{r\geqslant 0} e_r\Big) = \sum_\lambda s_\lambda,$$

the sum on the right being over *all* partitions λ. Since $\sum e_r = \Pi(1+x_i)$, the result now follows from Ex. 4.

6. We have

$$\sum_{\nu' \text{ even}} s_\nu = \prod_{i<j} (1-x_ix_j)^{-1}.$$

The proof is dual to that of Ex. 5: each partition λ can be reduced to one with even columns by removing a horizontal strip, in exactly one way. From this observation and (5.16) it follows that

$$\left(\sum_{\nu' \text{ even}} s_\nu\right)\left(\sum_{r\geqslant 0} h_r\right) = \sum_\lambda s_\lambda,$$

and since $\sum_{r\geqslant 0} h_r = \Pi(1-x_i)^{-1}$, the result again follows from Ex. 4.

The involution ω interchanges the identities of Ex. 5 and Ex. 6.

7. The same argument as in Ex. 6 shows that

$$\sum_\lambda t^{c(\lambda)} s_\lambda = \prod_i (1-tx_i)^{-1} \prod_{i<j} (1-x_ix_j)^{-1}$$

where the sum is over all partitions λ, and $c(\lambda)$ is the number of columns of odd length in λ. This includes the identities of Ex. 4 (when $t=1$) and Ex. 6 (when $t=0$).

8. By applying the involution ω to Ex. 7 we obtain

$$\sum_\lambda t^{r(\lambda)} s_\lambda = \prod_i \frac{1+tx_i}{1-x_i^2} \prod_{i<j} \frac{1}{1-x_ix_j}$$

where the sum is over all partitions λ, and $r(\lambda)$ is the number of rows of odd length in λ. When $t=1$ this reduces to Ex. 4, and when $t=0$ it reduces to Ex. 5.

9. The products

$$\prod_{i<j}(1-x_ix_j), \quad \prod_i (1-x_i) \prod_{i<j} (1-x_ix_j), \quad \prod_i (1-x_i^2) \prod_{i<j} (1-x_ix_j).$$

(i.e., the reciprocals of those of Exx. 4,5,6) can also be expanded as series of S-functions. The expansions may be derived from Weyl's identity for the root-systems of types D_n, B_n, C_n respectively. (If R is a root system with Weyl group W, R^+ a system of positive roots, ρ half the sum of the positive roots, then Weyl's identity ([6], p. 185) is

$$(*) \qquad \sum_{w\in W} \varepsilon(w) e^{w\rho} = \prod_{\alpha\in R^+} (e^{\alpha/2} - e^{-\alpha/2})$$

where $\varepsilon(w)$ is the sign of $w \in W$, and the e's are formal exponentials.)

(a) When R is of type D_n, the identity $(*)$ leads to

$$\sum_\pi (-1)^{|\pi|/2} s_\pi(x_1,\ldots,x_n) = \prod_{i<j} (1-x_ix_j)$$

summed over all partitions $\pi = (\alpha_1-1,\ldots,\alpha_p-1 \mid \alpha_1,\ldots,\alpha_p)$ in Frobenius notation, where $\alpha_1 \leqslant n-1$.

(b) When R is of type C_n, we obtain from $(*)$

$$\sum_\rho (-1)^{|\rho|/2} s_\rho(x_1,\ldots,x_n) = \prod_i (1-x_i^2) \prod_{i<j} (1-x_ix_j)$$

summed over all partitions $\rho = (\alpha_1+1,\ldots,\alpha_p+1 \mid \alpha_1,\ldots,\alpha_p)$, where $\alpha_1 \leqslant n-1$.

(c) When R is of type B_n, we obtain from $(*)$

$$\sum_\sigma (-1)^{(|\sigma|+p(\sigma))/2} s_\sigma(x_1,\ldots,x_n) = \prod_i (1-x_i) \prod_{i<j} (1-x_ix_j)$$

summed over all self-conjugate partitions $\sigma = (\alpha_1,\ldots,\alpha_p \mid \alpha_1,\ldots,\alpha_p)$ such that $\alpha_1 \leqslant n-1$, where $p(\sigma)=p$.

10. In the language of λ-rings, the identities of Exs. 5, 6, and 9 give series expansions (in terms of S-operations) for $\sigma_t(\sigma^2(x))$, $\sigma_t(\lambda^2(x))$, $\lambda_t(\sigma^2(x))$ and $\lambda_t(\lambda^2(x))$, namely

$$\sigma_t(\sigma^2(x)) = \sum_{\mu \text{ even}} S^\mu(x) t^{|\mu|/2},$$

$$\sigma_t(\lambda^2(x)) = \sum_{\nu' \text{ even}} S^\nu(x) t^{|\nu|/2},$$

$$\lambda_t(\sigma^2(x)) = \sum_\rho S^\rho(x) t^{|\rho|/2},$$

$$\lambda_t(\lambda^2(x)) = \sum_\pi S^\pi(x) t^{|\pi|/2},$$

the last two summations being over partitions $\rho = (\alpha_1+1,\ldots,\alpha_p+1 \mid \alpha_1,\ldots,\alpha_p)$ and $\pi = (\alpha_1-1,\ldots,\alpha_p-1 \mid \alpha_1,\ldots,\alpha_p)$.

11. Let $x_1 = \ldots = x_N = t$, $x_{N+1} = x_{N+2} = \ldots = 0$ in the formula of Ex. 4. Then $s_\lambda = \binom{N}{\lambda'} t^{|\lambda|}$ (§3, Ex. 4) and hence, for each $n \geqslant 0$,

$$\begin{aligned}\sum_{|\lambda|=n} \binom{N}{\lambda} &= \text{coefficient of } t^n \text{ in } (1-t)^{-N}(1-t^2)^{-N(N-1)/2} \\ &= \text{coefficient of } t^n \text{ in } (1-t)^{-N(N+1)/2}(1+t)^{-N(N-1)/2}\end{aligned}$$

Since this is true for all positive integers N, it is a polynomial identity, i.e.

$$\sum_{|\lambda|=n} \binom{X}{\lambda} = \text{coefficient of } t^n \text{ in } (1-t)^{-X(X+1)/2}(1+t)^{-X(X-1)/2}.$$

12. Let $x_1 = \ldots = x_N = t/N$, $x_{N+1} = x_{N+2} = \ldots = 0$ in the identity of Ex. 4, and let $N \to \infty$. Then from Ex. 11 we obtain

$$\sum_{|\lambda|=n} h(\lambda)^{-1} = \text{coefficient of } t^n \text{ in } \exp(t+\tfrac{1}{2}t^2)$$

where (§4, Ex. 3) $h(\lambda)$ is the product of the hook lengths of λ. From Ex. 2 it follows that the total number of standard tableaux of weight (1^n) is equal to $n!$ multiplied by the coefficient of t^n in $\exp(t+\frac{1}{2}t^2)$. This number is also the number of permutations $w \in S_n$ such that $w^2 = 1$.

13. Let λ be a partition. A *plane partition of shape* λ is a mapping π from (the diagram of) λ to the positive integers such that $\pi(x_1) \geqslant \pi(x_2)$ whenever x_2 lies below or to the right of x_1 in λ. The numbers $\pi(x)$ are the *parts* of π, and

$$|\pi| = \sum_{x \in \lambda} \pi(x)$$

is called the *weight* of π. Any plane partition π determines a sequence $\lambda = \lambda^{(0)} \supset \lambda^{(1)} \supset \ldots$ of (linear) partitions such that $\pi^{-1}(i) = \lambda^{(i-1)} - \lambda^{(i)}$ for each $i \geqslant 1$.

If $\pi(x_1) > \pi(x_2)$ whenever x_2 lies directly below x_1 (i.e., if the parts of π decrease strictly down each column) then π is said to be *column-strict*. Clearly π is column-strict if and only if each skew diagram $\pi^{-1}(i) = \lambda^{(i-1)} - \lambda^{(i)}$ is a horizontal strip.

A plane partition π has a 3-dimensional *diagram*, consisting of the points (i, j, k) with integer coordinates such that $(i, j) \in \lambda$ and $1 \leqslant k \leqslant \pi(i, j)$. Alternatively, we may think of the diagram of π as a set of unit cubes, such that $\pi(x)$ cubes are stacked vertically on each square $x \in \lambda$. As in the case of ordinary (linear) partitions, we shall use the same symbol π to denote a plane partition and its diagram.

If S is any set of plane partitions, the *generating function* of S is the polynomial or formal power series

$$\sum_{\pi \in S} q^{|\pi|}$$

in which the coefficient of q^n is the number of plane partitions of weight n which belong to S.

(a) Consider column-strict plane partitions of shape λ, with all parts $\leqslant n$. By (5.12) the generating function for these is $s_\lambda(q^n, q^{n-1}, \ldots, q)$, which by §3, Ex. 1 is

$$q^{|\lambda|+n(\lambda)} \prod_{x \in \lambda} \frac{1-q^{n+c(x)}}{1-q^{h(x)}}.$$

(b) Let l, m, n be three positive integers, and consider the set of plane partitions π with all parts $\leqslant n$ and shape λ such that $l(\lambda) \leqslant l$ and $l(\lambda') \leqslant m$: that is, the set of three-dimensional diagrams π which fit inside a box B with side-lengths l, m, n. By adding $l+1-i$ to each part in the ith now of π, for $1 \leqslant i \leqslant l$, we convert π into a column-strict plane partition of shape $(m, \ldots, m) = (m^l)$ and largest part $\leqslant l+n$. From (a) above, the generating function for the plane partitions $\pi \subset B$ is therefore

$$\prod_{x \in (m^l)} \frac{1-q^{l+n+c(x)}}{1-q^{h(x)}}. \tag{1}$$

In this form the result does not display the symmetry which it must have as a function of l, m, and n. It may be rewritten as follows: for each $y = (i, j, k) \in B$, define the *height* of y to be $\mathrm{ht}(y) = i+j+k-2$ (so that the point $(1, 1, 1)$ has height 1). Then the generating function (1) may be written in the form

$$\sum_{\pi \subset B} q^{|\pi|} = \prod_{y \in B} \frac{1-q^{1+\mathrm{ht}(y)}}{1-q^{\mathrm{ht}(y)}}. \tag{2}$$

(c) We may now let any or all of l, m, n become infinite. The most striking result is obtained by letting all of l, m, n tend to ∞: the box B is then replaced by the positive octant, and for each $n \geqslant 1$ the number of lattice points (i, j, k) with $i+j+k-2 = n$ and $i, j, k \geqslant 1$ is equal to the coefficient of t^{n-1} in $(1-t)^{-3}$, hence to $\frac{1}{2}n(n+1)$. It follows that the generating function for *all* plane partitions is

$$\prod_{n=1}^{\infty} \left(\frac{1-q^{n+1}}{1-q^n}\right)^{n(n+1)/2} = \prod_{n=1}^{\infty} (1-q^n)^{-n}. \tag{3}$$

(d) Likewise, the generating function for all plane partitions with largest part $\leqslant m$ is

$$\prod_{n=1}^{\infty} (1-q^n)^{-\min(m,n)}. \tag{4}$$

14. From Ex. 13(a), by letting $n \to \infty$, the generating function for all column-strict plane partitions of shape λ is

$$q^{|\lambda|+n(\lambda)} H_\lambda(q)^{-1} \tag{1}$$

where $H_\lambda(q) = \prod_{x \in \lambda} (1-q^{h(x)})$.

Another way of obtaining this generating function is as follows. Let π be a column-strict plane partition of shape λ, and let S be the set of pairs $(\pi(i, j), j)$ where $(i, j) \in \lambda$. The elements of S are all distinct, because π is column-strict. We order S as follows: (r, j) precedes (r', j') if either $r > r'$, or $r = r'$ and $j < j'$. This is a linear ordering of S. Define a standard tableau $T(\pi)$ of shape λ as follows: $T(i, j) = k \Leftrightarrow (\pi(i, j), j)$ is the kth element of S in the linear ordering defined above. Then $T(\pi)$ is a standard tableau. For example, if π is

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then S is the ordered set

$$(3, 1), (3, 2), (2, 1), (2, 2), (2, 3), (1, 1), (1, 4), (1, 5),$$

and $T(\pi)$ is the standard tableau

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Conversely, let T be a standard tableau of shape λ, and let π be a column-strict plane partition such that $T(\pi) = T$. Let $|\lambda| = n$, and for $1 \leqslant k \leqslant n$ let a_k be the part of π in the square occupied by k in T. Then $a_1 \geqslant \ldots \geqslant a_n \geqslant 1$ and

$$a_k > a_{k+1} \Leftrightarrow k \in R(T) \qquad \longleftarrow$$

where $R(T)$ is the set of integers $k \in [1, n-1]$ such that $k+1$ lies in a lower row than k in the tableau T. Now let

$$b_k = \begin{cases} a_k - a_{k+1} & \text{if} \quad k \notin R(T) \quad \text{and} \quad k \neq n \\ a_k - a_{k+1} - 1 & \text{if} \quad k \in R(T) \\ a_n - 1 & \text{if} \quad k = n \end{cases}$$

so that $b_k \geqslant 0$ for $k = 1, 2, \dots, n$. Then we have

$$\sum_{k=1}^{n} a_k = n + r(T) + \sum_{k=1}^{n} kb_k$$

where

$$r(T) = \sum \{k : k+1 \text{ lies in a lower row than } k \text{ in } T\}$$

and therefore the generating function for the column-strict plane partitions π such that $T(\pi) = T$ is

$$q^{n+r(T)}\varphi_n(q)^{-1}$$

where as usual $\varphi_n(q) = (1-q)\dots(1-q^n)$.

Hence the generating function for column-strict plane partitions of shape λ is

$$q^n\Big(\sum_T q^{r(T)}\Big)\Big/\varphi_n(q) \tag{2}$$

summed over all standard tableaux T of shape λ.

From (1) and (2) it follows that

$$\sum_T q^{r(T)} = q^{n(\lambda)}\varphi_n(q)/H_\lambda(q). \tag{3}$$

15. Let S be any set of positive integers. From (5.12) and Ex. 4 it follows that the generating function for column-strict plane partitions all of whose parts belong to S is

$$\prod_{i \in S}(1-q^i)^{-1} \prod_{\substack{i,j \in S \\ i<j}} (1-q^{i+j})^{-1}.$$

(a) Take S to consist of all the positive integers. Then the generating function for *all* column-strict plane partitions, of arbitrary shape, is

$$\prod_{n=1}^{\infty} (1-t^n)^{-[(n+1)/2]}. \tag{1}$$

(b) Take S to consist of all the *odd* positive integers. We obtain the generating function

$$\prod_{n=1}^{\infty} ((1-t^{2n-1})^{-1}(1-t^{2n})^{-[n/2]}. \tag{2}$$

Now the column-strict plane partitions with all parts odd are in one-to-one correspondence with the *symmetrical* plane partitions π (i.e. such that $\pi(i,j) = \pi(j,i)$). For the diagram of a symmetrical plane partition may be thought of as a sequence of diagrams of symmetrical (linear) partitions $\pi^{(1)} \supset \pi^{(2)} \supset \dots$, piled one on top of the other; each $\pi^{(i)}$ is of the form $(\alpha_1, \dots, \alpha_p \mid \alpha_1, \dots, \alpha_p)$ in Frobenius notation, and hence determines a linear partition $\sigma^{(i)} = (2\alpha_1 + 1, \dots, 2\alpha_p + 1)$ with odd parts, all distinct; and the $\sigma^{(i)}$ can be taken as the columns of a column-strict plane partition with odd parts. It follows that (2) is the generating function for the set of all symmetrical plane partitions.

16. Let $\Phi(x_1, \dots, x_n) = \prod_i (1-x_i)^{-1} \prod_{i<j} (1-x_ix_j)^{-1}$ as in Ex. 4.

By setting $t = 0$ in the identity of III, §5, Ex. 5 we obtain

$$\sum_{m,\lambda} u^m s_\lambda(x_1, \dots, x_n) = \sum_\varepsilon \Phi(x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n})/(1 - u \prod x_i^{(1-\varepsilon_i)/2}) \tag{1}$$

where the sum on the left is over all partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ of length $\leqslant n$, and integers $m \geqslant \lambda_1$; and the sum on the right is over all $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ with each $\varepsilon_i = \pm 1$.

We shall rewrite (1) in the notation of root-systems. Let $v_1, \dots, v_n$ be the standard basis of $\mathbf{R}^n$. Then the set of vectors

$$R = \{\pm v_i (1 \leqslant i \leqslant n), \pm v_i \pm v_j (1 \leqslant i < j \leqslant n)\}$$

is a root-system of type B_n, for which

$$R^+ = \{v_i (1 \leqslant i \leqslant n), v_i \pm v_j (1 \leqslant i < j \leqslant n)\}$$

is a system of positive roots, so that

$$\rho = \tfrac{1}{2}((2n-1)v_1 + (2n-3)v_2 + \dots + v_n)$$

is half the sum of the positive roots. The subset R_0 of R defined by

$$R_0 = \{v_i - v_j : i \neq j\}$$

is a subsystem of R of type A_{n-1}, and $R_0^+ = R^+ \cap R_0$ is a system of positive roots for R_0. The Weyl group W_0 of R_0 is the symmetric group S_n, acting by permutations of $v_1, \dots, v_n$, and the Weyl group W of R is the semidirect product of W_0 with the group (of order 2^n) of transformations $w_\varepsilon : v_i \mapsto \varepsilon_i v_i (1 \leqslant i \leqslant n)$, where as before $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ and each ε_i is ± 1. In this notation,

$$\begin{aligned}\Phi(e^{-v_1}, \dots, e^{-v_n}) &= \prod_{\alpha \in R_0^+} (1-e^{-\alpha}) \Big/ \prod_{\alpha \in R^+} (1-e^{-\alpha}) \\ &= \Big(\sum_{w \in W_0} \varepsilon(w)e^{w\rho}\Big) \Big/ \Big(\sum_{w \in W} \varepsilon(w)e^{w\rho}\Big),\end{aligned}$$

by virtue of Weyl's identity (Ex. 9). It follows that the right-hand side of (1) (with x_i replaced by e^{-v_i}) may be written as a sum over W, and by equating the coefficients of u^m on either side of (1) we arrive at the identity

$$\sum_\lambda s_\lambda(e^{-v_1}, \dots, e^{-v_n}) = e^{-m\theta}J(m\theta + \rho)/J(\rho) \tag{2}$$

where $\theta = \tfrac{1}{2}(v_1 + \dots + v_n)$, and for any vector v

$$J(v) = \sum_{w \in W} \varepsilon(w)e^{wv},$$

and the sum on the left is over all partitions λ such that $l(\lambda) \leqslant n$ and $l(\lambda') \leqslant m$ (i.e. such that $\lambda \subset (m^n)$).

(If preferred, the right-hand side of (2) can be written as a quotient of determinants:

$$\sum_\lambda s_\lambda(x_1, \dots, x_n) = D_m/D_0 \tag{2'}$$

where $D_m = \det(x_j^{m+2n-i} - x_j^{i-1})_{1\leqslant i,j\leqslant n}$, and the summation is as before over partitions $\lambda \subset (m^n)$.)

This identity (2) is a polynomial identity in n independent variables e^{-v_i}. We may therefore specialize it to obtain identities in one variable q, by replacing each e^{-v_i} by q^{f_i}, where the f_i are arbitrary integers. This means that each exponential e^{-v} is replaced by $q^{\langle v,f\rangle}$, where $f = \sum f_i v_i$ and $\langle v, f\rangle$ is the standard scalar product on $\mathbf{R}^n$. In this way we obtain

$$(3) \qquad \sum_{\lambda} s_\lambda(q^{f_1}, \ldots, q^{f_n}) = q^{m\langle\theta,f\rangle} \frac{\sum \varepsilon(w) q^{-\langle m\theta+\rho,wf\rangle}}{\sum \varepsilon(w) q^{-\langle\rho,wf\rangle}},$$

the sum on the left being over all partitions $\lambda \subset (m^n)$.

17 In formula (3) of Ex. 16 let us take $f = 2\rho$, the sum of the positive roots of R, so that $f_i = 2n - 2i + 1$. On the right-hand side, the alternating sum

$$\sum \varepsilon(w) q^{-\langle m\theta+\rho, 2w\rho\rangle}$$

is by Weyl's identity (Ex. 9) equal to the product

$$\prod_{\alpha\in R^+} (q^{-\langle m\theta+\rho,\alpha\rangle} - q^{\langle m\theta+\rho,\alpha\rangle})$$

and therefore the right-hand side of (3) is equal to

$$\prod_{\alpha\in R^+} \frac{q^{\langle 2m\theta+2\rho,\alpha\rangle}-1}{q^{\langle 2\rho,\alpha\rangle}-1}.$$

In this product the positive roots $v_i - v_j (i<j)$ make no contribution, because they are orthogonal to $\theta = \frac{1}{2}\sum v_i$. Hence we obtain the identity

$$(4) \qquad \sum_{\lambda\subset(m^n)} s_\lambda(q^{2n-1}, q^{2n-3}, \ldots, q) = \prod_{i=1}^{n} \frac{q^{m+2i-1}-1}{q^{2i-1}-1} \prod_{1\leqslant i<j\leqslant n} \frac{q^{2(m+i+j-1)}-1}{q^{2(i+j-1)}-1}.$$

The left-hand side of (4) is the generating function for column-strict plane partitions with odd parts $\leqslant 2n-1$, and with at most m columns and at most n rows; or equivalently (Ex. 15) it is the generating function for *symmetrical* plane partitions π whose diagrams are contained in the box $B = B_{(n,n,m)} = \{(i,j,k): 1\leqslant i,j\leqslant n, 1\leqslant k\leqslant m\}$.

The right-hand side of (4) can be rewritten in a form analogous to that of Ex. 13, formula (2), as follows. Let G_2 be the group of two elements consisting of the identity and the mapping $(i,j,k)\mapsto(j,i,k)$, so that the box B is stable under G_2. For each orbit η of G_2 in B let $|\eta|$ ($=1$ or 2) be the number of elements of η, and let

$$\operatorname{ht}(\eta) = \sum_{y\in\eta} \operatorname{ht}(y)$$

where $\operatorname{ht}(i,j,k) = i+j+k-2$ as in Ex. 13. Then the generating function for symmetrical plane partitions $\pi \subset B$ is

$$(5) \qquad \prod_{\eta\in B/G_2} \frac{1-q^{\operatorname{ht}(\eta)+|\eta|}}{1-q^{\operatorname{ht}(\eta)}}.$$

18. Let G_3 be the group of three elements generated by $(i,j,k)\mapsto(j,k,i)$ and let C_n be the cube $\{(i,j,k): 1\leqslant i,j,k\leqslant n\}$. The formula (5) of Ex. 17 suggests the following conjecture: the generating function for *cyclically symmetric* plane partitions π (i.e. those whose diagrams are stable under G_3) contained in the cube C_n should be

$$(6) \qquad \prod_{\eta\in C_n/G_3} \frac{1-q^{\operatorname{ht}(\eta)+|\eta|}}{1-q^{\operatorname{ht}(\eta)}}.$$

This has been checked numerically for $n\leqslant 10$ (J. McKay) and has recently been proved for arbitrary n and $q=1$ by G. E. Andrews [4].

19. With the notation of Ex. 16, the set of vectors

$$R_1 = \{\pm 2v_i (1\leqslant i\leqslant n), \pm v_i \pm v_j (1\leqslant i<j\leqslant n)\}$$

is a root system of type C_n, for which

$$R_1^+ = \{2v_i (1\leqslant i\leqslant n), v_i \pm v_j (1\leqslant i<j\leqslant n)\}$$

is a system of positive roots, so that

$$\rho_1 = nv_1 + (n-1)v_2 + \ldots + v_n$$

is half the sum of the positive roots. The Weyl group is the same group W as in Ex. 16.

We shall take $f = \rho_1$ in formula (3) of Ex. 16, so that e^{-v_i} is replaced by q^{n-i+1}. As in Ex. 17, by virtue of Weyl's identity we have

$$\sum \varepsilon(w) q^{-\langle m\theta+\rho, w\rho_1\rangle} = \prod_{\alpha\in R_1^+} (q^{-\langle m\theta+\rho,\alpha/2\rangle} - q^{\langle m\theta+\rho,\alpha/2\rangle})$$

and therefore the right-hand side of (3) is equal to

$$\prod_{\alpha\in R_1^+} \frac{q^{\langle m\theta+\rho,\alpha\rangle}-1}{q^{\langle\rho,\alpha\rangle}-1}.$$

Again the roots $v_i - v_j (i<j)$ make no contribution to this product, and hence we obtain

$$(7) \qquad \sum_{\lambda\subset(m^n)} s_\lambda(q^n, \ldots, q) = \prod_{1\leqslant i\leqslant j\leqslant n} \frac{q^{m+i+j-1}-1}{q^{i+j-1}-1}.$$

The left-hand side of (7) is the generating function for column-strict plane partitions with largest part $\leqslant n$ and at most m columns, and the right-hand side can be written in terms of the height function introduced in Ex. 13, namely as

$$\prod_{y\in D} \frac{1-q^{\operatorname{ht}(y)+1}}{1-q^{\operatorname{ht}(y)}}$$

where D is the prism $\{(i,j,k): 1\leqslant i\leqslant j\leqslant n, 1\leqslant k\leqslant m\}$.

Notes and references

Ex. 2. The fact that the number of standard tableaux of shape λ is equal to $n!/h(\lambda)$ is due to Frame, Robinson, and Thrall [11]. No simple direct combinatorial proof seems to be known.

Ex. 3. The operators $D(e_n)$, $D(h_n)$ were introduced by Hammond [18] and the $D(s_\lambda)$ by Foulkes [9], in both cases defined as differential operators. See also Foulkes [10].

Ex. 4. This identity is usually ascribed to Littlewood [29], p. 238; however, in an equivalent form it was stated by Schur in 1918 (Ges. Abhandlungen, Vol. 3, p. 456). Bender & Knuth found an elegant combinatorial proof (reproduced in [51], p. 177), using the properties of Knuth's correspondence.

Exx. 5, 6, 9. These identities are all due to Littlewood (*loc. cit.*) The observation that the identities of Ex. 9 follow naturally from Weyl's identity for the classical root systems is, I believe, new.

Exx. 13, 14, 15. Plane partitions were first investigated by MacMahon [35], and the generating functions (1), (3), and (4) of Ex. 13 are due originally to him, but proved differently. The application of S-functions to these problems is due to Stanley [51], who gives more details and references.

Exx. 16, 17. The results in these examples are new. MacMahon [35] conjectured the generating function (Ex. 17, (4)) for symmetrical plane partitions, but was unable to prove it. It remained a conjecture until proved recently by Andrews [3]. His proof is quite different from that given here.

Ex. 19. The generating function (7) was established by Gordon (see Stanley, *loc. cit.* p. 265) who did not publish his proof. It too was proved recently by Andrews [3].

6. Transition matrices

In this section we shall be dealing with matrices whose rows and columns are indexed by the partitions of a positive integer n. We shall regard the partitions of n as arranged in reverse lexicographical order (§1), so that (n) comes first and (1^n) comes last. It follows from (1.10) that λ precedes μ if $\lambda \geqslant \mu$ (but not conversely). A matrix $(M_{\lambda\mu})$ indexed by the partitions of n will be said to be *strictly upper triangular* if $M_{\lambda\mu} = 0$ unless $\lambda \geqslant \mu$, and *strictly upper unitriangular* if also $M_{\lambda\lambda} = 1$ for all λ. Likewise we define *strictly lower triangular* and *strictly lower unitriangular*.

Let U_n (resp. U'_n) denote the set of strictly upper (resp. lower) unitriangular matrices with integer entries, indexed by the partitions of n.

(6.1) U_n, U'_n *are groups (with respect to matrix multiplication).*

Proof. Suppose $M, N \in U_n$. Then $(MN)_{\lambda\mu} = \sum_\nu M_{\lambda\nu} N_{\nu\mu}$ is zero unless there exists a partition ν such that $\lambda \geqslant \nu \geqslant \mu$, i.e. unless $\lambda \geqslant \mu$. For the same reason, $(MN)_{\lambda\lambda} = M_{\lambda\lambda} N_{\lambda\lambda} = 1$. Hence $MN \in U_n$.

Now let $M \in U_n$. The set of equations

$$\sum_\mu M_{\lambda\mu} x_\mu = y_\lambda \tag{1}$$

is equivalent to

$$\sum (M^{-1})_{\lambda\mu} y_\mu = x_\lambda. \tag{2}$$

For a fixed λ, the equations (1) for y_ν where $\nu \leqslant \lambda$, involve only the x_μ for $\mu \leqslant \nu$, hence for $\mu \leqslant \lambda$. Hence the same is true of the equations (2), and therefore $(M^{-1})_{\lambda\mu} = 0$ unless $\mu \leqslant \lambda$. It follows that $M^{-1} \in U_n$. |

Let J denote the transposition matrix:

$$J_{\lambda\mu} = \begin{cases} 1 & \text{if } \lambda' = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

(6.2) *M is strictly upper triangular* (resp. *unitriangular*) *if and only if JMJ is strictly lower triangular* (resp. *unitriangular*).

Proof. If $N = JMJ$, we have $N_{\lambda\mu} = M_{\lambda'\mu'}$. By (1.11), $\lambda' \geqslant \mu'$ if and only if $\mu \geqslant \lambda$, whence the result. |

If (u_λ), (v_λ) are any two $\mathbf{Z}$-bases of Λ^n, each indexed by the partitions of n, we denote by $M(u, v)$ the matrix $(M_{\lambda\mu})$ of coefficients in the equations

$$u_\lambda = \sum_\mu M_{\lambda\mu} v_\mu ;$$

$M(u, v)$ is called the *transition matrix* from the basis (u_λ) to the basis (v_λ). It is a non-singular matrix of integers.

(6.3) *Let (u_λ), (v_λ), (w_λ) be $\mathbf{Z}$-bases of Λ^n. Then*

$$M(u, v)M(v, w) = M(u, w), \tag{1}$$

$$M(v, u) = M(u, v)^{-1}. \tag{2}$$

Let (u'_λ), (v'_λ) be the bases dual to (u_λ), (u_λ) respectively (with respect to the scalar product of §4). *Then*

$$M(u', v') = M(v, u)' = M(u, v)^* \tag{3}$$

(where M' denotes the transpose and M^* the transposed inverse of a matrix M).

$$M(\omega u, \omega v) = M(u, v) \tag{4}$$

where $\omega : \Lambda \to \Lambda$ is the involution defined in §2.

All of these assertions are obvious.

Consider now the five **Z**-bases of Λ^n defined in §2 and §3: (e_λ), (f_λ), (h_λ), (m_λ), (s_λ). We shall show that all the transition matrices relating pairs of these bases can be expressed in terms of the matrix

$$K = M(s, m)$$

and the transposition matrix J.

Since (m_λ) and (h_λ) are dual bases, and the basis (s_λ) is self-dual (4.8), we have

$$M(s, h) = K^*$$

by (6.3)(3). If we now apply the involution ω and observe that

$$M(\omega s, s) = J,$$

by virtue of (3.8), we have

$$M(s, e) = M(\omega s, h) = M(\omega s, s)M(s, h) = JK^*$$

by (6.3)(1) and (4). Finally, by (6.3)(3) again,

$$M(s, f) = M(s, e)^* = (JK^*)^* = JK.$$

We can now use (6.3)(1) and (2) to complete the following table of transition matrices, in which the entry in row u and column v is $M(u, v)$:

TABLE 1

	e	h	m	f	s
e	1	$K'JK^*$	$K'JK$	$K'K$	$K'J$
h	$K'JK^*$	1	$K'K$	$K'JK$	K'
m	$K^{-1}JK^*$	$K^{-1}K^*$	1	$K^{-1}JK$	K^{-1}
f	$K^{-1}K^*$	$K^{-1}JK^*$	$K^{-1}JK$	1	$K^{-1}J$
s	JK^*	K^*	K	JK	1

Some of the transition matrices in Table 1 have combinatorial interpretations. From (5.13) it follows that

(6.4) $K_{\lambda\mu}$ *is the number of tableaux of shape* λ *and weight* μ. |

The numbers $K_{\lambda\mu}$ are sometimes called *Kostka numbers*. By (6.4) they are non-negative. Moreover,

(6.5) *The matrix* $(K_{\lambda\mu})$ *is strictly upper unitriangular.*

Proof. If T is a tableau of shape λ and weight μ, then for each $r \geqslant 1$ there are altogether $\mu_1 + \ldots + \mu_r$ symbols $\leqslant r$ in T, which must all be located in the top r rows of T (because of the condition of strict increase down the columns of a tableau). Hence $\mu_1 + \ldots + \mu_r \leqslant \lambda_1 + \ldots \lambda_r$ for all $r \geqslant 1$, i.e. $\mu \leqslant \lambda$. Hence $K_{\lambda\mu} = 0$ unless $\lambda \geqslant \mu$, and for the same reason $K_{\lambda\lambda} = 1$. |

(6.6) (i) $M(e, m)_{\lambda\mu} = \sum_\nu K_{\nu\lambda}K_{\nu'\mu}$ *is the number of matrices of* 0*'s and* 1*'s with row sums* λ_i *and column sums* μ_j.
(ii) $M(h, m)_{\lambda\mu} = \sum_\nu K_{\nu\lambda}K_{\nu\mu}$ *is the number of matrices of non-negative integers with row sums* λ_i *and column sums* μ_j.

Proof. (i) Consider the coefficient of a monomial x^μ (where μ is a partition of n) in $e_\lambda = e_{\lambda_1}e_{\lambda_2}\ldots$. Each monomial in e_{λ_i} is of the form $\prod_j x_j^{a_{ij}}$, where each a_{ij} is 0 or 1, and $\sum_j a_{ij} = \lambda_i$; hence we must have

$$\prod_{i,j} x_j^{a_{ij}} = \prod_j x_j^{\mu_j},$$

so that $\sum_i a_{ij} = \mu_j$. Hence the matrix (a_{ij}) has row sums λ_i and column sums μ_j.

For (ii) the proof is similar: the only difference is that e_λ is replaced by h_λ, and consequently the exponents a_{ij} can now be arbitrary integers $\geqslant 0$. |

From Table 1 and (6.5) we can read off:

(6.7) (i) *$M(s, h)$ and $M(h, s)$ are strictly lower unitriangular.*
(ii) *$M(s, m)$ and $M(m, s)$ are strictly upper unitriangular.*
(iii) *$M(e, m) = M(h, f)$ and is symmetric.*
(iv) *$M(e, f) = M(h, m)$ and is symmetric.*
(v) *$M(e, h) = M(h, e)$.*
(vi) *$M(m, f) = M(f, m)$.*
(vii) *$M(h, s) = M(s, m)'$.*
(viii) *$M(e, s) = M(s, f)'$.*

Remarks. 1. From (6.4) and (6.6)(i) it follows that the number of (0, 1)-matrices with row sums λ_i and column sums μ_j is equal to the number of pairs of tableaux of conjugate shapes and weights λ, μ. In fact one can set up an explicit one–one correspondence between these two sets of objects (Knuth's dual correspondence [22], [51]).

Likewise, there is an explicit one–one correspondence, also due to Knuth, between the set of matrices of non-negative integers with row sums λ_i and column sums μ_j, and pairs of tableaux of the same shape and weights λ, μ.

2. We have not considered transition matrices involving the **Q**-basis (p_λ). In fact, as we shall show in §7, the matrix $M(p, s)$ is the character table of the symmetric group S_n.

Examples

1. $M(h, m)_{\lambda\mu}$ is equal to the number of double cosets $S_\lambda w S_\mu$ in S_n, where $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \dots$, $S_\mu = S_{\mu_1} \times S_{\mu_2} \times \dots$.

2. $M(e, m)_{\lambda\mu} \neq 0$ if and only if $\lambda \leqslant \mu'$. (This is the Gale–Ryser theorem: for a proof see [42].)

3. If R is a raising operator (§1) let $R(h_\lambda)$ denote $h_{R(\lambda)}$, with the usual convention that $h_{R(\lambda)} = 0$ if any component of $R(\lambda)$ is negative. Then we have

$$s_\lambda = \prod_{i<j} (1 - R_{ij}) h_\lambda \tag{1}$$

for any partition λ, and hence also

$$h_\lambda = \left(\prod_{i<j} (1 - R_{ij})^{-1} \right) . \, s_\lambda \tag{2}$$

where $R(s_\lambda) = s_{R(\lambda)}$ for any raising operator R.

The formula (1) is just another way of writing (3.4): if $l(\lambda) \leqslant n$,

$$\begin{aligned} s_\lambda &= \det(h_{\lambda_i - i + j})_{1 \leqslant i, j \leqslant n} \\ &= \sum_{w \in S_n} \varepsilon(w) h_{\lambda + \rho - w\rho} \end{aligned}$$

where $\rho = \frac{1}{2}(n-1, n-3, \dots, 1-n)$, and this is equal to $\prod_{i<j} (1 - R_{ij}) h_\lambda$ by Weyl's identity (§5, Ex. 9).

4. For any partition λ of n, $K_{\lambda,(1^n)}$ is by (6.4) the number of standard tableaux of shape λ, so that by §5, Ex. 2 we have $K_{\lambda,(1^n)} = n!/h(\lambda)$, where $h(\lambda)$ is the product of the hook-lengths of λ.

5. The (infinite) matrix K is the diagonal sum of matrices $K_n (n \geqslant 0)$ where $K_n = (K_{\lambda\mu})_{\lambda,\mu \in \mathscr{P}_n}$. We have $K_0 = K_1 = (1)$, and for $n = 2, 3, \dots, 6$ the matrices K_n are given on the next page.

Notes and references

The relations between the various transition matrices contained in Table 1 and (6.7) were known to Kostka [25]. See also Foulkes [10].

	2	1^2
2	1	1
1^2		1

	3	21	1^3
3	1	1	1
21		1	2
1^3			1

$\lambda\backslash\mu$	4	31	2^2	21^2	1^4
4	1	1	1	1	1
31		1	1	2	3
2^2			1	1	2
21^2				1	3
1^4					1

	5	41	32	31^2	2^21	21^3	1^5
5	1	1	1	1	1	1	1
41		1	1	2	2	3	4
32			1	1	2	3	5
31^2				1	1	3	6
2^21					1	2	5
21^3						1	4
1^5							1

	6	51	42	41^2	3^2	321	31^3	2^3	2^21^2	21^4	1^6
6	1	1	1	1	1	1	1	1	1	1	1
51		1	1	2	1	2	3	2	3	4	5
42			1	1	1	2	3	3	4	6	9
41^2				1	0	1	3	1	3	6	10
3^2					1	1	1	1	2	3	5
321						1	2	2	4	8	16
31^3							1	0	1	4	10
2^3								1	1	2	5
2^21^2									1	3	9
21^4										1	5
1^6											1

7. The characters of the symmetric groups

In this section we shall take for granted the elementary facts about representations and characters of finite groups.

If G is a finite group and f, g are functions on G with values in a commutative **Q**-algebra, the *scalar product* of f and g is defined by

$$\langle f, g\rangle_G = \frac{1}{|G|}\sum_{x\in G} f(x)g(x^{-1}).$$

If H is a subgroup of G and f is a character of H, the induced character of G will be denoted by $\operatorname{ind}_H^G(f)$. If g is a character of G, its restriction to H will be denoted by $\operatorname{res}_G^H(g)$.

Each permutation $w\in S_n$ factorizes uniquely as a product of disjoint cycles. If the orders of these cycles are $\rho_1, \rho_2, \dots$, where $\rho_1 \geqslant \rho_2 \geqslant \dots$, then $\rho(w) = (\rho_1, \rho_2, \dots)$ is a partitition of n called the *cycle-type* of w. It determines w up to conjugacy in S_n, and the conjugacy classes of S_n are indexed in this way by the partitions of n.

We define a mapping $\psi : S_n \to \Lambda^n$ as follows:

$$\psi(w) = p_{\rho(w)}.$$

If m, n are positive integers, we may embed $S_m \times S_n$ in S_{m+n} by making S_m and S_n act on complementary subsets of $\{1, 2, \dots, m+n\}$. Of course there are many different ways of doing this, but the resulting subgroups of S_{m+n} are all conjugate. Hence if $v\in S_m$ and $w \in S_n$, $v\times w\in S_{m+n}$ is well-defined up to conjugacy in S_{m+n}, with cycle-type $\rho(v\times w) = \rho(v)\cup\rho(w)$, so that

$$\psi(v\times w) = \psi(v)\psi(w). \tag{7.1}$$

Let R^n denote the **Z**-module generated by the irreducible characters of S_n, and let

$$R = \bigoplus_{n\geqslant 0} R^n,$$

with the understanding that $S_0 = \{1\}$, so that $R^0 = \mathbf{Z}$. The **Z**-module R has a ring structure, defined as follows. Let $f\in R^m$, $g \in R^n$, and embed $S_m\times S_n$ in S_{m+n}. Then $f\times g$ is a character of $S_m\times S_n$, and we define

$$f\,.\,g = \operatorname{ind}_{S_m\times S_n}^{S_{m+n}}(f\times g),$$

which is a character of S_{m+n}, i.e. an element of R^{m+n}. Thus we have defined a bilinear multiplication $R^m\times R^n \to R^{m+n}$, and it is not difficult to verify that with this multiplication R is a commutative, associative, graded ring with identity element.

Moreover, R carries a scalar product: if f, $g\in R$, say $f = \sum f_n$, $g = \sum g_n$

with f_n, $g_n \in R^n$, we define

$$\langle f, g\rangle = \sum_{n\geqslant 0} \langle f_n, g_n\rangle_{S_n}.$$

Next we define a **Z**-linear mapping

$$\mathrm{ch} : R \to \Lambda_{\mathbf{Q}} = \Lambda\otimes_{\mathbf{Z}}\mathbf{Q}$$

as follows: if $f\in R^n$, then

$$\mathrm{ch}(f) = \langle f, \psi\rangle_{S_n} = \frac{1}{n!}\sum_{w\in S_n} f(w)\psi(w)$$

(since $\psi(w) = \psi(w^{-1})$). If f_ρ is the value of f at elements of cycle type ρ, we have

$$\mathrm{ch}(f) = \sum_{|\rho|=n} z_\rho^{-1} f_\rho p_\rho. \tag{7.2}$$

$\mathrm{ch}(f)$ is called the *characteristic* of f, and ch is the *characteristic map.* From (7.2) and (4.7) it follows that, for f and g in R^n,

$$\langle \mathrm{ch}(f), \mathrm{ch}(g)\rangle = \sum_{|\rho|=n} z_\rho^{-1} f_\rho g_\rho = \langle f, g\rangle_{S_n}$$

and hence that ch is an *isometry.*

The basic fact is now

(7.3) *The characteristic map is an isometric isomorphism of R onto Λ.*

Proof. Let us first verify that ch is a ring homomorphism. If $f\in R^m$ and $g\in R^n$, we have

$$\begin{aligned}\mathrm{ch}(f\,.\,g) &= \langle \operatorname{ind}_{S_m\times S_n}^{S_{m+n}}(f\times g), \psi\rangle_{S_{m+n}}\\ &= \langle f\times g, \operatorname{res}_{S_{m+n}}^{S_m\times S_n}(\psi)\rangle_{S_m\times S_n}\end{aligned}$$

by Frobenius reciprocity,

$$= \langle f, \psi\rangle_{S_m}\langle g, \psi\rangle_{S_n} = \mathrm{ch}(f)\,.\,\mathrm{ch}(g)$$

by (7.1).

Next, let η_n be the identity character of S_n. Then

$$\mathrm{ch}(\eta_n) = \sum_{|\rho|=n} z_\rho^{-1} p_\rho = h_n$$

by (7.2) and (2.14′). If now $\lambda = (\lambda_1, \lambda_2, \dots)$ is any partition of n, let η_λ denote $\eta_{\lambda_1}\,.\,\eta_{\lambda_2}\,.\dots$. Then η_λ is a character of S_n, namely the character induced by the identity character of $S_\lambda = S_{\lambda_1}\times S_{\lambda_2}\times\dots$, and we have $\mathrm{ch}(\eta_\lambda) = h_\lambda$.

Now define, for each partition λ of n,

$$\chi^\lambda = \det(\eta_{\lambda_i - i + j})_{1\leqslant i,j\leqslant n} \in R^n, \tag{7.4}$$

i.e. χ^λ is a (possibly virtual) character of S_n, and by (3.4) we have

(7.5) $$\mathrm{ch}(\chi^\lambda) = s_\lambda.$$

Since ch is an isometry, it follows from (4.8) that

$$\langle \chi^\lambda, \chi^\mu \rangle = \delta_{\lambda\mu}$$

for any two partitions λ, μ, and hence in particular that the χ^λ are, up to sign, irreducible characters of S_n. Since the number of conjugacy classes in S_n is equal to the number of partitions of n, these characters exhaust all the irreducible characters of S_n; hence the χ^λ for $|\lambda| = n$ form a basis of R^n, and hence ch is an isomorphism of R^n onto Λ^n for each n, hence of R onto Λ. |

(7.6) *The irreducible characters of S_n are $\chi^\lambda(|\lambda| = n)$ defined by* (7.4).

Proof. From the proof of (7.3), we have only to show that χ^λ and not $-\chi^\lambda$ is an irreducible character; for this purpose it will suffice to show that $\chi^\lambda(1) > 0$. Now we have from (7.5) and (7.2)

$$s_\lambda = \mathrm{ch}(\chi^\lambda) = \sum_\rho z_\rho^{-1} \chi_\rho^\lambda p_\rho$$

where χ_ρ^λ is the value of χ^λ at elements of cycle-type ρ. Hence

(7.7) $$\chi_\rho^\lambda = \langle s_\lambda, p_\rho \rangle$$

by (4.7), and in particular

$$\chi^\lambda(1) = \chi_{(1^n)}^\lambda = \langle s_\lambda, p_1^n \rangle$$

so that

$$h_1^n = p_1^n = \sum_{|\lambda| = n} \chi^\lambda(1) s_\lambda$$

and therefore

$$\chi^\lambda(1) = M(h, s)_{(1^n),\lambda} = K_{\lambda,(1^n)}$$

from Table 1: hence $\chi^\lambda(1)$ is the number of standard tableaux of shape λ, hence is a positive integer. |

(7.8) *The transition matrix $M(p, s)$ is the character table of S_n, i.e.*

$$p_\rho = \sum_\lambda \chi_\rho^\lambda s_\lambda.$$

This is a restatement of (7.7). |

Let λ, μ, ν be partitions of n, and let

$$\gamma_{\mu\nu}^\lambda = \langle \chi^\lambda, \chi^\mu \chi^\nu \rangle_{S_n} = \frac{1}{n!} \sum_{w \in S_n} \chi^\lambda(w) \chi^\mu(w) \chi^\nu(w)$$

which is symmetrical in λ, μ, ν. Then we have, for two sets of variables $x = (x_1, x_2, ..)$ and $y = (y_1, y_2, \ldots)$,

(7.9) $$s_\lambda(xy) = \sum_{\mu,\nu} \gamma_{\mu\nu}^\lambda s_\mu(x) s_\nu(y).$$

(Compare (5.9).)

Proof. For all partitions ρ we have $p_\rho(xy) = p_\rho(x) p_\rho(y)$ and hence from (7.8)

$$\sum_\lambda \chi^\lambda s_\lambda(xy) = \sum_{\mu,\nu} \chi^\mu \chi^\nu s_\mu(x) s_\nu(y)$$

so that $s_\lambda(xy)$ is the coefficient of χ^λ in the right-hand side. |

Finally, we remark that (7.8) is equivalent to

$$\chi_\rho^\lambda = \langle s_\lambda, p_\rho \rangle$$

and hence also by virtue of (4.7) to

(7.10) $$s_\lambda = \sum_\rho z_\rho^{-1} \chi_\rho^\lambda p_\rho.$$

Examples

1. $\chi^{(n)} = \eta_n$ is the trivial character of S_n, and $\chi^{(1^n)} = \varepsilon_n = \varepsilon$ is the sign character. (Compare (7.10) with (2.14′).)

2. For any partition λ of n, $\chi^{\lambda'} = \varepsilon_n \chi^\lambda$. For

$$\chi_\rho^{\lambda'} = \langle s_{\lambda'}, p_\rho \rangle = \langle s_\lambda, \varepsilon_\rho p_\rho \rangle = \varepsilon_\rho \chi_\rho^\lambda$$

since $\omega(s_{\lambda'}) = s_\lambda$ and $\omega(p_\rho) = \varepsilon_\rho p_\rho$. Hence the involution ω on Λ corresponds to multiplication by ε_n in R^n.

3. Corresponding to each skew diagram $\lambda - \mu$ of weight n, there is a character $\chi^{\lambda/\mu}$ of S_n defined by $\mathrm{ch}(\chi^{\lambda/\mu}) = s_{\lambda/\mu}$. If $|\mu| = m$ we have from (5.1) and (7.3)

$$\begin{aligned} \langle \chi^{\lambda/\mu}, \chi^\nu \rangle_{S_n} &= \langle \chi^\lambda, \chi^\mu \cdot \chi^\nu \rangle_{S_{m+n}} \\ &= \langle \mathrm{res}_{S_{m+n}}^{S_m \times S_n} \chi^\lambda, \chi^\mu \times \chi^\nu \rangle_{S_m \times S_n} \end{aligned}$$

by Frobenius reciprocity, and therefore the restriction of χ^λ to $S_m \times S_n$ is $\sum_{|\mu| = m} \chi^\mu \times \chi^{\lambda/\mu}$.

The *degree* of $\chi^{\lambda/\mu}$ is equal to $\langle s_{\lambda/\mu}, e_1^n \rangle = K_{\lambda/\mu,(1^n)}$, i.e. to the number of standard tableaux of shape $\lambda - \mu$.

4. Let G be a subgroup of S_n, and let $c(G)$ be the cycle indicator of G (§2, Ex. 9). Then $c(G) = \mathrm{ch}(\chi_G)$, where χ_G is the character of S_n induced by the trivial character 1_G of G. For $\mathrm{ch}(\chi_G) = \langle \chi_G, \psi \rangle_{S_n} = \langle 1_G, \psi \mid G \rangle_G$ (by Frobenius reciprocity) $= c(G)$.

If G, H are subgroups of S_n, $\langle c(G), c(H) \rangle$ is the number of (G, H) double cosets in S_n.

5. From §3, Ex. 11 and (7.8) we obtain the following combinatorial rule for computing χ^λ_ρ:

$$\chi^\lambda_\rho = \sum_S (-1)^{\mathrm{ht}(S)}$$

summed over all sequences of partitions $S = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(m)})$ such that $m = l(\rho)$, $0 = \lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(m)} = \lambda$, and such that each $\lambda^{(i)} - \lambda^{(i-1)}$ is a border strip (§3, Ex. 11) of length ρ_i, and $\mathrm{ht}(S) = \sum_i \mathrm{ht}(\lambda^{(i)} - \lambda^{(i-1)})$.

6. The degree $f^\lambda = \chi^\lambda(1)$ of χ^λ may also be computed as follows. By (7.8), it is the coefficient of $x^{\lambda+\delta}$ in $(\sum x_i)^n \sum_{w \in S_n} \varepsilon(w) x^{w\delta}$. If we put $\mu = \lambda + \delta$ (so that $\mu_i = \lambda_i + n - i$, $1 \leqslant i \leqslant n$), this coefficient is

$$\sum_{w \in S_n} \varepsilon(w) n! \Big/ \prod_{i=1}^{n} (\mu_i - n + w(i))!$$

which is the determinant $n! \det(1/(\mu_i - n + j)!)$, hence equal to

$$\frac{n!}{\mu!} \det(\mu_i(\mu_i - 1) \dots (\mu_i - n + j + 1))$$

$$= \frac{n!}{\mu!} \det(\mu_i^{j-1}) = \frac{n!}{\mu!} \Delta(\mu_1, \dots, \mu_n)$$

where $\mu! = \prod_i \mu_i!$ and $\Delta(\mu_1, \dots, \mu_n) = \prod_{i<j} (\mu_i - \mu_j)$.

7. Let $\rho = (r, 1^{n-r})$, so that χ^λ_ρ is the value of the character χ^λ of S_n at an r-cycle $(1 \leqslant r \leqslant n)$. By (7.8), χ^λ_ρ is the coefficient of $x^\mu = x^{\lambda+\delta}$ in $(\sum x_i^r)(\sum x_i)^{n-r} \sum_{w \in S_n} \varepsilon(w) x^{w\delta}$. From the result of Ex. 6, this coefficient is

$$\sum_i \frac{(n-r)! \Delta(\mu_1, \dots, \mu_i - r, \dots, \mu_n)}{\mu_1! \dots (\mu_i - r)! \dots \mu_n!}$$

and therefore

$$\chi^\lambda_\rho / f^\lambda = \frac{(n-r)!}{n!} \sum_{i=1}^{n} \frac{\mu_i!}{(\mu_i - r)!} \prod_{j \neq i} \frac{\mu_i - \mu_j - r}{\mu_i - \mu_j}.$$

If we put $\varphi(x) = \prod (x - \mu_i)$ and $h_\rho = n!/z_\rho = n!/(n-r)!r$, this formula becomes

$$-r^2 h_\rho \chi^\lambda_\rho / f^\lambda = \sum_{i=1}^{n} \mu_i(\mu_i - 1) \dots (\mu_i - r + 1) \varphi(\mu_i - r)/\varphi'(\mu_i)$$

which is equal to the coefficient of x^{-1} in the expansion of

$$x(x-1) \dots (x - r + 1) \varphi(x - r)/\varphi(x)$$

in *descending* powers of x.

In particular, when $r = 2$ we obtain

$$h_\rho \chi^\lambda_\rho / f^\lambda = n(\lambda') - n(\lambda).$$

Notes and references

The representation theory of finite groups was founded by Frobenius in a series of papers published in the last years of the nineteenth century, and reproduced in Vol. 3 of his collected works; in particular, he obtained the irreducible characters of the symmetric groups in 1900 [12], and our exposition does not differ substantially from his.

Ex. 5 is due to Littlewood and Richardson [28]; Exx. 6 and 7 to Frobenius (*loc. cit.*)

8. Plethysm

In this section we shall study briefly another sort of multiplication in Λ, called *plethysm* or *composition*, and defined as follows. Let f, $g \in \Lambda$, and write g as a sum of monomials:

$$g = \sum_\alpha u_\alpha x^\alpha.$$

Now introduce the set of fictitious variables y_i defined by

$$\prod (1 + y_i t) = \prod_\alpha (1 + x^\alpha t)^{u_\alpha} \tag{8.1}$$

and define

$$f \circ g = f(y_1, y_2, \dots). \tag{8.2}$$

If $f \in \Lambda^m$ and $g \in \Lambda^n$, then clearly $f \circ g \in \Lambda^{mn}$. Also e_1 acts as a two sided identity: $f \circ e_1 = e_1 \circ f = f$ for all $f \in \Lambda$.

From the definition (8.2) it is clear that

(8.3) *For each $g \in \Lambda$, the mapping $f \mapsto f \circ g$ is an endomorphism of the ring Λ.* |

By taking logarithms of both sides of (8.1) we obtain

$$p_n(y) = \sum_\alpha u_\alpha (x^\alpha)^n \qquad (n \geqslant 1)$$

so that

$$p_n \circ g = g \circ p_n = g(x_1^n, x_2^n, \dots) \tag{8.4}$$

for all $g \in \Lambda$. In particular,

$$p_n \circ p_m = p_m \circ p_n = p_{mn}. \tag{8.5}$$

From (8.4) it follows that

(8.6) *For each $n \geqslant 1$, the mapping $g \mapsto p_n \circ g$ is an endomorphism of the ring Λ.* |

Plethysm is associative: for all f, g, $h \in \Lambda$ we have

$$(f \circ g) \circ h = f \circ (g \circ h). \tag{8.7}$$

Proof. Since the p_n generate $\Lambda_{\mathbf{Q}}$ (2.12), by virtue of (8.3) and (8.6) it is enough to verify associativity when $f = p_m$ and $g = p_n$, in which case it is obvious from (8.4) and (8.5). |

For plethysm involving S-functions, there are the following formulas: from (5.9) it follows that

$$\begin{aligned} s_\lambda \circ (g+h) &= \sum_{\mu,\nu} c^\lambda_{\mu\nu}(s_\mu \circ g)(s_\nu \circ h) \\ &= \sum_\mu (s_{\lambda/\mu} \circ g)(s_\mu \circ h) \end{aligned} \tag{8.8}$$

and from (7.9) that

$$s_\lambda \circ (gh) = \sum_{\mu,\nu} \gamma^\lambda_{\mu\nu}(s_\mu \circ g)(s_\nu \circ h). \tag{8.9}$$

The sum in (8.8) is over pairs of partitions $\mu, \nu \subset \lambda$, and in (8.9) over pairs of partitions μ, ν such that $|\mu| = |\nu| = |\lambda|$.

Finally, let λ, μ be partitions. Then $s_\lambda \circ s_\mu$ is an integral linear combination of S-functions, say

$$s_\lambda \circ s_\mu = \sum_\pi a^\pi_{\lambda\mu} s_\pi \tag{8.10}$$

summed over partitions π such that $|\pi| = |\lambda| \,.\, |\mu|$. We shall prove in the Appendix that *the coefficients* $a^\pi_{\lambda\mu}$ *are all* $\geqslant 0$.

Remarks. 1. We have observed in (3.10) that to each $f \in \Lambda$ there corresponds a natural operation F on the category of λ-rings. In this correspondence, plethysm corresponds to composition of operations: if f, $g \in \Lambda$ correspond to the natural operations F, G, then $f \circ g$ corresponds to $F \circ G$.

2. Plethysm is defined in the ring R of §7 via the characteristic map: for u, $v \in R$, $u \circ v$ is defined to be $\mathrm{ch}^{-1}(\mathrm{ch}\, u \circ \mathrm{ch}\, v)$. If u (resp. v) is an irreducible character of S_m (resp. S_n), then $u \circ v$ is a character of S_{mn} which may be described as follows: if U (resp. V) is an S_m-module with character u (resp. an S_n-module with character v), the wreath product $S_n \sim S_m$ (which is the normalizer of $S_n^m = S_n \times \ldots \times S_n$ in S_{mn}) acts on U and on the mth tensor power $\mathrm{T}^m(V)$, hence also on $U \otimes \mathrm{T}^m(V)$; and $u \circ v$ is the character of the S_{mn}-module induced by $U \otimes \mathrm{T}^m(V)$. See the Appendix to this chapter.

Examples

1. Let $\tilde\omega : \Lambda \to \Lambda$ be the involution defined by

$$(\tilde\omega f)(x_1, x_2, \ldots) = (\omega f)(-x_1, -x_2, \ldots)$$

(so that $\tilde\omega = (-1)^n \omega$ on Λ^n). Then for any f, $g \in \Lambda$ we have $f \circ (-g) = (\tilde\omega f) \circ g$.

2. Since $s_{\lambda/\mu} = D_\mu(s_\lambda)$, where D_μ is the differential operator defined in §5, Ex. 3(a), it follows that from (8.8) that

$$f \circ (g+h) = \sum_\mu ((D_\mu f) \circ g)(s_\mu \circ h)$$

for all f, g, $h \in \Lambda$.

3.
$$\begin{aligned} h_n \circ (gh) &= \sum_{|\lambda|=n} (s_\lambda \circ g)(s_\lambda \circ h), \\ e_n \circ (gh) &= \sum_{|\lambda|=n} (s_\lambda \circ g)(s_{\lambda'} \circ h). \end{aligned}$$

These formulas are particular cases of (8.9) (and are consequences of (4.3) and (4.3′).)

4. Let λ be a partition of length $\leqslant n$, and consider $(s_\lambda \circ s_{(n-1)})(x_1, x_2)$. By definition this is equal to $s_\lambda(x_1^{n-1}, x_1^{n-2}x_2, \ldots, x_2^{n-1})$, i.e. to $x_2^{(n-1)|\lambda|} s_\lambda(q^{n-1}, q^{n-2}, \ldots, 1)$, where $q = x_1 x_2^{-1}$. On the other hand, by the positivity of the coefficients in (8.10), $(s_\lambda \circ s_{n-1})(x_1, x_2)$ is a linear combination of the $s_\pi(x_1, x_2)$ with non-negative integer coefficients, where $\pi = (\pi_1, \pi_2)$ and $\pi_1 + \pi_2 = (n-1)|\lambda| = d$ say. Now

$$\begin{aligned} s_\pi(x_1, x_2) &= x_1^{\pi_1} x_2^{\pi_2} + x_1^{\pi_1 - 1} x_2^{\pi_2 + 1} + \ldots + x_1^{\pi_2} x_2^{\pi_1} \\ &= x_2^d (q^{\pi_1} + q^{\pi_1 - 1} + \ldots + q^{\pi_2}) \end{aligned}$$

Hence $s_\lambda(q^{n-1}, q^{n-2}, \ldots, 1)$ is a non-negative linear combination of the polynomials $q^{\pi_1} + q^{\pi_1 - 1} + \ldots + q^{\pi_2}$, where $\pi_1 \geqslant \pi_2$ and $\pi_1 + \pi_2 = d$. It follows that $s_\lambda(q^{n-1}, q^{n-2}, \ldots, 1)$ is a *unimodal* symmetrical polynomial in q, i.e. that if a_i is the coefficient of q^i in this polynomial, for $0 \leqslant i \leqslant d$, then $a_i = a_{d-i}$ (symmetry) and

$$a_0 \leqslant a_1 \leqslant \ldots \leqslant a_{[d/2]}$$

(unimodality).

From §3. Ex. 1 it follows that the generalized Gaussian polynomial

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} = \prod_{x \in \lambda} \frac{1 - q^{n - c(x)}}{1 - q^{h(x)}}$$

is symmetrical and unimodal for all n and λ.

5. Let G be a subgroup of S_m and H a subgroup of S_n, so that $G \sim H$ is a subgroup of the wreath product $S_m \sim S_n \subset S_{mn}$. Then the cycle-indicator (§2, Ex. 9) of $G \sim H$ is

$$c(G \sim H) = c(H) \circ c(G).$$

Notes and references

Plethysm was introduced by D. E. Littlewood [29]. His notation for our $s_\lambda \circ s_\mu$ is $\{\mu\} \otimes \{\lambda\}$. Many authors have computed (or have described algorithms to compute) $s_\lambda \circ s_\mu$ for particular choices of either λ or μ. For their work we refer to the bibliographies in Littlewood [29] and Robinson [40].

9. The Littlewood–Richardson rule

If μ and ν are partitions, the product $s_\mu s_\nu$ is an integral linear combination of S-functions:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

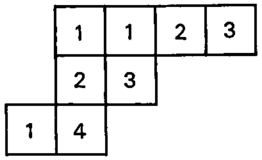
or equivalently

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu. \tag{9.1}$$

The coefficients $c^\lambda_{\mu\nu}$ are non-negative integers, because by (7.3) and (7.5) $c^\lambda_{\mu\nu} = \langle \chi^\lambda, \chi^\mu . \chi^\nu \rangle$ is the multiplicity of χ^λ in the character $\chi^\mu . \chi^\nu$; also we have $c^\lambda_{\mu\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$ and $\mu, \nu \subset \lambda$.

This section is devoted to the statement and proof of a combinatorial rule for computing $c^\lambda_{\mu\nu}$, due to Littlewood and Richardson [28].

Let T be a tableau. From T we derive a *word* or sequence $w(T)$ by reading the symbols in T from right to left (as in Arabic) in successive rows, starting with the top row. For example, if T is the tableau



$w(T)$ is the word 32113241.

If a word w arises in this way from a tableau of shape $\lambda - \mu$, we shall say that w is *compatible* with $\lambda - \mu$.

A word $w = a_1 a_2 \ldots a_N$ in the symbols $1, 2, \ldots, n$ is said to be a *lattice permutation* if for $1 \leqslant r \leqslant N$ and $1 \leqslant i \leqslant n-1$, the number of occurrences of the symbol i in $a_1 a_2 \ldots a_r$ is not less than the number of occurrences of $i+1$.

We can now state the Littlewood–Richardson rule:

(9.2) *Let λ, μ, ν be partitions. Then $c^\lambda_{\mu\nu}$ is equal to the number of tableaux T of shape $\lambda - \mu$ and weight ν such that $w(T)$ is a lattice permutation.*

The proof we shall give of (9.2) depends on the following proposition. For any partitions λ, μ, π such that $\lambda \supset \mu$, let $\mathrm{Tab}(\lambda - \mu, \pi)$ denote the set of tableaux T of shape $\lambda - \mu$ and weight π, and let $\mathrm{Tab}^0(\lambda - \mu, \pi)$ denote the subset of those T such that $w(T)$ is a lattice permutation. From (5.14) we have

$$|\mathrm{Tab}(\lambda - \mu, \pi)| = K_{\lambda-\mu,\pi} = \langle s_{\lambda/\mu}, h_\pi \rangle. \tag{9.3}$$

We shall prove that

(9.4) *There exists a bijection*

$$\mathrm{Tab}(\lambda - \mu, \pi) \overset{\sim}{\rightarrow} \coprod_\nu (\mathrm{Tab}^0(\lambda - \mu, \nu) \times \mathrm{Tab}(\nu, \pi)).$$

Before proving (9.4), let us deduce (9.2) from it. From (9.4) and (9.3), we have

$$\langle s_{\lambda/\mu}, h_\pi \rangle = \sum_\nu |\mathrm{Tab}^0(\lambda - \mu, \nu)| \langle s_\nu, h_\pi \rangle$$

for all partitions π, and therefore

$$s_{\lambda/\mu} = \sum_\nu |\mathrm{Tab}^0(\lambda - \mu, \nu)| \, s_\nu.$$

Comparison of this identity with (9.1) shows that $c^\lambda_{\mu\nu} = |\mathrm{Tab}^0(\lambda - \mu, \nu)|$.

To construct a bijection as required for (9.4), we shall follow the method of Littlewood and Robinson [39], which consists in starting with a tableau T of shape $\lambda - \mu$ and successively modifying it until the word $w(T)$ becomes a lattice permutation, and simultaneously building up a tableau M, which serves to record the sequence of moves made.

If $w = a_1 a_2 \ldots a_N$ is any word in the symbols $1, 2, \ldots$, let $m_r(w)$ denote the number of occurrences of the symbol r in w. For $1 \leqslant p \leqslant N$ and $r \geqslant 2$, the difference $m_r(a_1 \ldots a_p) - m_{r-1}(a_1 \ldots a_p)$ is called the *r-index* of a_p in w. Observe that w is a lattice permutation if and only if all indices are $\leqslant 0$.

Let m be the maximum value of the r-indices in w, and suppose that $m > 0$. Take the first element of w at which this maximum is attained (clearly this element will be an r), and replace it by $r-1$. Denote the result of this operation by $S_{r-1,r}(w)$ (substitution of $r-1$ for r). Observe that $S_{r-1,r}(w)$ has maximum r-index $m-1$ (unless $m = 1$, in which case it can be -1).

(9.5) *The operation $S_{r-1,r}$ is one-to-one.*

Proof. Let $w' = S_{r-1,r}(w)$. To reconstruct w from w', let m' be the maximum r-index in w'. If $m' \geqslant 0$, take the last symbol in w' with r-index m', and convert the next symbol (which must be an $r-1$) into r. If $m' < 0$, the first symbol in w' must be an $r-1$, and this is converted into r. In either case the result is w, which is therefore uniquely determined by w' and r. |

(9.6) *Let $w' = S_{r-1,r}(w)$. Then w' is compatible with $\lambda - \mu$ if and only if w is compatible with $\lambda - \mu$.*

Proof. Let $w = w(T)$, $w' = w(T')$, where T and T' are arrays of shape $\lambda - \mu$. They differ in only one square, say x, which in T is occupied by r and in T' by $r-1$.

Suppose that T is a tableau. If T' is not a tableau there are two possibilities: either (a) the square y immediately to the left of x in T is occupied by r, or (b) the square immediately above x is occupied by $r-1$.

In case (a) the symbol r in square y would have a higher r-index in $w(T)$ than the r in square x, which is impossible. In case (b) the square x in T will be the left-hand end of a string of say s squares occupied by the symbol r, and immediately above this string there will be a string of s squares occupied by the symbol $r-1$. It follows that $w(T)$ contains a segment of the form

$$(r-1)^s \ldots r^s$$

where the unwritten symbols in between the two strings are all either $>r$ or $<r-1$, and the last r is the one to be replaced by $r-1$ to form w'. But the r-index of this r is equal to that of the element of w immediately preceding the first of the string of $r-1$'s, and this again is impossible. Hence if T is a tableau, so also is T'.

The reverse implication is proved similarly, using the recipe of (9.5) for passing back from w' to w. |

Suppose now that the word w has the lattice permutation property with respect to $(1, 2, \ldots, r-1)$ but not with respect to $(r-1, r)$, or in other words that all the s-indices are $\leqslant 0$ for $2 \leqslant s \leqslant r-1$ but not for $s = r$. This is the only situation in which we shall use the operator $S_{r-1,r}$. The effect of replacing r by $r-1$ in w as required by $S_{r-1,r}$ may destroy the lattice permutation property with respect to $(r-2, r-1)$, i.e. it may produce some $(r-1)$-indices equal to $+1$. In this case we operate with $S_{r-2,r-1}$ to produce

$$S_{r-2,r}(w) = S_{r-2,r-1}S_{r-1,r}(w).$$

At this stage the $(r-1)$-indices will all be $\leqslant 0$, but there may be some $(r-2)$-indices equal to $+1$, and so on. Eventually this process will stop, and we have then say

$$S_{a,r}(w) = S_{a,a+1} \ldots S_{r-1,r}(w)$$

for some a such that $1 \leqslant a \leqslant r-1$, and the word $S_{a,r}(w)$ again has the lattice property with respect to $(1, 2, \ldots, r-1)$, and maximal r-index strictly less than that of w.

At this point the following lemma is crucial:

(9.7) *If w, $S_{a,r}(w) = w'$ and $S_{b,r}(w') = w''$ all have the lattice property with respect to $(1, 2, \ldots, r-1)$, then $b \leqslant a$.*

Proof. Let $w = x_1x_2x_3\ldots$. We have to study in detail the process of passing from w to w'. This starts by applying $S_{r-1,r}$, i.e. by replacing the first symbol r in w with r-index m, where m is the maximum of the r-indices, by $r-1$. Suppose that this happens at x_{p_0}. Then for each $s \geqslant 1$, the $(r-1)$-index of x_s is unaltered if $s < p_0$, and is increased by 1 if $s \geqslant p_0$. The element on which $S_{r-2,r-1}$ operates is therefore in the p_1th place, where p_1 is the first integer $\geqslant p_0$ for which x_{p_1} has $(r-1)$-index in w equal to 0. Likewise the element on which $S_{r-3,r-2}$ operates is in the p_2th place, where p_2 is the first integer $\geqslant p_1$ for which x_{p_2} has $(r-2)$-index zero, and so on.

In this way we obtain a sequence

$$p_0 \leqslant p_1 \leqslant \ldots \leqslant p_{r-a-1}$$

with the property that, for each $i \geqslant 1$, x_{p_i} is the first element not preceding $x_{p_{i-1}}$ for which the $(r-i)$-index is 0. Observe that in w' the element in the p_ith place still has $(r-i)$-index zero, for each $i \geqslant 1$ (though it will no longer be the first with this property).

Now consider the passage from $w' = y_1y_2y_3\ldots$ to w''. In w' the maximum r-index is $m-1$ (which by assumption is still positive) and occurs first at say y_{q_0}, where $q_0 < p_0$. (This is because the r-index can by its definition only go up or down in single steps, and therefore the r-index $m-1$ occurs first in w at some element to the left of x_{p_0}; and the elements to the left of the p_0th are the same in w' as in w.) In w' the $(r-1)$-index of y_{p_1} is zero, and is therefore $+1$ in $S_{r-1,r}(w')$. Hence $S_{r-1,r}(w')$ admits the substitution $S_{r-2,r-1}$, which will operate on the element in the q_1th place, where q_1 is the first integer $\geqslant q_0$ for which the $(r-1)$-index of y_{q_1} in w' is 0, so that $q_0 \leqslant q_1 < p_1$. Continuing in this way we get a sequence

$$q_0 \leqslant q_1 \leqslant q_2 \leqslant \ldots \leqslant q_{r-a-1}$$

with $q_i < p_i$ for all $i \geqslant 0$, and w' admits the operator $S_{a,r}$.

If $S_{a,r}(w') = w''$, then $b = a$; if not, then $S_{a,r}(w')$ admits further substitutions $S_{a-1,a}, \ldots$, until $w'' = S_{b,r}(w')$ is attained, so that $b < a$ in this case. In either case we have $b \leqslant a$, as required. |

We shall now describe the algorithm of Littlewood and Robinson which constructs from a tableau T of shape $\lambda - \mu$ and weight π, where λ, μ, π are partitions, a pair (L, M) where $L \in \mathrm{Tab}^0(\lambda - \mu, \nu)$ for some partition ν, and $M \in \mathrm{Tab}(\nu, \pi)$.

If A is any array—not necessarily a tableau—and a, r are positive integers such that $a < r$, we denote by $R_{a,r}(A)$ the result of raising the right-hand element of the rth row of A up to the right-hand end of the ath row.

The algorithm begins with the word $w_1 = w(T)$ and the array M_1 consisting of π_1 1's in the first row, π_2 2's in the second row, and so on (i.e. M_1 is the unique tableau of shape π and weight π).

Operate on w_1 with S_{12} until there are no positive 2-indices, and simultaneously on M_1 with R_{12} the same number of times: say

$$w_2 = S_{12}^m(w_1), \qquad M_2 = R_{12}^m(M_1).$$

Next operate on w_2 with S_{23} or S_{13} as appropriate until there are no positive 2- or 3-indices, and simultaneously operate on M_2 with R_{23} or R_{13}: say

$$w_3 = \ldots S_{a_2,3}S_{a_1,3}(w_2), \qquad M_3 = \ldots R_{a_2,3}R_{a_1,3}(M_2)$$

where each a_1, a_2 is 1 or 2.

Continue in this way until we reach (w_l, M_l), where $l = l(\pi)$. Clearly from our construction w_l is a lattice permutation. From (9.6) it follows that w_l is compatible with $\lambda - \mu$, so that $w_l = w(L)$ where $L \in \mathrm{Tab}^0(\lambda - \mu, \nu)$ for some partition ν. Next, it is clear from the construction that at each stage the length $l_i(M_r)$ of the ith row of the array M_r is equal to the multiplicity $m_i(w_r)$ of the symbol i in the corresponding word w_r, so that the final array $M = M_l$ has shape ν and weight π.

We have to show moreover that M_l is a *tableau*. For this, we shall prove by induction on r that the first r rows of M_r form a tableau. This is clear if $r = 1$, so assume that $r > 1$ and the result is true for $r - 1$.

Consider the steps that lead from M_{r-1} to M_r: we have, say,

$$M_r = R_{a_m,r}\ldots R_{a_1,r}(M_{r-1});$$

let us put

$$M_{r-1,i} = R_{a_i,r}\ldots R_{a_1,r}(M_{r-1}) \qquad (1 \leqslant i \leqslant m)$$

and likewise

$$w_{r-1,i} = S_{a_i,r}\ldots S_{a_1,r}(w_{r-1}),$$

where each word $w_{r-1,i}$ has the lattice property with respect to $(1, 2, \ldots, r-1)$. Each array $M_{r-1,i}$ is obtained from its predecessor $M_{r-1,i-1}$ (or M_{r-1} if $i = 1$) by moving up a single symbol r from the rth row to the a_ith row. By our construction the length $l_j(M_{r-1,i})$ of the jth row of $M_{r-1,i}$ is equal to the multiplicity $m_j(w_{r-1,i})$ of j in $w_{r-1,i}$, for each $j \geqslant 1$; and since each word $w_{r-1,i}$ has the lattice property with respect to $(1, 2, \ldots, r-1)$, it follows that

$$l_1(M_{r-1,i}) \geqslant \ldots \geqslant l_{r-1}(M_{r-1,i}).$$

Also, by (9.7), the integers a_i satisfy $a_1 \geqslant \ldots \geqslant a_m$. It follows that no two symbols r can appear in the same column at any stage, and consequently the first r rows of M_r form a tableau.

The algorithm therefore provides a mapping

$$\mathrm{Tab}(\lambda - \mu, \pi) \to \coprod_{\nu} \mathrm{Tab}^0(\lambda - \mu, \nu) \times \mathrm{Tab}(\nu, \pi).$$

To complete the proof of (9.4) we have to show that this mapping is a bijection. For this purpose it is enough to show that, for each $r \geqslant 1$, we can unambiguously trace our steps back from (w_r, M_r) to (w_{r-1}, M_{r-1}). With the notation used above, we have

$$w_r = S_{a_m,r}\ldots S_{a_1,r}(w_{r-1}),$$

and the sequence $(a_1, \ldots, a_m)$ can be read off from the array M_r, since the a_i are the indices $< r$ of the rows in which the symbols r are located in M_r, arranged in descending order: $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_m$ (by virtue of (9.7)). Since by (9.5) each $S_{a,r}$ is reversible, it follows that (w_{r-1}, M_{r-1}) is uniquely determined by (w_r, M_r). Finally, by (9.6), if w_r is compatible with $\lambda - \mu$, then so also is w_{r-1}, and the proof is complete. Q.E.D.

Remark. A lattice permutation $w = a_1 a_2 \ldots a_N$ of weight ν may be described by a standard tableau $T(w)$ of shape ν, in which the symbol r occurs in the a_rth now, for $1 \leqslant r \leqslant N$ (the fact that w is lattice ensures that $T(w)$ is a tableau). Hence the algorithm described above constructs from a word w a pair of tableaux $T(w_l)$ and M_l of the same shape ν, the first of which is standard and the second of weight π. It may be verified that this algorithm coincides with one described by Burge [7] (see also Gansner [13]).

Notes and references

The Littlewood–Richardson rule (9.2) was first stated, but not proved, by Littlewood and Richardson in [28] (p. 119). The proof subsequently published by Robinson [39], and reproduced in Littlewood's book ([29], pp. 94–6) is incomplete, and it is this proof that we have endeavoured to complete. Apparently, complete proofs of the rule were first published only recently, due to Lascoux and Schützenberger [46] and Thomas [53].