# The Condorcet Paradox

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## 1 Introduction

Consider an election between $n$ candidates with $v$ voters. Rather than a single vote, each voter casts a *ranking* of all $n$ candidates. The "Condorcet winner" of an election is the candidate which is higher ranked than any other voter when compared in a head-to-head contest. For example, if there are three candidates ($n = 3$) and three voters ($v = 3$), then the ballots

$$(1, 2, 3)$$
$$(3, 1, 2)$$
$$(2, 1, 3)$$

declare candidate 1 to be the Condorcet winner. Across all three ballots, candidate 1 is ranked above candidate 2 twice and the opposite situation only happens once, therefore candidate 1 "wins" this match-up. Similarly, candidate 1 beats candidate 3, so candidate 1 is the Condorcet winner of the election.

There can only be one Condorcet winner of an election—two potential winners would have to beat each other—but some elections have *no* Condorcet winner. For example, consider the ballots

$$(1, 2, 3)$$
$$(2, 3, 1)$$
$$(3, 1, 2)$$

In this case, 1 beats 2, 2 beats 3, and 3 beats 1. Every candidate loses to at least one other, so there's no Condorcet winner.

The *Condorcet paradox* is the observation that, in this head-to-head election method, majorities can be in disagreement with each other. In the above hypothetical election, a majority preferred candidate 1 to 2, a majority preferred candidate 2 to 3, and a majority preferred candidate 3 to 1.

The classical "game" is to determine how often the Condorcet paradox occurs. Given an election with $n$ candidates and $v$ voters, exactly how many possible ballots would *not* elect a Condorcet winner? This is computationally difficult to answer. With $n$ candidates and $v$ voters, there are $n!^v$ distinct collections of ballots. Going over all of these is nearly impossible for even moderately large values of $n$ and $v$. We can alleviate this if we declare voters to be indistinguishable, and instead go over all possible ways to assign $v$ votes to the $n!$ possible candidate rankings. This is equivalent to enumerating nonnegative compositions of $v$ into $n!$ parts, a set of size $\binom{v+n!}{n!} \approx v^{n!}/n!$. This is both much smaller than $n!^v$ and quite large. Naturally, much of the existing literature is dedicated to avoiding this brute-force search.

**Definition.** Let $C(n, v)$ be the number of collections of ballots with $n$ candidates and $v$ voters which do *not* elect a Condorcet winner. Let $P(n, v) = C(n, v)/m!^v$ be the probability that a collection of $v$ ballots for $n$ candidates chosen uniformly at random does not elect a Condorcet winner. Let $E(n, v)$ be the number of ballots with $n$ candidates and $v$ *indistinguishable* voters which do not elect a Condorcet winner.

The bulk of previous results have focused on finding exact formulas, recurrences, and asymptotic estimates for the functions $C$ and $P$. Useful overviews can be found in [3] and [1]. For asymptotics, we have the general result

$$P(n, \infty) = \frac{n!}{2^{n+1}} \sum_{k=0}^{(n-1)/2} \frac{1}{(n-1-2k)!4^k \theta^{\underline{k}}}$$

where $\theta = (4 \sin^{-1}(1/3))^{-1}$. For recurrences, the equation

$$P(4, v) = 2P(3, v) - 1$$

is known as *May's theorem*. More generally, there is a recurrence of the form

$$P(2n, v) = \sum_{0 \le k < n} c_k P(2k + 1, v)$$

where the $c_k$ are computed through an inclusion-exclusion argument.

There is a comparatively smaller literature on the function $E$. The sequence $E(3, 2k-1)$ is A277935 in the OEIS, and begins

$$0, 2, 12, 42, 112, 252, 504, 924, 1584, 2574, 4004, 6006, 8736, \ldots$$

Amazingly, it turns out that

$$E(3, 2k - 1) = 2\binom{k+3}{5}. \tag{1}$$

Using sledgehammers from the theory of diophantine equations, this is "routine" to prove, but in this paper we will provide an elegant bijective proof which extends to other cases. The sequences $E(n, 2k - 1)$ do not appear in the OEIS for $n > 3$, though the diophantine sledgehammers suggest that they *might* have simple representations.

In the following sections we will give our bijective proof of (1), give an overview of a Maple package to explore the Condorcet paradox, and provide some early thoughts on the sequence $E(4, 2k - 1)$.

## 2 Counting Condorcet scenarios with 3 candidates

We first present an efficient method for counting the number of Condorcet voting-profiles with 3 candidates and $2n - 1$ voters. This method was established in [2] and the relevant results are repeated here.

Denote the three candidates 1, 2, and 3. Every voter must decide on some complete ranking of the three candidates. Let $x_{123}$ denote the number of voters who choose the ranking $1,2,3$, and define $x_{132}$, $x_{213}$, $x_{231}$, $x_{312}$, and $x_{321}$ similarly, where $x_{123}+x_{132}+x_{213}+x_{231}+x_{312}+x_{321}=2n-1$. We call the resulting 6-tuple,

$$[x_{123},x_{132},x_{213},x_{231},x_{312},x_{321}],$$

a vote-count profile with 3 candidates, $2n-1$ voters.

For a vote-count profile to be Condorcet, it must admit a cycle: 1 beats 2, 2 beats 3, 3 beats 1 or vice-versa, 1 beats 3, 3 beats 2, 2 beats 1.

Given a vote-count profile, this profile admits the cycle 1 beats 2, 2 beats 3, 3 beats 1 if and only if it satisfies the following three inequalities:

$$x_{123}+x_{132}+x_{312} > x_{213}+x_{231}+x_{321} \quad \text{(1 beats 2)}$$

$$x_{123}+x_{213}+x_{231} > x_{132}+x_{312}+x_{321} \quad \text{(2 beats 3)}$$

$$x_{231}+x_{321}+x_{312} > x_{213}+x_{123}+x_{132} \quad \text{(3 beats 1)}$$

We then establish a bijection between compositions of $n-2$ into 6 non-negative parts and Condorcet vote-count profiles admitting a 1 beats 2, 2 beats 3, 3 beats 1 cycle. Let $[x_1,x_2,x_3,x_4,x_5,x_6]$ be a composition of $n-2$ into 6 non-negative parts.

We can define an affine-linear mapping from compositions of $n-2$ into 6 non-negative parts to Condorcet vote-count profiles with 3 candidates, $2n-1$ voters as follows:

$$[x_1,x_2,x_3,x_4,x_5,x_6] \mapsto [x_1+x_4+x_6+1, x_2, x_3, x_2+x_4+x_5+1, x_3+x_5+x_6+1, x_1]$$

It can be verified that the sum of the entries in this vote-count profile is $2n-1$, and that they satisfy the three given inequalities.

The inverse of the mapping is given by:

$$[x_{123},x_{132},x_{213},x_{231},x_{312},x_{321}] \mapsto [x_1,x_2,x_3,x_4,x_5,x_6]$$

Where,

$$x_1=x_{321}, \quad x_2=x_{132}, \quad x_3=x_{213},$$

$$x_4=\frac{(x_{123}+x_{213}+x_{231})-(x_{132}+x_{312}+x_{321})-1}{2}$$

$$x_5=\frac{(x_{231}+x_{321}+x_{312})-(x_{213}+x_{123}+x_{132})-1}{2}$$

$$x_6=\frac{(x_{312}+x_{132}+x_{123})-(x_{321}+x_{231}+x_{213})-1}{2}$$

It can be checked that $x_1+x_2+x_3+x_4+x_5+x_6=n-2$, and the three inequalities imply that each of $x_4,x_5,x_6$ are non-negative.

This bijection gives rise to the following theorem.

**Theorem 1.** *The number of Condorcet vote-count profiles with three candidates and $2n-1$ voters is*

$$2\binom{n+3}{5}.$$

A voting-profile is defined to be a mapping of $\{1,\ldots,2n-1\}$ into $S_3$, i.e., each voter chooses a complete ranking of the three candidates. Our bijection above then gives rise to the following theorem.

**Theorem 2.** *The number of Condorcet voting-profiles with three candidates and $2n-1$ voters is given by the following 5-fold sum:*

$$2\sum_{i1=0}^{n-2}\sum_{i2=0}^{n-2-i1-i2}\sum_{i3=0}^{n-2-i_1-i_2}\sum_{i_4=0}^{n-2-i_1-i_2-i_3}\sum_{i_5=0}^{n-2-i_1-i_2-i_3-i_4}$$

$$\frac{(2n-1)!}{(n-1-i_2-i_3-i_5)!i_2!i_3!(i_2+i_4+i_5+1)!(n-1-i_1-i_2-i_4)!i_1!}.$$

[Should have a note about the recurrence relation here, but I don't fully understand the math behind it so not sure how to type it up.]

## 3 $E(4,2k-1)$

As mentioned in the introduction and proven in the previous section,

$$E(3,2k-1)=2\binom{k+3}{5}.$$

The theory of diophantine equations tells us that $E(n,k)$, as a sequence of $k$, will always be a quasipolynomial. That is, there exists some $m$ such that $j\mapsto E(n,mj+i)$ is a polynomial in $j$ for each $i\in\{0,1,\ldots,m-1\}$. In the case of $n=3$, it turns out $m=2$ and $E(3,2k-1)$ is the *genuine polynomial* $2\binom{k+3}{5}$. It remains open to determine whether $E(4,2k-1)$ is a polynomial, a quasipolynomial, or even to give a hypergeometric representation of the sequence.

The sequence $E(4,2k-1)$ begins as follows:

$$0, 12480, 4081200, 351006480, 13752612000, 315501790560, \ldots$$

This does not appear in the OEIS. The sequence $E(4,k)$ begins as follows:

$$0, 256, 12480, 283200, 4081200, 42731088, 351006480, 2377644912, 13752612000, \ldots$$

The difference is that the latter includes even as well as odd numbers of voters. Based on computer experiments, we suspect that either $E(4,k)$ or $E(4,2k-1)$ are themselves polynomials, but we have been unable to prove this so far. We have computer $E(4,k)$ up to $k=28$, but we need more terms to be confident in what Maple is giving us. In particular, Maple can produce an explicit polynomial of degree 23 that it thinks fits $E(4,k)$ up to $k=28$. This is appealing, since the polynomial for three candidates had degree $5=3!-1$ and $23=4!-1$, but the polynomial contains a lot of nonsense. We need more terms to be sure of anything.

# 4 Maple Package

```
with(combinat):

Tour := proc(P)
    local n, i, j, T, ballot, x, y, sgn:
    n := nops(P[1]):
    for i from 1 to n do
    for j from 1 to n do
        T[i, j] := 0:
    od:
    od:

    for ballot in P do
        for i from 1 to nops(ballot) do
            for j from i + 1 to nops(ballot) do
                x := ballot[i]:
                y := ballot[j]:
                T[x, y] := T[x, y] + 1:
                T[y, x] := T[y, x] - 1:
            od:
        od:
    od:

    sgn := x -> ifelse(x = 0, 0, sign(x)):

    [seq([seq(sgn(T[i,j]),j=1..n)],i=1..n)]:
end:

IsCondorcet := proc(P) local i:
    local T:
    T := Tour(P):
    not TourWinner(T):
end:

TourWinner := proc(T)
    local size, row:
    size := nops(T[1]):
    for row in T do
        if add(row) = size - 1 then
            return true:
        fi:
    od:

    false:
end:
```

```
ExactCond := proc(n, v)
    option remember:
    local S, count, d, comp, R, i:
    S := permute(n):

    count := 0:
    for d in Iterator[Composition](v + n!, parts=n!) do
        d := convert(d, list) - [1 $ n!]:
        R := [seq(S[i] $ d[i], i=1..n!)]:
        if IsCondorcet(R) then
            count := count + 1:
        fi:
    od:

    count:
end:

CompToCond:=proc(c):
([1,0,0,1,1,0]+
[c[1]+c[4]+c[6],c[2],c[3],c[2]+c[4]+c[5],c[3]+c[5]+c[6],c[1]]):
end:

NuVC4 := proc(n) local C,Cnew,c:
    option remember:
    total := 0:
    for c in Iterator[Composition](n - 2 + 6, parts=6) do
        c := CompToCond(convert(c, list) - [1 $ 6]):
        total := total + 4*(binomial(c[1]+3,3)*binomial(c[2]+3,3)*
        binomial(c[3]+3,3)*binomial(c[4]+3,3)*binomial(c[5]+3,3)*
        binomial(c[6]+3,3)):
    od:

    total:
end:

#NuCo(N): The first N terms of the sequence
# "number of Condorcet vote-profiles" with 2v-1
# voters and three candidates.
NuCo:=proc(N) local L,n,kha:
    L:=[0,12,540]:
    if N<=3 then
        RETURN(L[N]):
    fi:

    for n from 4 to N do
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```
        kha := (4*(19*n^2-57*n+45)/(n-1)^2*L[-1]-36*(2*n-3)*
        (22*n^2-99*n+111)/(n-2)/(n-1)^2*L[-2]
        +1296*(n-3)*(2*n-3)*(2*n-5)/(n-2)/(n-1)^2*L[-3]):
        L:=[L[2],L[3],kha]:
    od:

    L[-1]:
end:

#NuCo4: given an integer n, outputs the number of Condorcet
# voting profiles with 4 candidates, 2*n-1 voters using May's theorem
NuCo4:=proc(n)
    2 * 4^(2*n-1) * NuCo(n):
end:
```

## References


[1] Steven J Brams and Peter C Fishburn. *Handbook of Social Choice and Welfare*. Elsevier, 2002.

[2] Rebecca Embar and Doron Zeilberger. Counting condorcet. 2022.

[3] William V Gehrlein. Condorcet's paradox. *Theory and Decision*, 15(2):161–197, 1983.