A VALUE FOR n-PERSON GAMES

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1. Introduction.

At the foundation of the theory of games is the assumption that the players in a game can evaluate, in their utility scales, every "prospect" that might arise as a result of a play. In attempting to apply the theory to any field, one would normally expect to be permitted to include, in the class of "prospects", the prospect of having to play a game. The possibility of evaluating games is therefore of critical importance. So long as the theory is unable to assign values to the games typically found in application, only relatively simple situations - where games do not depend on other games - will be susceptible to analysis and solution.

In the finite theory of von Neumann and Morgenstern[1] difficulty in evaluation persists for the "essential" games, and for only those. In this note we deduce a value for the "essential" case and examine a number of its elementary properties. We proceed from a set of three axioms, having simple intuitive interpretations, which suffice to determine the value uniquely.

Our present work, though mathematically self-contained, is founded conceptually on the von Neumann-Morgenstern theory as far as their introduction of characteristic functions. We thereby inherit certain important underlying assumptions: (a) that utility is objective and transferable; (b) that games are cooperative affairs; (c) that games, granting (a) and (b), are adequately represented by their characteristic functions. However, we are not committed to the assumptions regarding rational behavior embodied in the von Neumann-Morgenstern notion of "solution".

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[1] Reference [1] at the end of this paper. Examples of infinite games without values may be found in [2], pages 58-9, and in [3], page 110. See also Karlin [2], pages 152-3.

We shall think of a "game" as a set of rules with specified players in the playing positions. The rules alone describe what we shall call an "abstract game". Abstract games are played by roles - such as "dealer", or "visiting team" - rather than players external to the game. The theory of games deals mainly with abstract games[1]. The distinction will be useful in enabling us to state in a precise way that the value of a "game" depends only on its abstract properties. (Axiom 1 below).

2. Definitions.

Let U denote the universe of players, and define a game to be any superadditive set-function v from the subsets of U to the real numbers, thus:

$$(1) \qquad v(\emptyset) = 0 ,$$

$$(2) \qquad v(S) \geq v(S \cap T) + v(S - T) \qquad (\text{all } S, T \subseteq U) .$$

A carrier of v is any set $N \subseteq U$ with

$$(3) \qquad v(S) = v(N \cap S) \qquad (\text{all } S \subseteq U) .$$

Any superset of a carrier of v is again a carrier of v. The use of carriers obviates the usual classification of games according to the number of players. The players outside any carrier have no direct influence on the play since they contribute nothing to any coalition. We shall restrict our attention to games which possess finite carriers.

The sum ("superposition") of two games is again a game. Intuitively it is the game obtained when two games, with independent rules but possibly overlapping sets of players, are regarded as one. If the games happen to posses disjunct carriers,

[1] An exception is found in the matter of symmetrization (see for example [2], pages 81-3), in which the players must be distinguished from their roles.

then their sum is their "composition".[1]

Let $\Pi(U)$ denote the set of permutations of U - that is, the one to one mappings of U onto itself. If $\pi \in \Pi(U)$, then, writing πS for the image of S under π, we may define the function πv by

$$\pi v(\pi S) = v(S) \qquad (\text{all } S \subseteq U) .$$

If v is a game, then the class of games πv, $\pi \in \Pi(U)$, may be regarded as the "abstract game" corresponding to v. Unlike composition, the operation of addition of games can not be extended to abstract games.

By the <u>value</u> $\phi[v]$ of the game v we shall mean a function which associates with each i in U a real number $\phi_i[v]$, and which satisfies the conditions of the following axioms. The value will thus provide an additive set-function (an <u>inessential game</u>) $\bar{v}$:

$$\bar{v}(S) = \sum_S \phi_i[v] \qquad (\text{all } S \subseteq U) , \tag{5}$$

to take the place of the superadditive function v.

AXIOM 1. For each π in $\Pi(U)$,

$$\phi_{\pi i}[\pi v] = \phi_i[v] \qquad (\text{all } i \in U) .$$

AXIOM 2. For each carrier N of v,

$$\sum_N \phi_i[v] = v(N) .$$

AXIOM 3. For any two games v and w,

$$\phi[v + w] = \phi[v] + \phi[w] .$$

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[1] See [1], §§ 26.7.2 and 41.3.

Comments. The first axiom ("symmetry") states that the value is essentially a property of the abstract game. The second axiom ("efficiency") states that the value represents a distribution of the full yield of the game. This excludes, for example, the evaluation $\phi_i[v] = v((i))$, in which each player pessimistically assumes that the rest will all cooperate and combine against him. The third axiom ("law of aggregation") states that when two independent games are combined, their values must be added player by player. This is a prime requisite for any evaluation scheme designed to be applied eventually to systems of interdependent games.

It is remarkable that no further conditions are required to determine the value uniquely.[1]

3. Determination of the value function.

LEMMA 1. If N is a finite carrier of v , then, for $i \notin N$,

$$\phi_i[v] = 0 .$$

Proof. Take $i \notin N$. Both N and $N \cup (i)$ are carriers of v ; and $v(N) = v(N \cup (i))$. Hence $\phi_i[v] = 0$ by Axiom 2, as was to be shown.

We first consider certain symmetric games. For any $R \subseteq U$, $R \neq \emptyset$ define v_R :

$$v_R(S) = \begin{cases} 1 & \text{if } S \supseteq R , \\ 0 & \text{if } S \not\supseteq R . \end{cases} \tag{6}$$

The function cv_R is a game, for any nonnegative c , and R is a carrier.

In what follows, we shall use r , s , n , ... for the numbers of elements in R , S , N , ... respectively.

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[1] Three further properties of the value which might suggest themselves as suitable axioms will be proved as Lemma 1 and Corollaries 1 and 3 below.

LEMMA 2. For $c \geq 0$, $0 < r < \infty$, we have

$$\phi_i[cv_R] = \begin{cases} c/r & \text{if } i \in R , \\ 0 & \text{if } i \notin R . \end{cases}$$

<u>Proof</u>. Take i and j in R , and choose $\pi \in \Pi(U)$ so that $\pi R = R$ and $\pi i = j$. Then we have $\pi v_R = v_R$, and hence, by Axiom 1,

$$\phi_j[cv_R] = \phi_i[cv_R] .$$

By Axiom 2,

$$c = cv_R(R) = \sum_{j \in R} \phi_j[cv_R] = r\phi_i[cv_R] ,$$

for any $i \in R$. This, with Lemma 1, completes the proof.

LEMMA 3.[1] Any game with finite carrier is a linear combination of symmetric games v_R :

(7) $$v = \sum_{\substack{R \subset N \\ R \neq \emptyset}} c_R(v) v_R ,$$

N being any finite carrier of v . The coefficients are independent of N , and are given by

(8) $$c_R(v) = \sum_{T \subseteq R} (-1)^{r-t} v(T) \qquad (0 < r < \infty) .$$

<u>Proof</u>. We must verify that

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[1] The use of this lemma was suggested by H. Rogers.

$$v(S) = \sum_{\substack{R \subset N \\ R \neq \emptyset}} c_R(v) v_R(S) \tag{9}$$

holds for all $S \subseteq U$, and for any finite carrier N of v. If $S \subseteq N$, then (9) reduces, by (6) and (8), to

$$v(S) = \sum_{R \subseteq S} \sum_{T \subseteq R} (-1)^{r-t} v(T)$$

$$= \sum_{T \subseteq S} \left[\sum_{r=t}^{s} (-1)^{r-t} \binom{s-t}{r-t} \right] v(T) .$$

The expression in brackets vanishes except for $s = t$, so we are left with the identity $v(S) = v(S)$. In general we have, by (3),

$$v(S) = v(N \cap S) = \sum_{R \subseteq N} c_R(v) v_R(N \cap S) = \sum_{R \subseteq N} c_R(v) v_R(S) .$$

This completes the proof.

Remark. It is easily shown that $c_R(v) = 0$ if R is not contained in every carrier of v.

An immediate corollary to Axiom 3 is that $\phi[v-w] = \phi[v] - \phi[w]$ if v, w, and $v - w$ are all games. We can therefore apply Lemma 2 to the representation of Lemma 3 and obtain the formula:

$$\phi_i[v] = \sum_{\substack{R \subseteq N \\ R \ni i}} c_R(v)/r \qquad (\text{all } i \in N) . \tag{10}$$

Inserting (8) and simplifying the result gives us

$$(11) \qquad \phi_i[v] = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{(s-1)!(n-s)!}{n!} v(S) - \sum_{\substack{S \subseteq N \\ S \not\ni i}} \frac{s!(n-s-1)!}{n!} v(S) \qquad (\text{all } i \in N) .$$

Introducing the quantities

$$(12) \qquad \gamma_n(s) = (s-1)!(n-s)!/n ,$$

we now assert:

THEOREM. A unique value function ϕ exists satisfying Axioms 1 - 3, for games with finite carriers; it is given by the formula

$$(13) \qquad \phi_i[v] = \sum_{S \subseteq N} \gamma_n(s) \left[v(S) - v(S-(i)) \right] \qquad (\text{all } i \in U) ,$$

where N is any finite carrier of v .

Proof. (13) follows from (11), (12), and Lemma 1. We note that (13), like (10), does not depend on the particular finite carrier N ; the ϕ of the theorem is therefore well defined. By its derivation it is clearly the only value function which could satisfy the axioms. That it does in fact satisfy the axioms is easily verified with the aid of Lemma 3.

4. Elementary properties of the value.

COROLLARY 1. We have

$$(14) \qquad \phi_i[v] \geq v((i)) \qquad (\text{all } i \in U) ,$$

with equality if and only if i is a dummy - i.e., if and only if

$$(15) \qquad v(S) = v(S - (i)) + v((i)) \qquad (\text{all } S \ni i) .$$

Proof. For any $i \in U$ we may take $N \ni i$ and obtain, by (2),

$$\phi_i[v] \geq \sum_{\substack{S \subseteq N \\ S \ni i}} \gamma_n(s) v((i)) ,$$

with equality if and only if (15), since none of the $\gamma_n(s)$ vanishes. The proof is completed by noting that

$$\sum_{\substack{S \subseteq N \\ S \ni i}} \gamma_n(s) = \sum_{s=1}^{n} \binom{n-1}{s-1} \gamma_n(s) = \sum_{s=1}^{n} \frac{1}{n} = 1 . \tag{16}$$

Only in this corollary have our results made use of the superadditive nature of the functions v.

COROLLARY 2. If v is decomposable - i.e., if games $w^{(1)}, w^{(2)}, \ldots, w^{(p)}$ having pairwise disjunct carriers $N^{(1)}, N^{(2)}, \ldots, N^{(p)}$ exist such that

$$v = \sum_{k=1}^{p} w^{(k)} ,$$

\- then, for each $K = 1, 2, \ldots, p$,

$$\phi_i[v] = \phi_i[w^{(k)}] \qquad (\text{all } i \in N^{(k)}) .$$

Proof. By Axiom 3.

COROLLARY 3. If v and w are strategically equivalent - i.e., if

$$w = cv + \bar{a} , \tag{17}$$

where c is a positive constant and $\bar{a}$ an additive set-function on U with

finite carrier[1] - then

$$\phi_i[w] = c\phi_i[v] + \bar{a}((i)) \qquad (\text{all} \ \ i \in U) \ .$$

Proof. By Axiom 3, Corollary 1 applied to the inessential game $\bar{a}$, and the fact that (13) is linear and homogeneous in v .

COROLLARY 4. If v is constant-sum - i.e., if

$$v(S) + v(U - S) = v(U) \qquad (\text{all} \ \ S \subseteq U) \ , \tag{18}$$

- then its value is given by the formula:

$$\phi_i[v] = 2 \sum_{\substack{S \_ N \\ S \ \ i}} {}_n(s)v(S) \ - v(N) \qquad (\text{all} \ \ i \quad N) \ , \tag{19}$$

where N is any finite carrier of v .

Proof. We have, for $i \quad N$,

$$\phi_i[v] = \sum_{\substack{S \subseteq N \\ S \ni i}} \gamma_n(s)v(S) - \sum_{\substack{T \subseteq N \\ T \not\ni i}} \gamma_n(t+1)v(T)$$

$$= \sum_{\substack{S \subseteq N \\ S \ni i}} \gamma_n(s)v(S) - \sum_{\substack{S \subseteq N \\ S \ni i}} \gamma_n(n-s+1)[v(N) - v(S)] \ .$$

But $\gamma_n(n-s+1) = \gamma_n(s)$; hence (18) follows with the aid of (16).

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[1] This is McKinsey's "S-equivalence" (see [2], page 120), wider than the "strategic equivalence" of von Neumann and Morgenstern ([1], §27.1).

5. Examples.

If N is a finite carrier of v, let A denote the set of n-vectors (α_i) satisfying

$$\left.\begin{aligned} &\textstyle\sum_N \alpha_i = v(N) , \\ &\alpha_i \geq v((i)) \end{aligned}\right. \qquad \text{(all } i \in N) .$$

If v is inessential A is a single-point; otherwise A is a regular simplex of dimension $n - 1$. The value of v may be regarded as a point ϕ in A, by Axiom 2 and Corollary 1. Denote the centroid of A by θ :

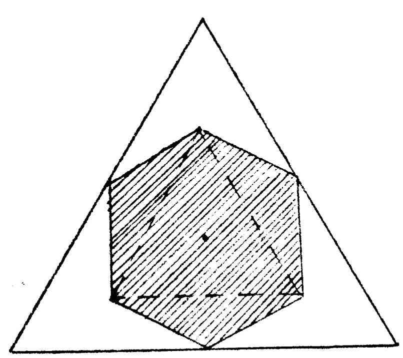
$$\theta_i = v((i)) + \frac{1}{n} \left[v(N) - \sum_{j \in N} v((j)) \right] .$$

Example 1. For two-person games, three-person constant-sum games, and inessential games, we have

$$\phi = \theta . \tag{20}$$

The same holds for arbitrary symmetric games - i.e., games which are invariant under a transitive group of permutations of N - and, most generally, games strategically equivalent to them. These results are demanded by symmetry, and do not depend on Axiom 3.

Example 2. For general three-person games the positions taken by ϕ in A cover a regular hexagon, touching the boundary at the midpoint of each 1-dimensional face (see figure). The latter cases are of course the decomposable games, with one player a dummy.



Example 3. The quota games[1] are characterized by the existence of constants ${}_i$ satisfying

$$\begin{cases} \omega_i + \omega_j = v((i,\ j)) \\ \sum_N \omega_i = v(N) \ . \end{cases} \qquad \text{(all } i,\ j \in N,\ i \neq j)$$

For $n = 3$, we have

$$\phi - \theta = \frac{\omega - \theta}{2} \ . \tag{21}$$

Since ω can assume any position in A the range of ϕ is a triangle, inscribed in the hexagon of the preceding example (see the figure).

Example 4. All four-person constant-sum games are quota games. For them we have

$$\phi - \theta = \frac{\omega - \theta}{3} \ . \tag{22}$$

The quota ω ranges over a certain cube[2], containing A . The value ϕ meanwhile ranges over a parallel, inscribed cube, touching the boundary of A at the midpoint of each 2-dimensional face. In higher quota games the points ϕ and ω are not so directly related.

Example 5. The weighted majority games[3] are characterized by the existence of "weights" w_i such that never $\sum_S w_i = \sum_{N-S} w_i$, and such that

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1 Discussed in [4].

2 Illustrated in [4], figure 1.

3 See [1], § 50.1.

$$\begin{cases} v(S) = n - s & \text{if} \quad \sum_S w_i > \sum_{N-S} w_i \,, \\ v(S) = -s & \text{if} \quad \sum_S w_i < \sum_{N-S} w_i \,. \end{cases}$$

The game is then denoted by the symbol $[w_1, w_2, \ldots, w_n]$. It is easily shown that

$$\phi_i < \phi_j \quad \text{implies} \quad w_i < w_j \qquad (\text{all} \quad i, j \in N) \tag{23}$$

in any weighted majority game $[w_1, w_2, \ldots, w_n]$. Hence "weight" and "value" rank the players in the same order.

The exact values can be computed without difficulty for particular cases. We have

$$\phi = \frac{n-3}{n-1} \; (-1, -1, \ldots, -1, n-1)$$

for the game $[1, 1, \ldots, 1, n-2]$[1], and

$$\phi = \frac{2}{5} \, (1, 1, 1, -1, -1, -1)$$

for the game $[2, 2, 2, 1, 1, 1]$[2], etc.

6. Derivation of the value from a bargaining model.

The deductive approach of the earlier sections has failed to suggest a bargaining procedure which would produce the value of the game as the (expected) outcome. We conclude this paper with a description of such a procedure. The form of our model,

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[1] Discussed at length in [1], §55.

[2] Discussed in [1], §53.2.2.

with its chance move, lends support to the view that the value is best regarded as an *a priori* assessment of the situation, based on either ignorance or disregard of the social organization of the players.

The players constituting a finite carrier N agree to play the game v in a grand coalition, formed in the following way: 1. Starting with a single member, the coalition adds one player at a time until everyone has been admitted. 2. The order in which the players are to join is determined by chance, with all arrangements equally probable. 3. Each player, on his admission, demands and is promised the amount which his adherence contributes to the value of the coalition (as determined by the function v). The grand coalition then plays the game "efficiently" so as to obtain the amount $v(N)$ - exactly enough to meet all the promises.

The expectations under this scheme are easily worked out. Let $T^{(i)}$ be the set of players preceding i . For any $S \ni i$ the payment to i if $S - (i) = T^{(i)}$ is $v(S) - v(S-(i))$, and the probability of that contingency is $\gamma_n(s)$. The total expectation of i is therefore just his value, (13), as was to be shown.

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