Analysis of Ramanujan Tau Function using J.C.P. Miller recurrence

Yuxuan Yang Doron Zeilberger Zidong Zhang

May 9, 2021

Abstract

Abstract: One of the famous open problems in number theory is Lehmer's Conjecture. It wants to prove that the Ramanujan's tau-function never vanishes. By adding Euler's pentagonal numbers theorem and J.C.P Miller recurrence as two new ingredients, we investigate several generalized problems with our experimental mathematics approach.

1 Introduction

Ramanujan introduced $\tau(n)$ in his famous paper "On certain arithmetical functions"

$$\sum_{n=1}^{\infty} \tau(n) q^n := q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots \tag{1.1}$$

Surprisingly, Lehmer's Conjecture[2] that $\tau(n)$ is never zero remains open. This has been verified for n less than 816212624008487344127999 by computing Galois representations and equations for modular curves, [1]. In this paper, we will investigate generalized Lehmer conjecture experimentally

This article is accompanied by a Maple package, `Lehmerconj.txt` , which can be found on the front of the article

`https://sites.math.rutgers.edu/ zeilberg/EM21/projs.html`.

Readers are encouraged to download the package and use the procedures to experiment.

2 Basic Algorithm

Since our algorithm does not rely on the degree number 24 in Ramanujan function (1.1), we can work on a larger family of functions

$$\sum_{n=1}^{\infty} \tau_r(n) q^n := q \prod_{n=1}^{\infty} (1-q^n)^r \tag{2.1}$$

In this section, we will show how we make the algorithm work to find the the first place where the coefficient $\tau_r(n)$ is zero looking at the first N coefficients of $q \cdot \eta(q)^r$ using the J.C.P. Miller recurrence, where $\eta(q) = \prod\limits_{n\geq 1}(1-q^n)$. Since by Euler Pentagonal number theorem

$$\eta(q) = \prod_{n\geq 1}(1-q^n) = 1 + \sum_{k\geq 1}(-1)^k (x^{k(3k+1)/2} + x^{k(3k-1)/2}) \tag{2.2}$$

The J.C.P Miller recurrence[5] is actually one way to compute the power of a polynomial. Given a polynomial

$$P(x) = \sum_{i=1}^{L} p_i x^i, \tag{2.3}$$

we can find the coefficients of its m-th power by recurrence.

$$P(x)^m = \sum_{i=1}^{mL} a(m,k)x^k, a(m,k) = \frac{1}{kp_0} \sum_{i=1}^{L} p_i[(m+1)i-k]a(m,k-i). \tag{2.4}$$

By applying the J.C.P Miller recurrence to $\eta(q)^r$, we can get $\tau_r(n)$. This is a very useful since it will allow us to calculate faster than just using Taylor expansion for the formula to find the coefficients. Also, it will allow us to calculate for any r-th power of $\eta(q)$, for example, if $r = 24$, we get the famous Ramanujan Tau Function. We can use this algorithm to see for various r how the coefficients act. Does it never vanish? We can compute first N coefficients of the $q \cdot \eta(q)^r$ by this algorithm.

Using the procedure `FZ(r,N)` in the Maple package, we examine the for $r \in [1, 100]$. Due to our performance of our computers, we examine the first 2 millions coefficients for $r \in [1, 40]$. The rest r, we only check for the first one million coefficients. And some of them vanish at some place.

r	1	2	3	4	5	6	7	8	10	14	15	26
n	4	8	3	10	1561	6	28018	4	7	5	54	10

Table 1: List of pairs (r,n) verifying $\tau_r(n) = 0$,$r \leq 40, n \leq 2 * 10^6$

Also, after $r > 26$, the coefficients won't vanish at least in their first 1 million terms. So we can conjecture that when $r > 26$, the coefficients of the function never vanish.

3 Congruence properties

With the genes of number theory, the tau function has many well-known results about the congruence. As a generalization of Lehmer's Conjecture, the question whether the equation $\tau(p) \equiv 0(mod\ p)$ has infinitely many solutions remains open. Also, we can ask what about $\tau_r(p)$?

3.1 Congruence of $\tau(p)$

To investigate congruence properties of original Ramanujan tau function, we can simplify our algorithm in the last section. This simplification comes from some well-known arithmetical properties proved by Mordell [4]:

$$\begin{aligned} &\tau(nm) = \tau(n)\tau(m) && \textit{for n,m relatively prime integers;} \\ &\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1}) && \textit{for p prime and r an integer} \geq \textit{1.} \end{aligned} \tag{3.1}$$

It turns out that the value of $\tau(n)$ for an integer n can be easily derived from the values $\tau(p)$ for all prime divisors p of n. Then we can use J.C.P. Miller recurrence for prime numbers only and use Mordell properties for the rest. Then, to compute $\tau(p)$ up to $p < N$, the time-complexity is roughly

$$\sum_{p\in\Pi, p\leq N} \sqrt{p} \sim \int_e^N \frac{\sqrt{t}dt}{\ln(t)}, \tag{3.2}$$

which is slightly smaller than $N^{3/2}$. The space-complexity is N, since we need the whole list of $\tau(p)$. The Maple code for this algorithm are given as `taup(N)` and `taupcheck(N,k)`.

The complexity here is similar to the work of Nik Lygeros and Olivier Rozier[3]. They discovered some new solutions to $\tau(p) \equiv q(mod\ p)$ for $|q| \leq 100$ and prime $p < 10^{10}$. The main result is the discovery of a new prime $p = 7758337633$ such that $\tau(p)$ is divisible by p. With similar complexity and different approach, we can use our algorithm to check their results.

3.2 Congruence of $\tau_r(p)$

If $r \neq 24$, there is no multiplicative property for $\tau_r(n)$, We will fully use J.C,P Miller recurrence again. To compute all $\tau_r(n)$ for n from 1 to N, The time- and space- complexity are $N^{3/2}$ and N respectively. Then, we can use this algorithm, the procedure `tauF(r,k)` in the Maple package, to investigate congruence of $\tau_r(p)$.

If we assume that the values $\tau_r(p)$ are randomly distributed modulo p for all prime numbers p, then we can evaluate the number of p less than n such that $\tau_r(p) \equiv 0(mod\ p)$:

$$\sum_{p<N, p\in\Pi} \frac{1}{p} \sim \int_e^N \frac{dt}{t\log t} = \log\log N \tag{3.3}$$

This is the log log philosophy[3]. A natural question is whether the number of prime $p \leq N$ such that $\tau_r(p) \equiv 0(mod\ p)$ is about $\log\log N$. We can use the procedure `pmodseqzero(r,N)` in the Maple package. According to log log philosophy, the number of prime $p \leq 10^5$ such that $\tau_r(p) \equiv 0(mod\ p)$ should be approximately

$$\log\log 10^5 \sim 2.44 \tag{3.4}$$

r	21	22	23	24	25	26	27	28	29	30
k	4	4	3	5	3	2927	3	5	6	6

Table 2: the number k of prime $p \le 10^5$ such that $\tau_r(p) \equiv 0 (mod\ p)$

A general question is here. For which r, does the following statement hold?

$$\sum_{p \in \Pi, p \le N, \tau_r(p) \equiv 0 (mod\ p)} 1 \sim \log\log N \tag{3.5}$$

According to the experiment above, for $r \in [21, 30]$, everyone except 26 is a good candidate. For $r = 26$, the reason is $\tau_{26}(n) = 0$ for a large family of n. See the sequence A322433 in OEIS.

Acknowledgment

We would like to thank Alison J Bu, Victoria Chayes, Robert Dougherty-Bliss, Mike Murr, Blair Seidler and George Spahn for inspiring conversations and useful remarks.

References

[1] Maarten Derickx, Mark van Hoeij, and Jinxiang Zeng. Computing galois representations and equations for modular curves xh (l). *arXiv preprint arXiv:1312.6819*, 2013.

[2] Derrick H Lehmer. The vanishing of ramanujan's function τ(n). *Duke Mathematical Journal*, 14(2):429–433, 1947.

[3] Nik Lygeros and Olivier Rozier. A new solution to the equation τ(p)≡0(modp). *Journal of Integer Sequences [electronic only]*, 13, 01 2010.

[4] Louis Joel Mordell and On Mr. On mr ramanujan's empirical expansions of modular functions. In *Proc. Cambridge Philos. Soc*, volume 19, pages 117–124, 1917.

[5] Doron Zeilberger. The j.c.p. miller recurrence for exponentiating a polynomial, and its q- analog. *Journal of Difference Equations and Applications*, 1(1):57–60, 1995.

Yuxuan Yang, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
Email: `yy458 at math dot rutgers dot edu` .

Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
Email: `DoronZeil at gmail dot com` .

Zidong Zhang, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
Email: `zz430 at math dot rutgers dot edu`