Banzhaf and Shapely-Shubik Power Indices

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This project is a follow up to the paper "On Banzhaf and Shapley-Shubik Fixed Points and Divisor Voting Systems" (`https://arxiv.org/pdf/2010.08672.pdf`) by Alex Arnell, Richard Chen, Evelyn Choi, Miroslav Marinov, Nastia Polina, and Aaryan Prakash.

Here, we are concerned with voting systems where each party can hold an arbitrary amount of votes. Parties must seek to group up with each other, and form coalitions, to combine their voting power and win the election. The threshold amount of voting power required is called the quota, and a winning coalition is any coalition that has combined voting power that strictly exceeds the quota. In a winning coalition, a player is called a critical player, if the coalition would not be winning without that player's votes.

The Banzhaf and Shapely-Shubik power indices are two ways of describing a player's strength in the election. Direct quoting the paper:

"The Banzhaf power index of a player is the number of times that player is a critical player in all winning coalitions divided by the number of total times any player is a critical player. The Shapley-Shubik index looks at permutations of all players in a system, called sequential coalitions. We sum each player's votes starting from the beginning of a sequential coalition, and see if the sum reaches the quota as we progress. The player whose votes first cause this sum to meet or exceed the quota is called a pivotal player. The Shapley-Shubik power index of a player is the number of times that player is a pivotal player divided by the total number sequential coalitions."

The paper was divided into 2 main sections. The first dealt with divisor games. For a fixed n, the divisor game for n has a player with voting power equal to d for each divisor d of n. The quota is equal to $\sigma(n)$, where $\sigma(n)/2$ is the sum of the divisors of n. The second section dealt with games with any number of players so long as the voting powers added to 1. These games had quota equal to $1/2$. The paper looked for fixed points, games where each player's power index was equal to their voting power.

1 Divisor Games

The first thing we did for this section was implement a clever way of calculating the Banzhaf Power Index for a given system. This follows the same generating function idea that Dr. Z. implemented in class for the Shapley-Shubik power indices, which we also included in the code, of course. We also implemented a quick function that outputs whether or not two lists are equal and at what indices the values differ. In the PROMYS paper, they saw that for integers n where $\sigma(n) = 2n + i$ for $i = 0, 1, \ldots, 5$, the SS and BB indices differ for some divisor, but not a uniform one. For three of the cases they saw that the power indices for the divisor 1 differed, but this will in general is likely not possible, indeed not even for small divisors. Experimentally, we found that there are many integers which equal SS and BB power indices for small divisors. These can be found by running the code, enumerated, those in the first 2000 integers were 18,100,162,196,738,748,846,954,968, 1062, 1098, 1206, 1278, 1314, 1352, 1422, 1458, 1494, 1602, 1746, 1818, 1854, 1926, 1962. Towards, the end, all of these have 12 divisors. Indeed for most of these only the 5 largest divisors differ in the power indices - but this is also because the smaller divisors actually have no power in either setup. It is not infeasible to look beyond $n = 2000$, we managed to calculate in sporadic intervals up to $n = 10000$, and further is possible. It seems that, if the conjecture is true, that if trying to prove it directly, one might want to look at the largest divisor's power indices, generally it seems those will be different. Unfortunately, it also seems a direct approach might be difficult, because counting winning coalitions including n is highly dependent on the factorization of n. By checking the differences in the power indices of n compared to n, it is not clear that there is a uniform lower bound on this difference.

We were also able to check for the first 719 integers that if the number was abundant, the SS and BB values differed, and if not, they were the same. The function BBvc takes very long to compute for highly abundant numbers like 240,360,480,540,600 so it is impractical to continue these calculations beyond how far we did it, without some algorithmic improvement.

2 Fixed Points

The paper looked for fixed points where all the players except for 2 had the same voting power. We deviated from this special case and opted to exhaustively search for fixed points when the number of players was small. The approach was to notice that the exact values of voting power does not influence the power indices. It only matters which subsets of voters can surpass the threshold (1/2). Once we fix which subsets are successful, we can compute the power indices, and then check if the indices satisfy the criteria specified by the subsets. This reduces the solution space from $[0, 1]^n$ to the set of subsets of the set of subsets of n, which can be reduced further by symmetry. Our code was able to completely enumerate the fixed points for up to 5 players. Here is the results:

We note that a primitive fixed point is one in which none of the players have 0 voting power. We only consider primitive fixed points, and note that the non-primitive ones must contain a primitive one on a subset of the players. For $n = 1, 2, 3$, the only fixed points are the trivial solutions

$$n = 1 : (1) \qquad n = 2 : (\frac{1}{2}, \frac{1}{2}) \qquad n = 3 : (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

For $n = 4$, in addition to the trivial $(1/4, 1/4, 1/4, 1/4)$, we found two nontrivial fixed points, that each worked for both the Banzhaf index and the Shapely-Shubik index at once!

$$n = 4 : \qquad (\frac{5}{12}, \frac{1}{4}, \frac{1}{4}, \frac{1}{12}) \qquad (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$$

For $n = 5$, in addition to the trivial $(1/5, 1/5, 1/5, 1/5, 1/5)$ the Banzhaf and Shapely-Shubik fixed points were different.

Shapely-Shubik:

$$n = 5 : \quad (\frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}) \qquad (\frac{3}{10}, \frac{3}{10}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}) \qquad (\frac{2}{5}, \frac{7}{30}, \frac{7}{30}, \frac{1}{15}, \frac{1}{15})$$

Banzhaf:

$$n = 5 : \quad (\frac{2}{7}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}) \qquad (\frac{2}{7}, \frac{3}{14}, \frac{3}{14}, \frac{1}{7}, \frac{1}{7}) \qquad (\frac{5}{13}, \frac{3}{13}, \frac{3}{13}, \frac{1}{13}, \frac{1}{13})$$

In the attached maple code, you can try it for yourself. The function fixed_points(n) will perform the exhaustive search. In the future we hope to optimize the code a little more so that it will work for $n = 6$, and also to neaten up the maple package.