The Reluctance of a Sequence

Guest Lecture, Math 640, Rutgers, May 2 2024

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Outline

- Recamán's sequence A5132
- Minimized Recamán A125717
- Definition of reluctance function
- EKG A064413
- Cup of Coffee sequence A280864
- Cald's sequence A6509

Recamán’s Sequence

A5132

Recamán’s Sequence

Bernardo Recamán Santos, 1991

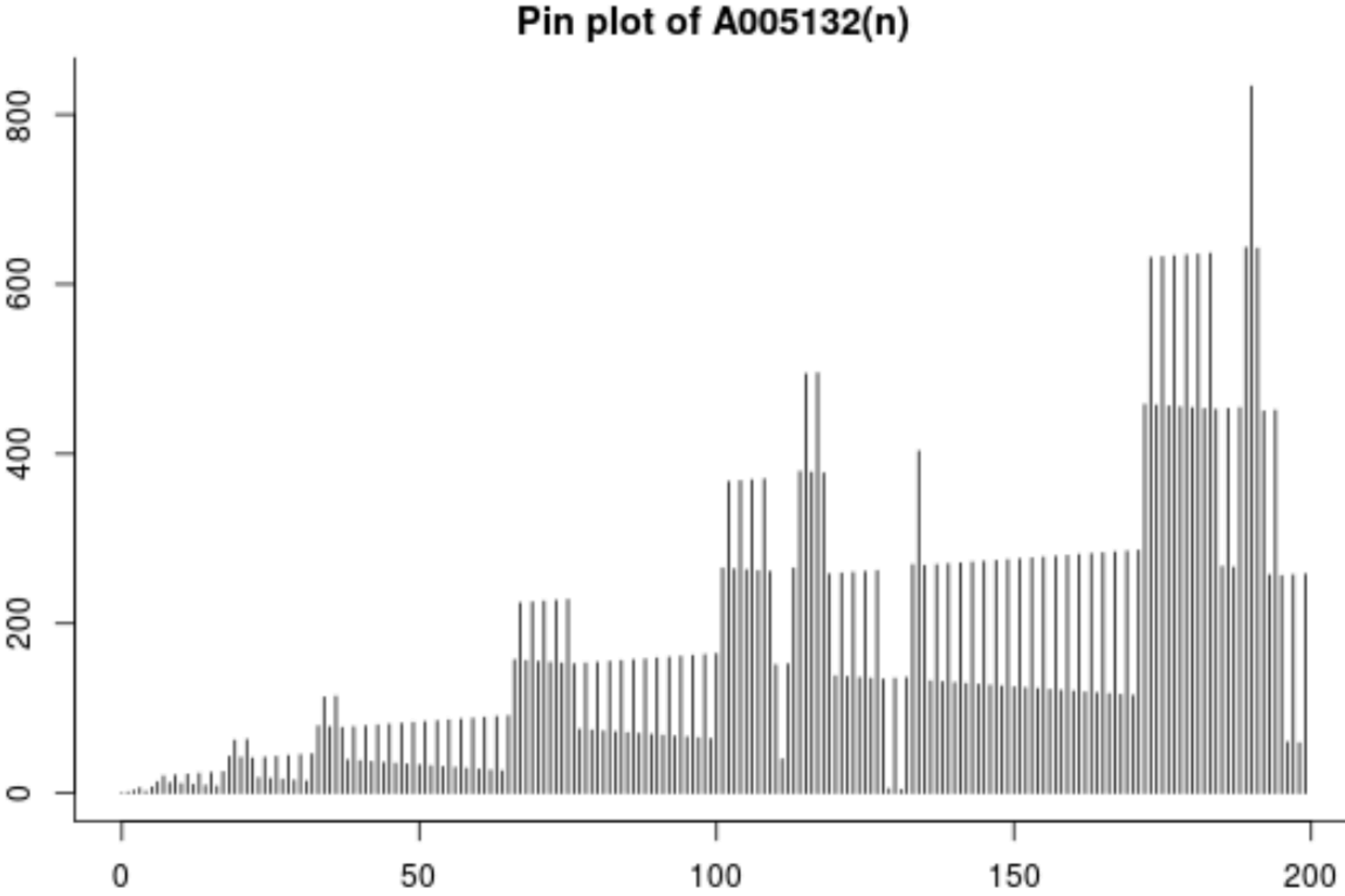
Subtract or add: 1, 2, 3, 4, 5, 6, …

No negative terms, no repeats (except when adding)

0, 1, 3, 6, 2, 7, 13, 20, 12, 21, 11, …

1 2 3 -4 5 6 7 -8 9 -10 …

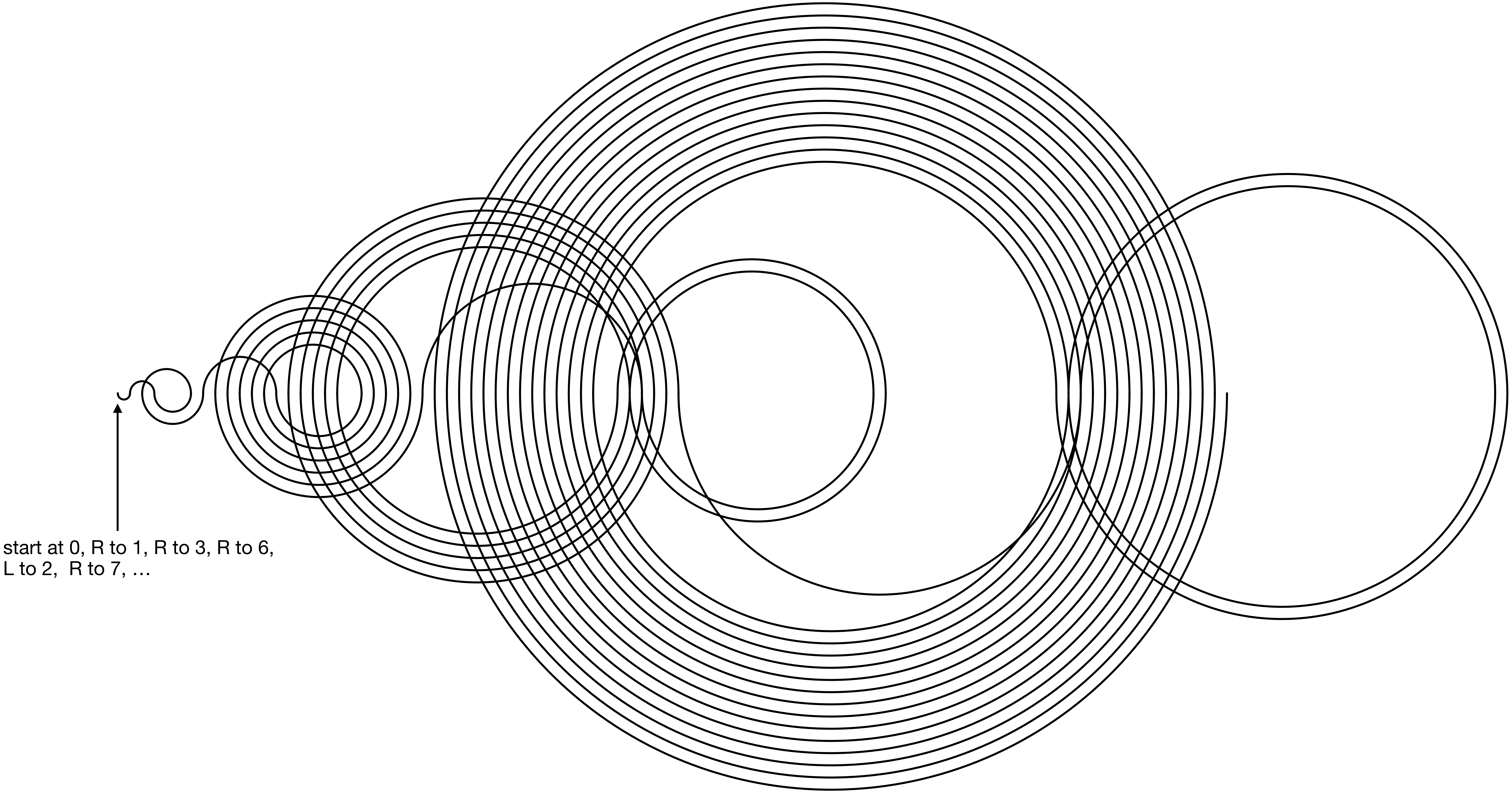
A5132



Recamán’s Sequence (2)

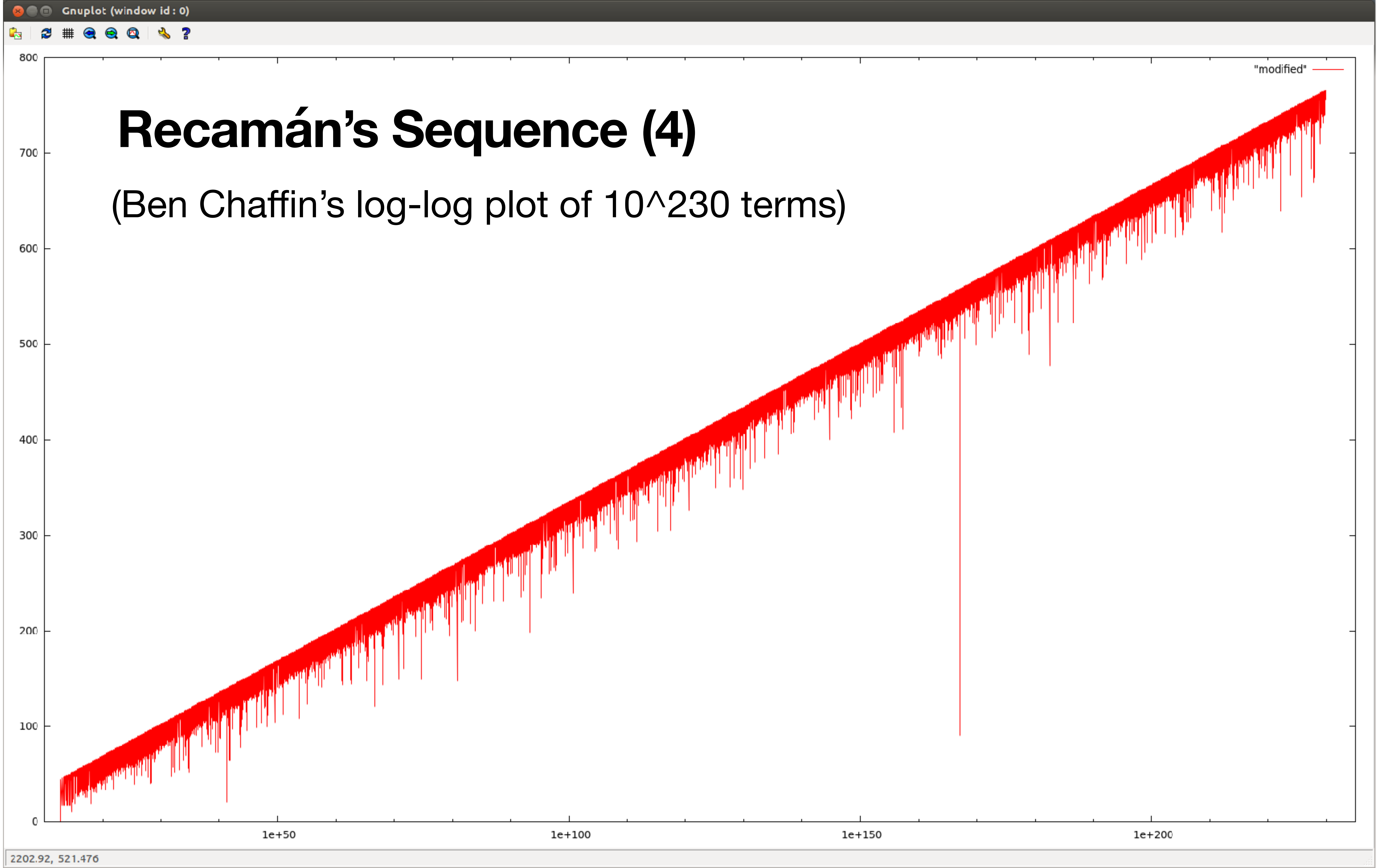
Edmund Harriss,

First 62 terms drawn as a spiral



Recamán’s Sequence (4)

(Ben Chaffin’s log-log plot of 10^230 terms)



Recamán's Sequence (5)

A5132

The Big Question: Does every number appear?

After 10^15 terms, 852655 = 5 x 31 x 5501 was missing (Allan Wilks, 2001)

After 10^230 terms, 852655 is still missing (Ben Chaffin, 2018)

30 years ago I believed that every number would eventually appear. Today I think that there are infinitely many missing terms, and 852655 just got lucky and is the first of many.

Why is 852655 so reluctant to appear? Can we define "reluctance"?

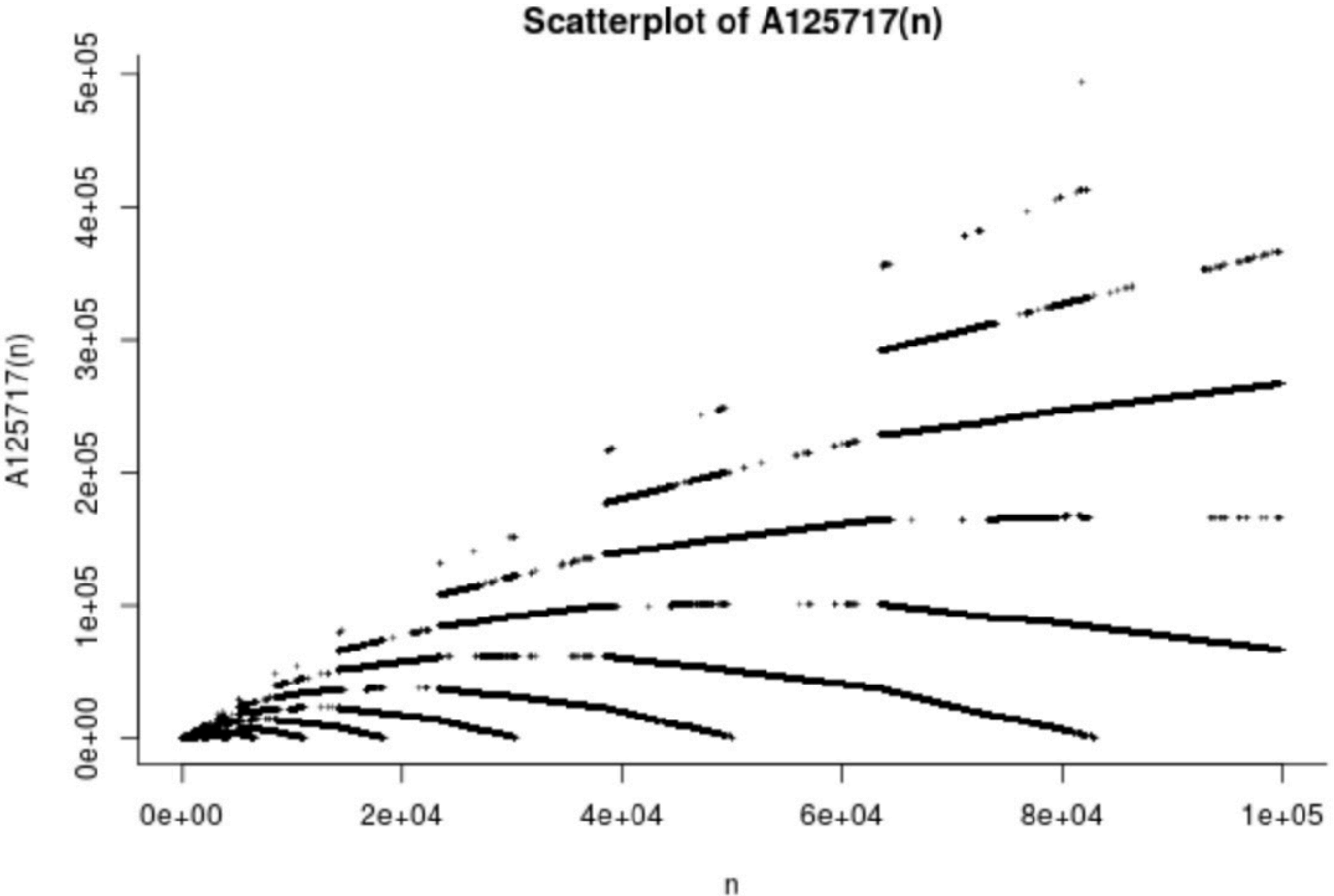
Minimized Recamán Sequence

A125717

Leroy Quet, 2007

Minimized Recamán Sequence (Leroy Quet, 2007)
A125717

a(0)=0; a(n) = min m >= 0 s.t. m is new and m == a(n-1) mod n



A125717 Minimized Recamán

$a_0 = 0$. $a_n = \min m \geq 0$ s.t. new &

$m \equiv a_{n-1} \bmod n$

When slowest appear

Which are slowest?

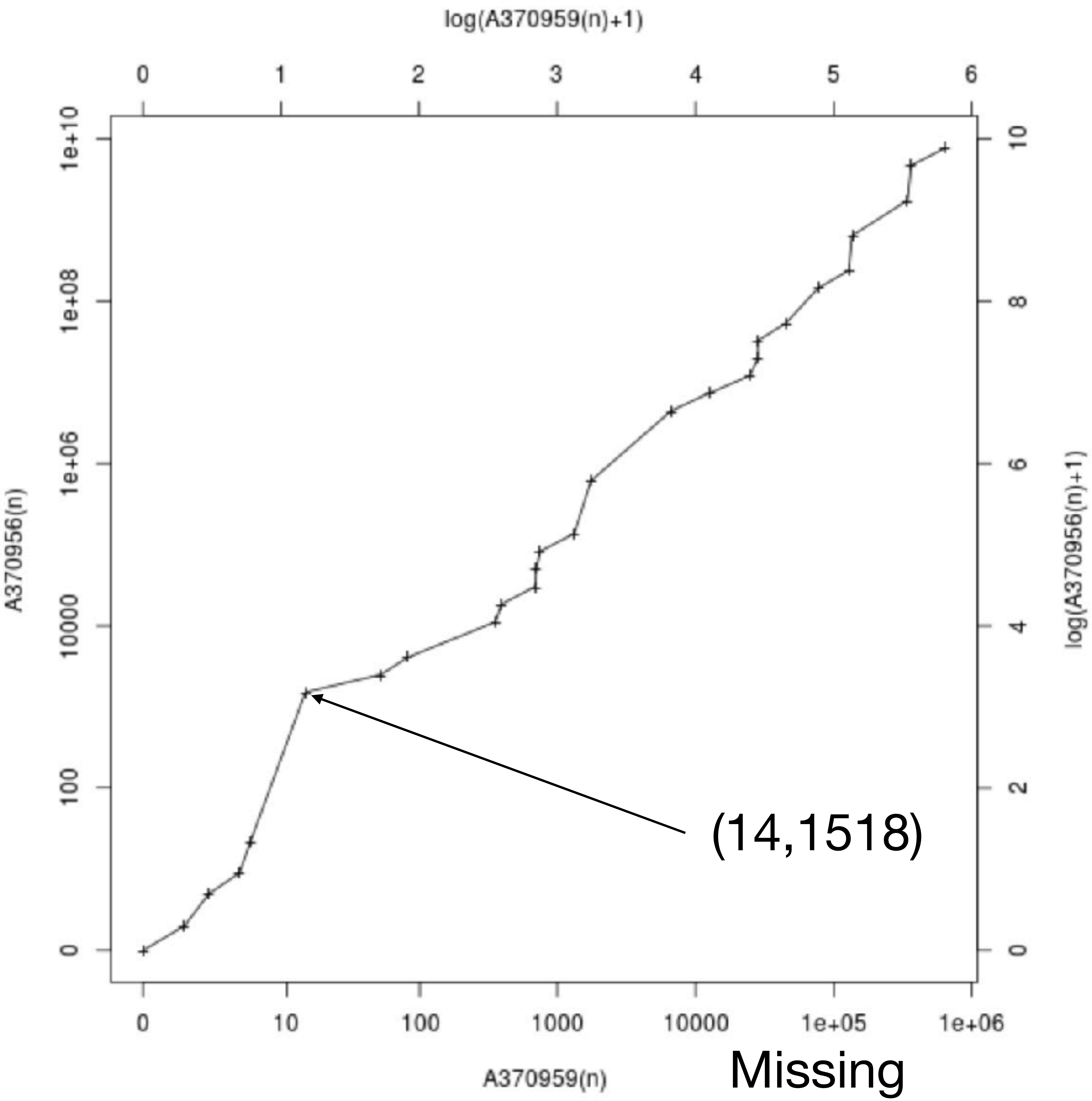
A125717 n	0	1	2	3	4	5	6	7	8	9
a_n	0	1	3	6	2	7	13	20	4(!)	22
action A370958	–	1	1	1	−1	1	1	1	−2	2
A245340 inverse	0	1	4	2	8	21	3	5	18	16
A370956 records	[0_1	1_2	4_3	8_4	21_5	1518_6	2510_7	4100_8	11181_9	18414_{10}
A370959 which?	0	1	2	4	5	14	52	82	356	392
smn A372057	1	2	2	2	4	4	4	4	5	5

- 5 doesn't appear until n = 21
14 " " " " 1518
52 " " " " 2510

n	10	11	12	13	14	15	16	17	18	19
a(n)	12	23	11	24	10	25	9	26	8	27
action A370958	−1	1	−1	1	−1	1	−1	1	−1	1
inverse A245340	14	12	10	6	1518	32	58	30	184	28
records A370956	30374	~~[illegible]~~	50121	82924						
which?	688		704	751						

A3709

A370959 vs A370956



When?

(14,1518)

Missing

$$\log_{10}(\text{when}) \approx \frac{10}{6}\log_{10}(\text{which})$$

$$\text{when} \approx (\text{which})^{5/3}$$

$$y = x^{5/3}$$

is reluctance function for A125717

Use Plot2 command in OEIS

Reluctance Function for a sequence a(n) believed to be a permutation of positive integers

Let b(n) = inverse perrmutation to a(n),

compute h(n) = record high-points in b(n),

and w(n) = indices of these high-points in b(n)

Then w(n) are numbers that take a record number of steps to appear in a(n), and h(n) = how long it takes fot w(n) to appear in a(n).

If h(n) = f(w(n)), we call y = f(x) the reluctance function.

The EKG Sequence

A064413

Jonathan Ayres, 2001

EKG Sequence (A64413)

1, 2, 4, 6, 3, 9, 12, 8, 10, 5, 15, ...

$a(1)=1$, $a(2)=2$,

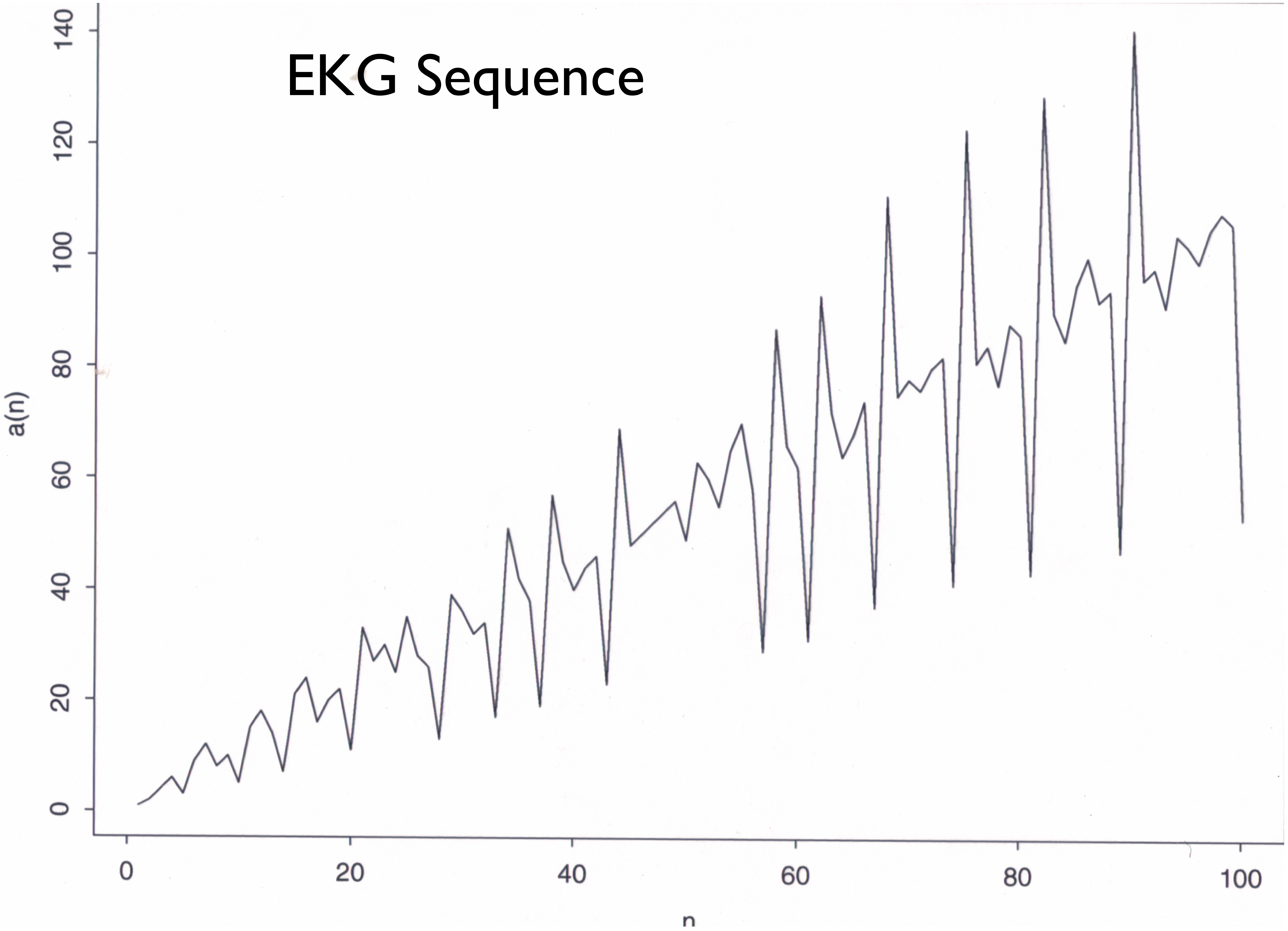
$a(n) = \min k$ such that

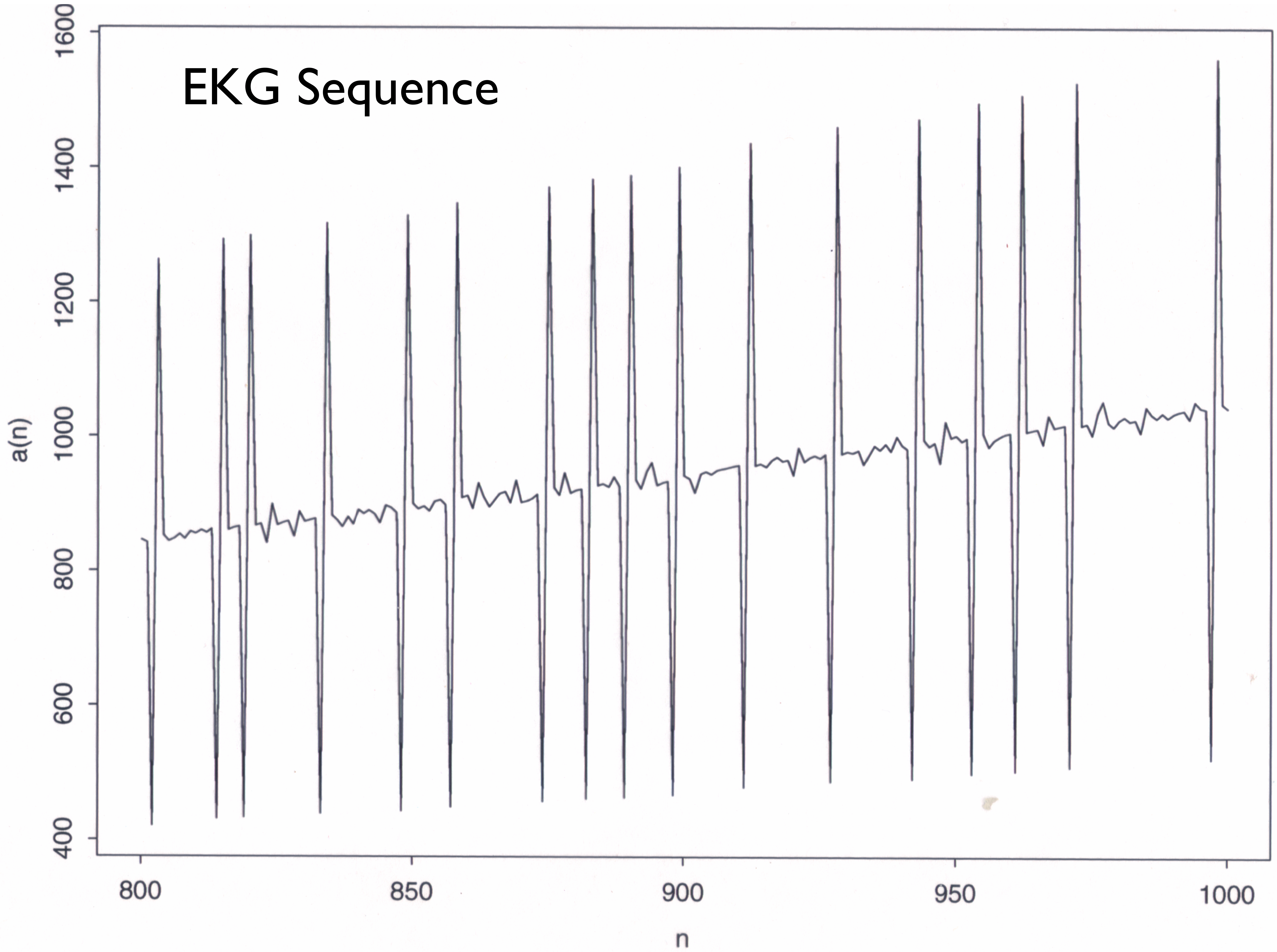
- $\mathrm{GCD}\{ a(n-1), k \} > 1$
- k not already in sequence

- Jonathan Ayres, 2001
- Analyzed by Lagarias, Rains, NJAS, Exper. Math., 2002
- Gordon Hamilton, Videos related to this sequence:

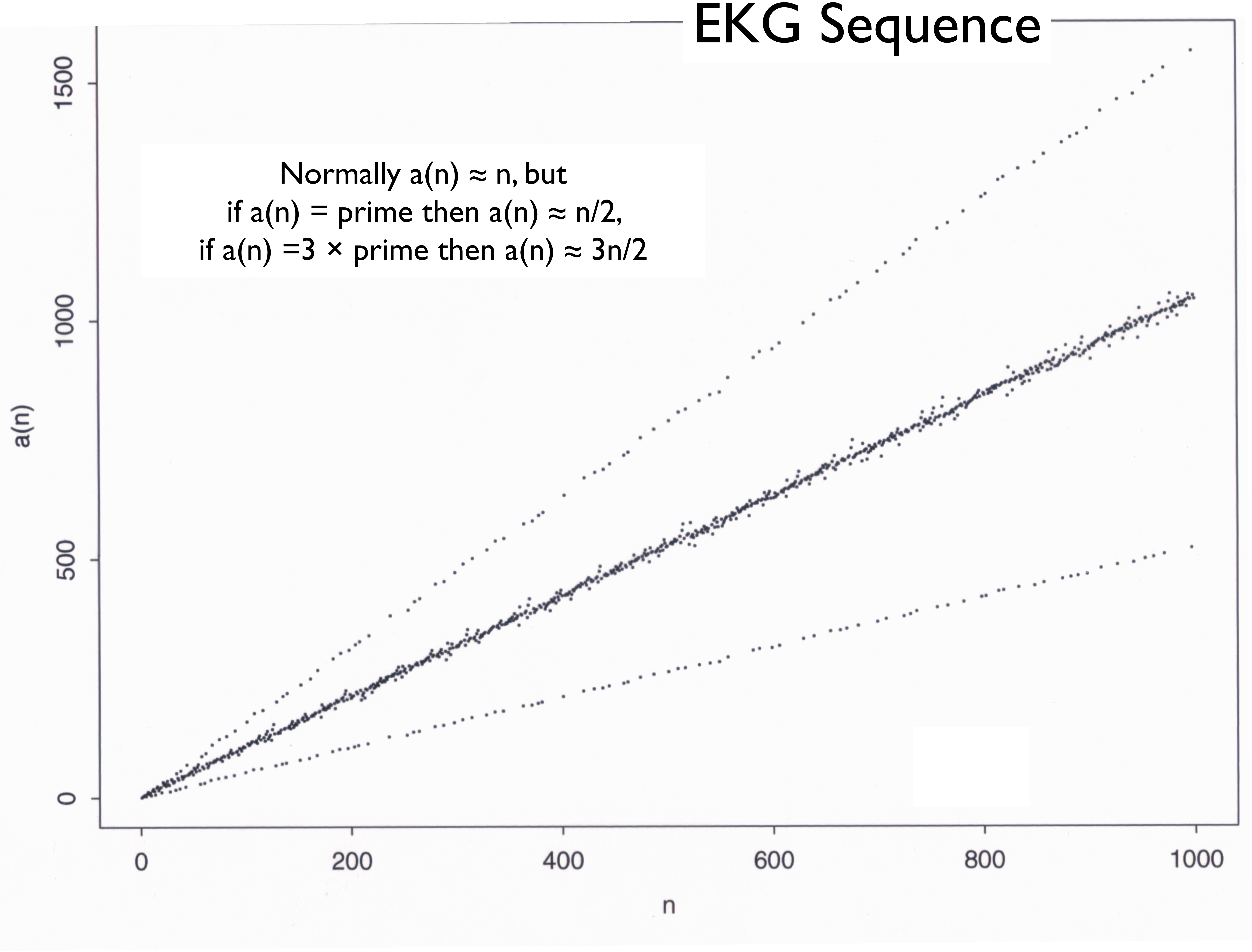
https://www.youtube.com/watch?v=yd2jr30K2R4&feature=youtu.be http://m.youtube.com/watch?v=Y2KhEW9CSOA

EKG Sequence




EKG Sequence
a(n)
n
400
600
800
1000
1200
1400
1600
800
850
900
950
1000

EKG Sequence



Normally a(n) ≈ n, but
if a(n) = prime then a(n) ≈ n/2,
if a(n) =3 × prime then a(n) ≈ 3n/2

EKG cont. A64413

Question: Does every number appear?

High school student:
That’s obvious!

Me: I don’t think so!

The EKG sequence (cont)

A64413

Theorem: Every positive number appears

Proof:

There are several steps. (i) Sequence is infinite (easy).
(ii) Let T(m) = n such that a(n)=m, or -1 if m is missing from sequence.
Let W(m) = max T(i), i <= m. Then if n > W(m), a(n) > m.

(iii) Let p = prime. Exists n such that p | a(n). If not, no prime q>p can divide
any term either, because if a(n) = qk then pk would be a smaller choice.
So all terms are products just of primes < p.
Choose n>W(p^2), say a(n) = qk, for prime q<p, so qk > p^2.
Then pk < p^2 < qk was a smaller candidate for a(n), contradiction.

(iv) When p first divides a(n), say a(n) = kp, then k is a prime < p.
If k = 2 we have a(n)=2p, a(n+1)=p. Otherwise we have a(n)=kp,
a(n)=p, a(n+1)=2p. Either way we see adjacent terms p and 2p.

Proof (continued)

(v) If for some prime p there are infinitely many multiples of p,
then all multiples of p are in the sequence.
If not, let kp = smallest missing multiple of p.
Find n >W(kp) with a(n) = mp. Then kp < mp was a smaller
candidate for a(n), a contradiction.

(vi) If for some prime p all multiples of p are in the sequence
then all numbers appear. For suppose k is smallest missing number.
Find n > W(k) such that a(n) is multiple of kp. Then k was smaller
candidate for a(n), contradiction.

(vii) By (iii) and (iv) we see infinitely many multiples of 2,
and by (v) and (vi) we see all numbers.

QED

The three worst non-proofs

in order of increasing badness

It's obvious
It's true for the first 10000 terms
Here is the proof … [and it's wrong]

When you write your paper proving that the long-standing Gauss PQR conjecture is true, start off by describing the previous attempts at proof, and where they failed and then explain how your proof is better

EKG Reluctance A064413

	1	2	1	0	0	2	-2	1	1	3	1	1	7	8
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A64413 → EKG	1	2	4	6	3	9	12	8	10	5	15	18	14	7
A64664 → Inverse	1	2	5	3	10	4	14	8	6	9	20	7	28	13
-1× when?	1	2	5	10	14	20	28	33	37	48	57	61	67	74
2× slowest	1	2	3	5	7	11	13	17	19	23	29	31	A8578	
smn	2	3	3	3	5	5	5	5	5	7	7	7	7	11

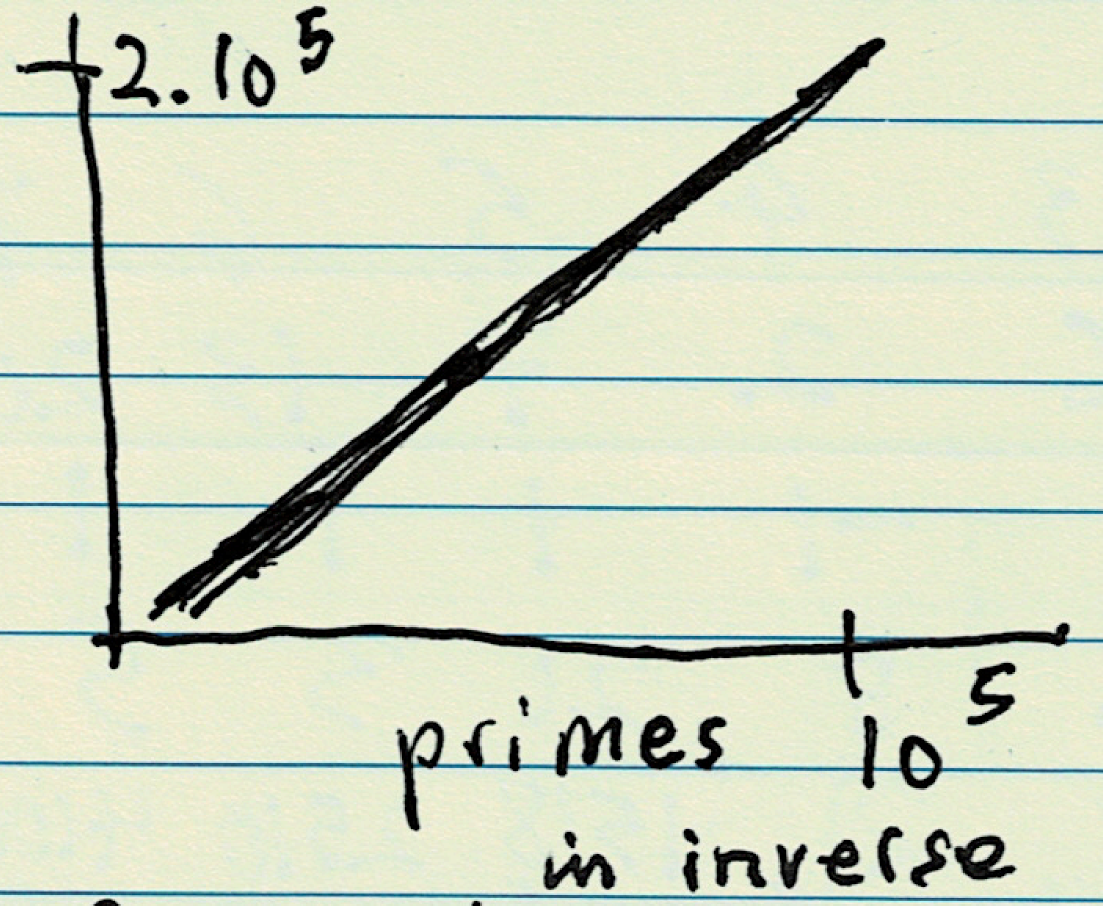
← A064955

← A137847

Open Problem

what are these numbers??

when?



heavy line
slope = 2

Reluctance: $y \div 2x$
conj!

k^{th} high point$_k$ = p_k = $k \log k$

When? index is about $2p_k$

Consistent with "Conjecture: if $a(n) = p$ then $a(n) \sim \frac{n}{2}\left(1 + \frac{1}{3 \log n}\right)$"

Open Problem

The “Cup of Coffee” Sequence

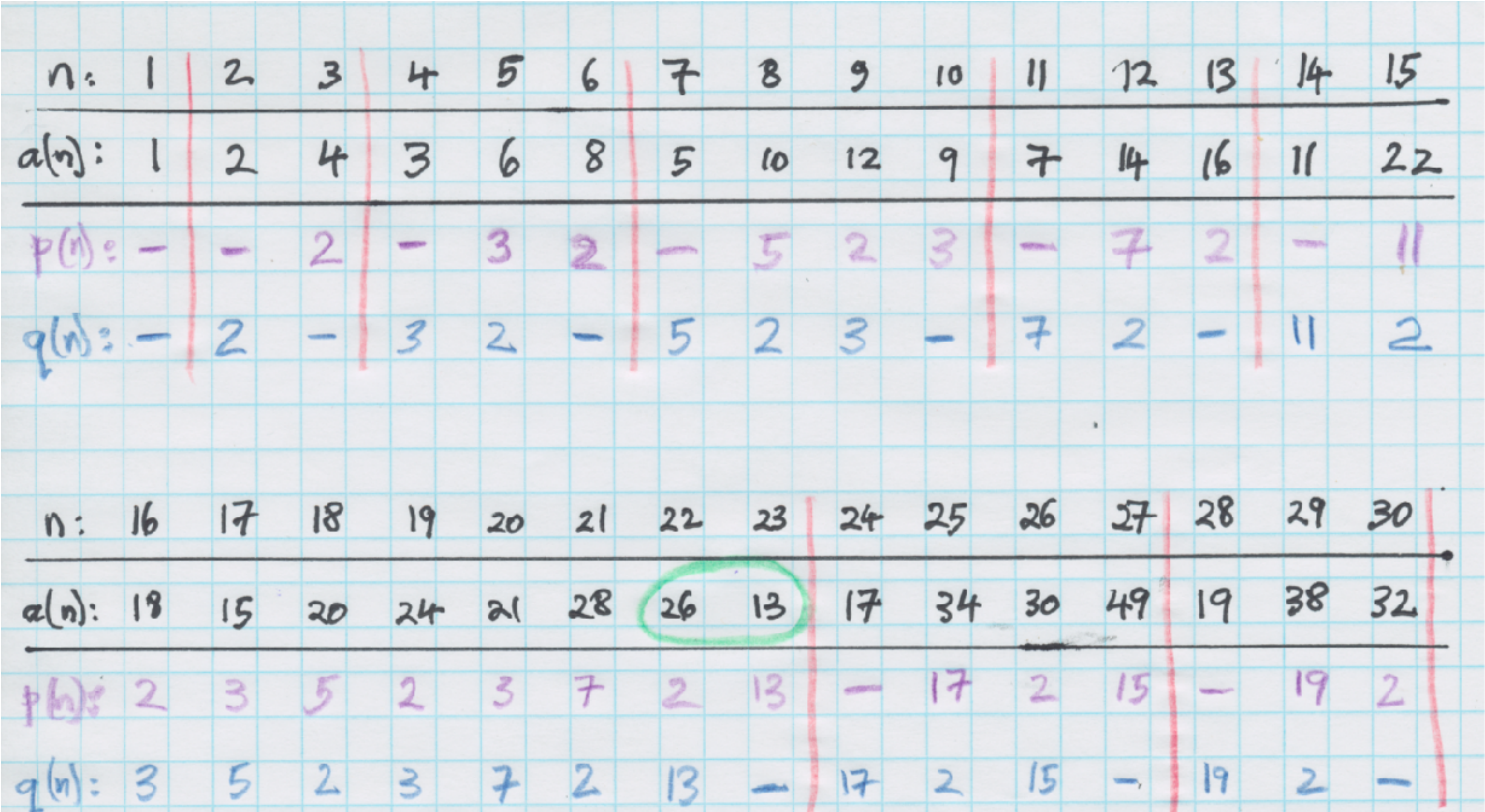
A280864

Rémy Sigrist, 2017

A280864

REMY SIGRIST'S SEQUENCE

LES of positive integers such that if a prime p divides a(n) then p divides a(n-1) or a(n+1) but not both



n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a(n):	1	2	4	3	6	8	5	10	12	9	7	14	16	11	22
p(n):	–	–	2	–	3	2	–	5	2	3	–	7	2	–	11
q(n):	–	2	–	3	2	–	5	2	3	–	7	2	–	11	2

n:	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
a(n):	18	15	20	24	21	28	26	13	17	34	30	49	19	38	32
p(n):	2	3	5	2	3	7	2	13	–	17	2	15	–	19	2
q(n):	3	5	2	3	7	2	13	–	17	2	15	–	19	2	–

REMY SIGRIST'S SEQUENCE (cont.)

A280864

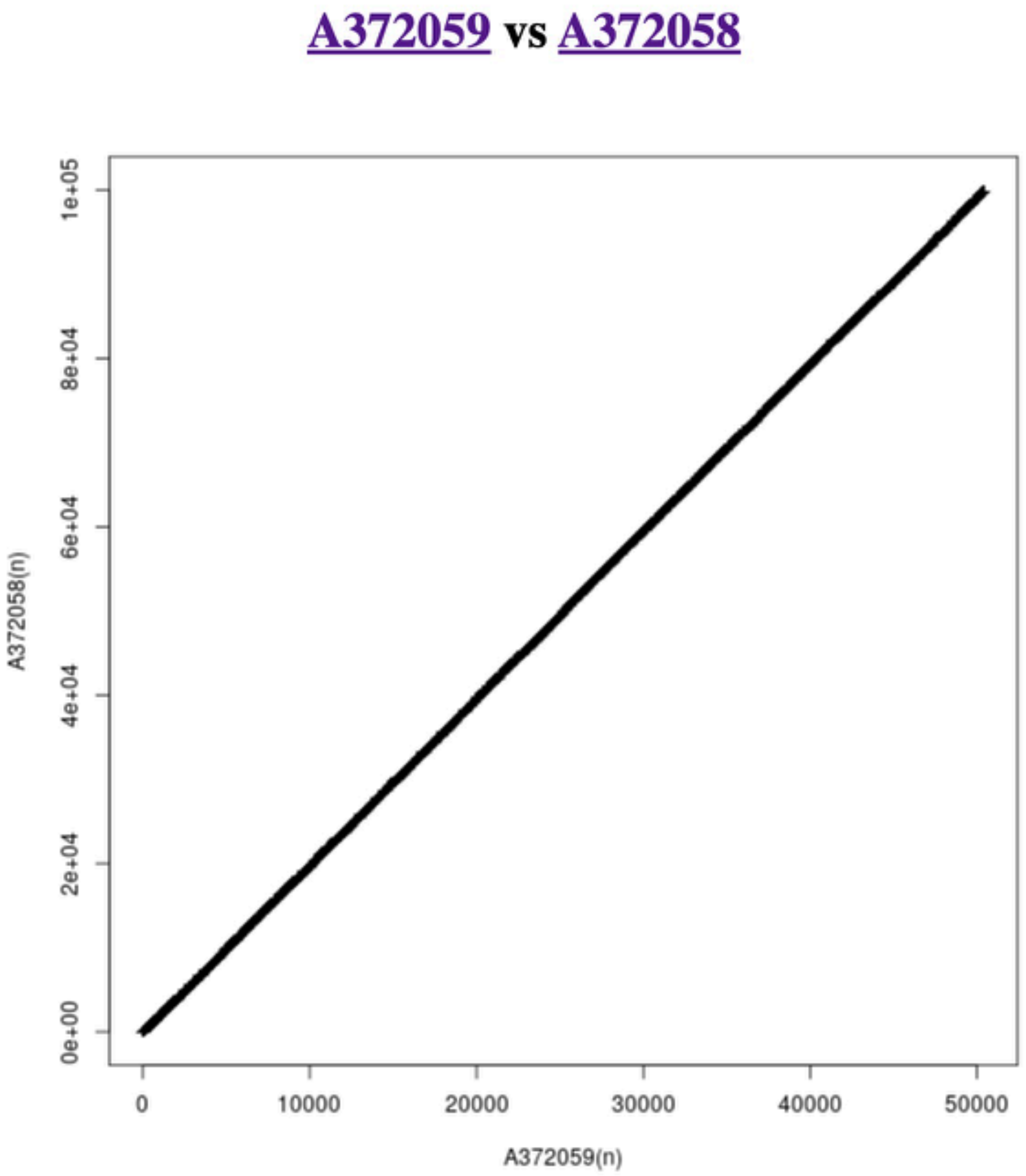
Conjecture: This is a permutation of the positive integers.

I can prove:
- every prime appears
- every even number appears
- infinitely many odd multiples of any odd prime p
- every number appears iff every square appears

But I cannot prove that every odd number appears

Reluctance of Cup-of-Coffee Sequence A280864

A372058 When?


A372059 vs A372058

A372059 Missing

Conjecture: 1, 25, and all primes

Conjecture:

Reluctance function is

y = 2x

Essentially same reluctance

as EKG sequence

Other LES (Lexicographically Earliest Sequences)

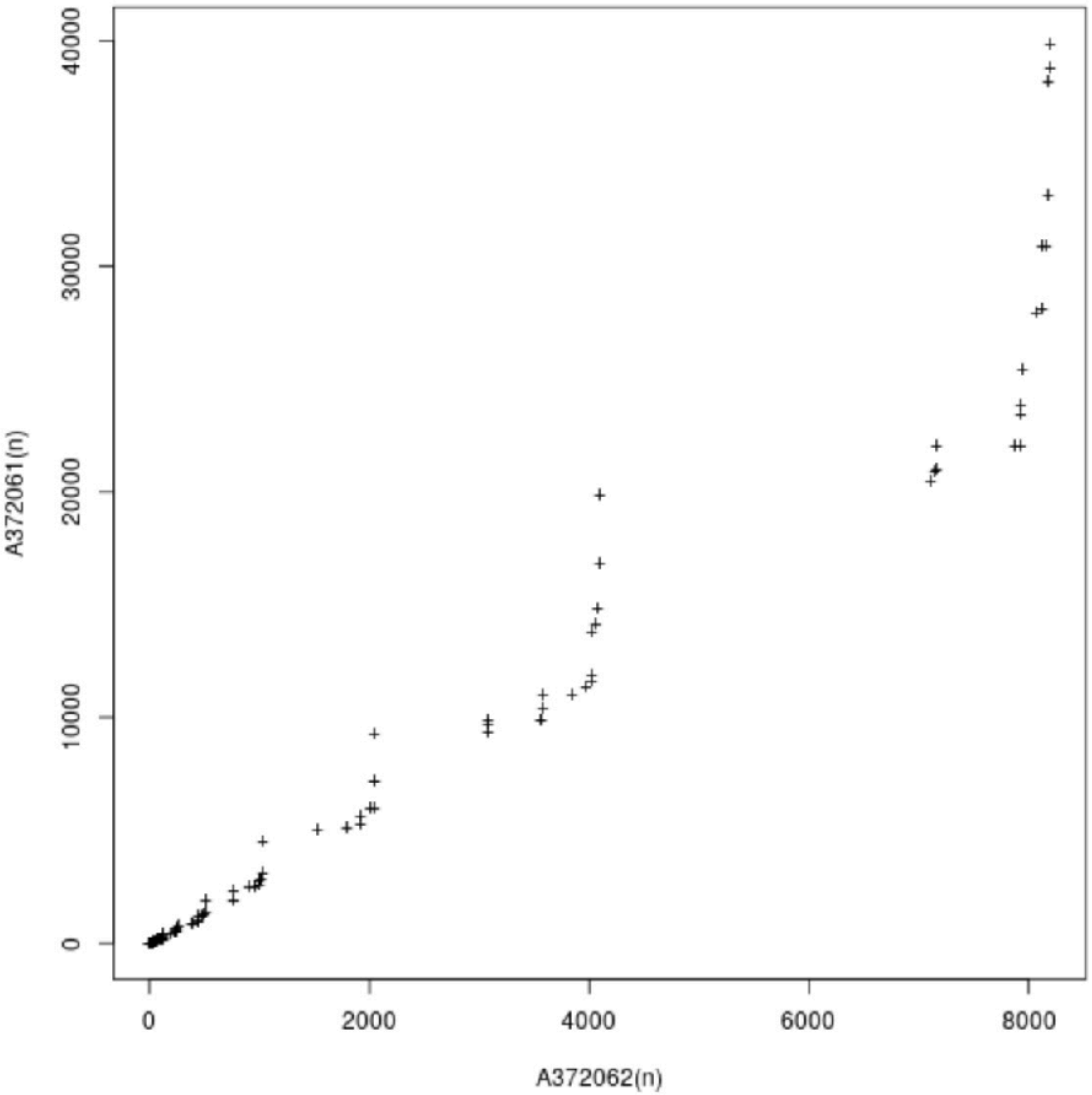
There are a great many! Tetris A109812,
Yellowstone Permutation, Enots Wolley.
Grant Olson, Two-Up, Cald's Sequence, Ali Sada, etc

Many are conjectured to be permutations of positive integers,
but there are very few proofs.

An inconclusive example
A252867

(Set-theory analog of Yellowstone permuation A098550)

A372062 vs A372061



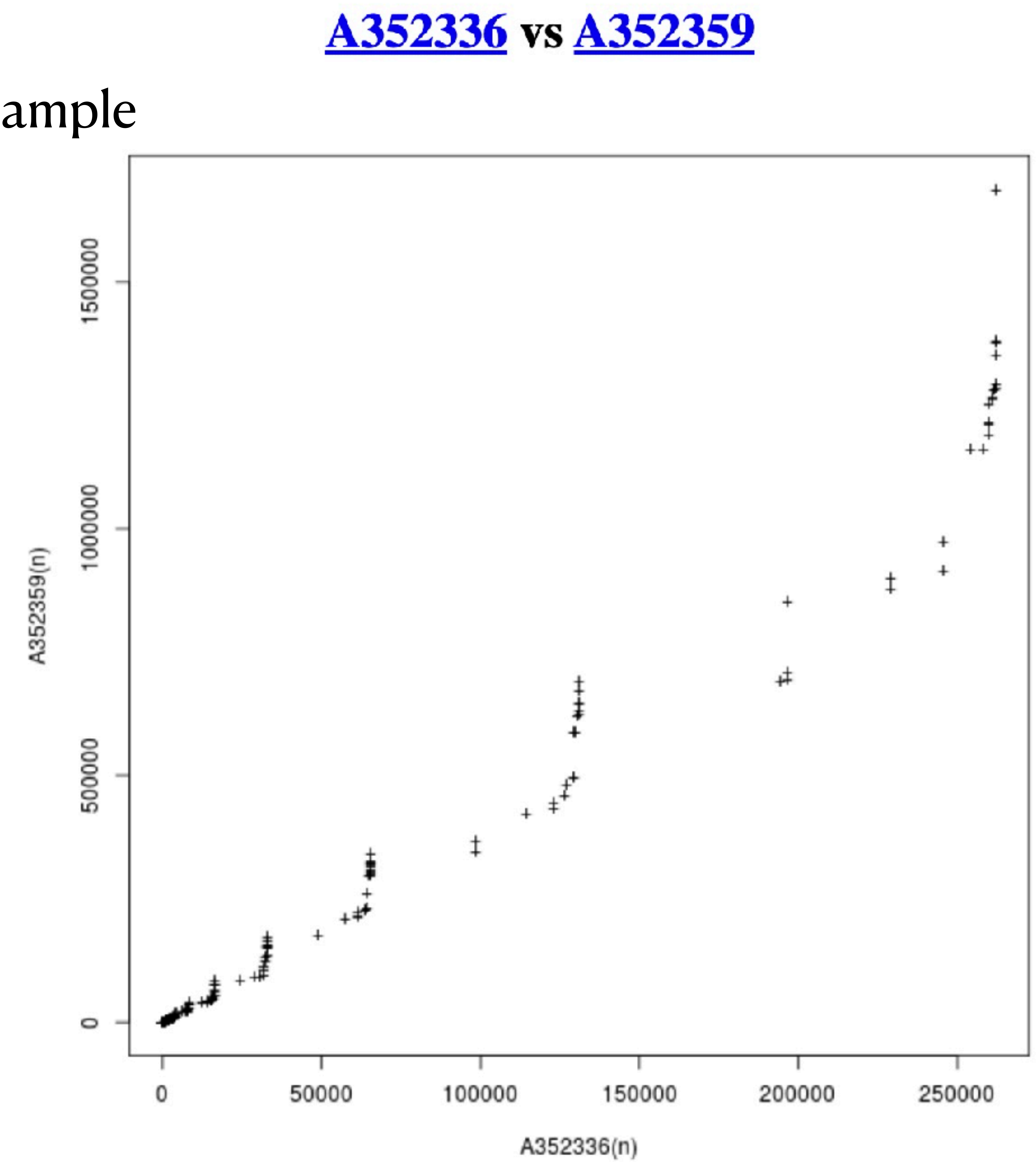
Another inconclusive example

A109812

The Tetris sequence

Set-theory analog of natural numbers!

What is the reluctance function?

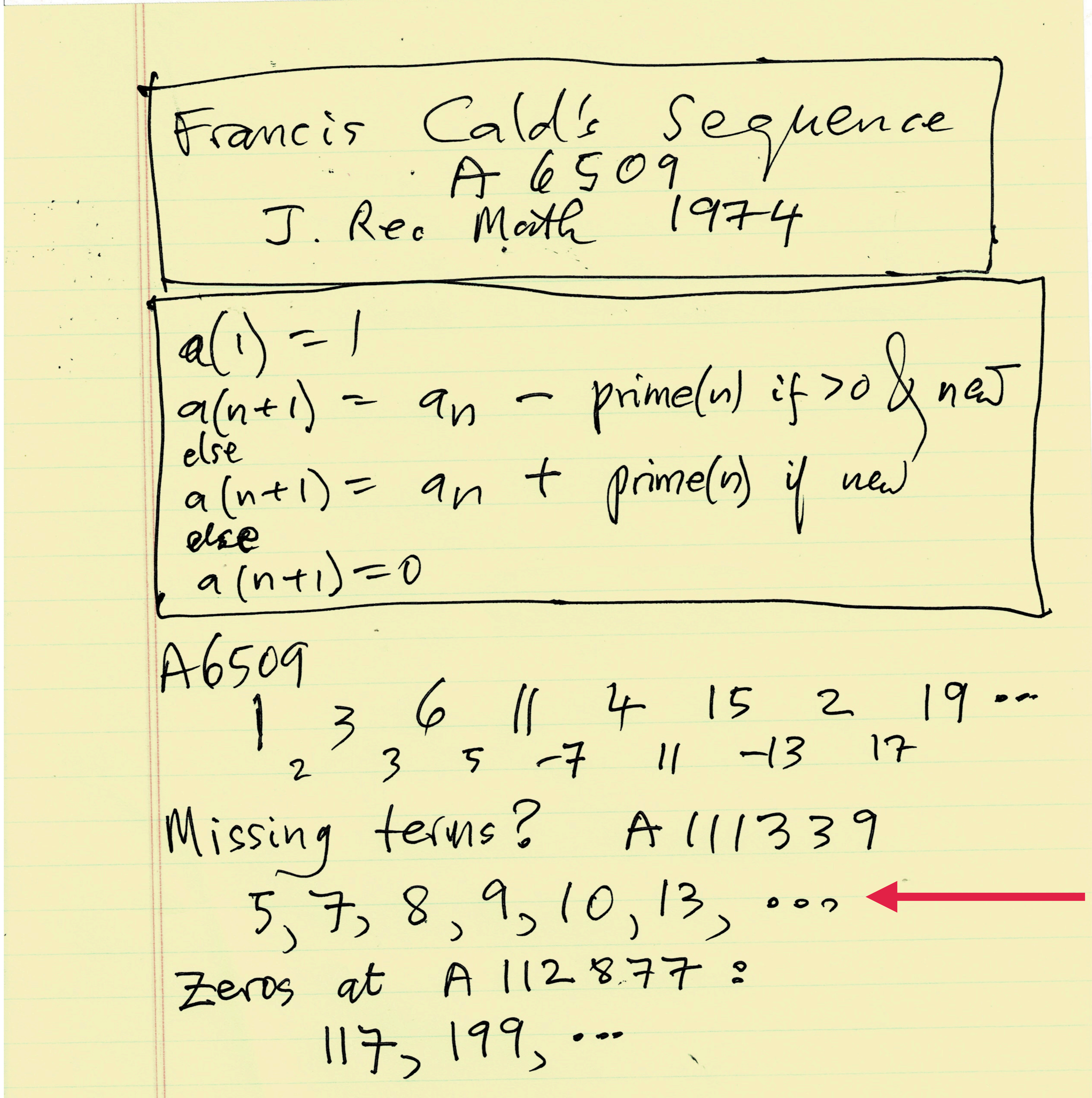


Cald’s Sequence

A6509

Jnl. Rec. Math., 1974

Cald's Sequence



Francis Cald's Sequence
A 6509
J. Rec Math 1974

$a(1) = 1$

$a(n+1) = a_n - \text{prime}(n)$ if >0 & new

else

$a(n+1) = a_n + \text{prime}(n)$ if new

else

$a(n+1) = 0$

A6509

1 3 6 11 4 15 2 19 ...

2 3 5 −7 11 −13 17

Missing terms? A111339

5, 7, 8, 9, 10, 13, ...

Zeros at A112877:

117, 199, ...

Can't define reluctance in same way, because don't know inverse function.

I offer $250 for first proof that 5 is missing from Cald's sequence

What are these numbers?