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① Prove that if a rational function of the form

$$\frac{A(t)}{(1-ty_1)(1-ty_2)\cdots(1-ty_n)} = \frac{A(t)}{B(t)}$$

with all y_i's distinct, is written in partial fractions

$$\frac{C_1}{1-ty_1} + \frac{C_2}{1-ty_2} + \cdots + \frac{C_n}{1-ty_n}$$

then $C_i = -\dfrac{y_i\, A\left(\frac{1}{y_i}\right)}{B'\left(\frac{1}{y_i}\right)}$ where $B(t) = (1-ty_1)\cdots(1-ty_n)$.

Proof:

Let $B(t) = \prod_{i=1}^{n}(1-ty_i) = (1-ty_1)(1-ty_2)\cdots(1-ty_n)$.

Set $S = \{1, 2, \cdots, n\}$.

Next, observe that

$$B'(t) = -y_1 \prod_{i\in S/\{1\}}(1-ty_i) - y_2 \prod_{i\in S/\{2\}}(1-ty_i) - \cdots - y_n \prod_{i\in S/\{y_n\}}(1-ty_i)$$

Now, consider the following:

$$\lim_{t\to\frac{1}{y_i}} \frac{(1-ty_i)A(t)}{B(t)}$$

Since $\lim_{t\to\frac{1}{y_i}} (1-ty_i)A(t) = 0$ and $\lim_{t\to\frac{1}{y_i}} B(t) = 0$

then we can apply L'Hopital's rule:

$$A(t) - ty_i A(t) = A'(t) - ty_i A'(t) - y_i A(t)$$

$$\lim_{t \to \frac{1}{y_i}} \frac{(1-ty_i)A(t)}{B(t)} = \lim_{t \to \frac{1}{y_i}} \frac{A(t) - ty_iA(t)}{B(t)}$$

$$= \lim_{t \to \frac{1}{y_i}} \frac{A'(t) - ty_iA'(t) - y_iA(t)}{B'(t)}$$

(Apply L'Hopitals)

$$= \frac{A'\left(\frac{1}{y_i}\right) - \frac{1}{y_i} y_i A'\left(\frac{1}{y_i}\right) - y_i A\left(\frac{1}{y_i}\right)}{B'\left(\frac{1}{y_i}\right)}$$

$$= \frac{-y_i A\left(\frac{1}{y_i}\right)}{B'\left(\frac{1}{y_i}\right)}$$

Thus, $\displaystyle\lim_{t \to \frac{1}{y_i}} \frac{(1-ty_i)A(t)}{B(t)} = \frac{-y_iA\left(\frac{1}{y_i}\right)}{B'\left(\frac{1}{y_i}\right)} = c_i$.

We will use Residue's theorem to deduce that given

$$\frac{A(t)}{B(t)} = \frac{A(t)}{(1-ty_1)\cdots(1-ty_n)}$$ with y_i's distinct, it

can be decomposed as follows:

$$\frac{A(t)}{B(t)} = \frac{c_1}{1-ty_1} + \cdots + \frac{c_n}{1-ty_n}$$ where

$$c_i = -\frac{y_i A\left(\frac{1}{y_i}\right)}{B'\left(\frac{1}{y_i}\right)}.$$

First, call $R(t) = \frac{A(t)}{B(t)}$. We need to find all c_i's such that

$$R(t) = \frac{c_1}{1-ty_1} + \cdots + \frac{c_n}{1-ty_n}.$$

So, take the Cauchy integral to both sides:

$$\oint R(t) = \oint \frac{c_1}{1-ty_1} + \cdots + \oint \frac{c_n}{1-ty_n}.$$

Observe that the right hand side has singularities at $t = \frac{1}{y_i}$. Even more, these singularities are single poles.

Hence, we can apply Residue's theorem to each integral to obtain the C_i's since C_i's are the residues at $t = \frac{1}{y_i}$'s.

Therefore,

$$C_i = \lim_{t \to \frac{1}{y_i}} \left(t - \frac{1}{y_i}\right) \frac{A(t)}{B(t)} = \lim_{t \to \frac{1}{y_i}} \frac{(ty_i - 1)\, A(t)}{B(t)}$$

$$= \frac{-y_i\, A\left(\frac{1}{y_i}\right)}{B'\left(\frac{1}{y_i}\right)}$$

as desired.

∎

$$\frac{t+1}{(1-t)(1-2t)} = R(t)$$

$$\frac{t+1}{(1-t)(1-2t)} = \frac{C_1}{1-t} + \frac{C_2}{1-2t}$$

$y_1 = 1 \qquad y_2 = 2$

$$t+1 = (1-2t)C_1 + C_2(1-t)$$

$$t+1 = -2C_1 t - C_2 t + C_1 + C_2$$

$$t+1 = -(2C_1+C_2)t + (C_1+C_2)$$

$C_1 + C_2 = 1 \qquad -2C_1 - C_2 = 1$

$C_1 = 1 - C_2 \qquad -2(1-C_2) - C_2 = 1$

$C_1 = 1 - 3 \qquad -2 + 2C_2 - C_2 = 1$

$C_1 = -2 \qquad -2 + C_2 = 1$

$C_2 = 3$

So

$$\frac{t+1}{(1-t)(1-2t)} = \frac{-2}{1-t} + \frac{3}{1-2t}$$

$$C_i = \frac{-y_i\, A\left(\frac{1}{y_i}\right)}{B'\left(\frac{1}{y_i}\right)}$$

$A(t) = t+1$

$B(t) = (1-t)(1-2t)$

$B'(t) = (1-t)(-2) + (1-2t)(-1)$

$= -2 + 2t - 1 + 2t$

$B'(t) = -3 + 4t$

$$C_1 = \frac{-1\left(\frac{1}{1}+1\right)}{-3+4} = \frac{-2}{1} = -2$$

$2\left(\frac{3}{2}\right)$

$$C_2 = \frac{-2\left(\frac{1}{2}+1\right)}{-3+4\left(\frac{1}{2}\right)} = \frac{-3}{-1} = 3$$

$B(t) = (1-t)(1-2t)(1-3t)$

$B'(t) = (-1)(1-2t)(1-3t) + -2(1-t)(1-3t)$

$-3(1-t)(1-2t)$