A Moment a Part

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May 2021

ROADMAP We have a lot of stuff here. We can automatically find generating functions for $p(n, k, m, C, DI)$ and we have some preliminary results on the moments and moments about the mean of $L(P)$, where L is the number of parts of a partition and P is a random partition. We would like to "glue" these together to automatically guess moments of restricted partitions, but this is ill-defined. We'll sort it out more directly soon.

1 Introduction

A *partition* of an integer n is a non-increasing sequence positive integers whose sum is n. The entries of the sequence are called the *parts* of the partition. For example, the three partitions of 3 are

$$3, \quad 2+1, \quad 1+1+1.$$

Some of the most difficult questions about partitions concern *partition statistics*. How many parts, on average, will a partition of n have? How large, on average, will the largest part of a partition be? What about the second largest part? What are the *standard deviations* of these quantities? As n grows, do these quantities have well-known distributions? These questions amount to asking about the behavior of the random variable $f(P_n)$, where f is some function and P_n is a partition sampled uniformly at random from some interesting collection of partitions of n.

The first object of study for partition statistics are *moments*. The mth moment of $f(P_n)$ is

$$E[f(P_n)^m] = \frac{1}{N} \sum_p f(p)^m,$$

where p ranges over all partitions of interest, and N is the number of partitions of interest. For example, $\mu = E[f(P_n)]$ is the *mean* of $f(P_n)$ and tells us how large $f(P_n)$ is, on average. The mth *moment about the mean* of $f(P_n)$ is

$$E[(f(P_n) - \mu)^m] = \frac{1}{N} \sum_p (f(p) - \mu)^m,$$

Moments are thus composed of two parts: The weighted sums "on top"

$$\sum_{p} f(p)^m \quad ; \quad \sum_{p} (f(p) - \mu)^m,$$

and the number of partitions N.

In the "classical" case, we sample P_n from *all* partitions of n. This gives $N = p(n)$ where p is the partition function. This is highly technical because the generating function for $p(n)$ is an analytically-complicated *infinite* product:

$$\sum_{n \geqslant 0} p(n)t^n = \prod_{n \geqslant 1} \frac{1}{1 - t^n}.$$

We are interested in statistics that we can explore *automatically and experimentally.* Generating functions expressed only as infinite products do not qualify. The most direct remedy to this is to limit the *largest part* of a partition. The generating function for number of partitions with largest part not exceeding K is

$$\prod_{n=1}^{K} \frac{1}{1 - t^n}.$$

This turns our infinite product into a finite object that we can automatically analyze.

Our project attempts to study moments automatically with computer algebra using this idea. Limit the number of parts, then look at interesting restrictions. The restrictions that we can handle *automatically* are as follows: Given positive integers k and m, and sets C and DI, let $\mathcal{P}_{k,m,C,DI}$ be the set of integer partitions where

- the largest part is k,
- all parts are congruent to elements of the set C mod m, and
- the difference between any two terms is not in the set DI.

2 Enumerating restricted partitions

The generating function of the sequence enumerating partitions of n in $\mathcal{P}_{k,m,C,DI}$ is

$$P(x) = \sum_{n=0}^{\infty} p(n, k, m, C, DI)x^n,$$

where $p(n, k, m, C, DI)$ is the number of partitions of n in our set. We can find $P(x)$ by looking at the "children" of the partitions, from which we form and solve a system of algebraic equations.

If we remove one part k—which has weight x^k—from any nonempty partition in $\mathcal{P}_{k,m,C,DI}$, what remains is either an empty partition or a partition in

$\mathcal{P}_{k_1,m,C,DI}$ for some $k_1 \in \{1,\ldots,k\}$ such that $k_1 \bmod m \in C$ and $k - k_1 \notin DI$. Thus, if S denotes the set of such k_1, then

$$\mathcal{P}_{k,m,C,DI} = x^k + x^k \sum_{k_1 \in S} \mathcal{P}_{k_1,m,C,DI}$$

and the "children" of our original set are $\mathcal{P}_{k_1,m,C,DI}$, for $k_1 \in S$. We apply this procedure for each of our children sets, and each of their children sets, and so on. Note that we will eventually remove all possible choices for k_1 and will therefore have finitely many "descendants". Moreover, since we have an equation to find the children for each "descendant" (i.e. variable), we have as many equations as variables. Moreover, the variables in each equation have degree one, and the last equation only has one variable. Given the way that we are generating our system of equations, we can eliminate every variable except the one representing our original family $\mathcal{P}_{k,m,C,DI}$. This gives us the polynomial equation satisfied by the generating function of $\mathcal{P}_{k,m,C,DI}$, and we can solve this equation to get our generating function.

This procedure is implemented in the Maple package `IntParts.txt` by the procedure `f_PnkRest(k,m,C,DI,x)`. For example, say we want the generating function of the sequence $a_n{}_{n=0}^{\infty}$, where a_n is the number of integer partitions of n with largest part 4 and only even parts. Then, running

`f_PnkRest(4,2,{0},{},x),`

outputs the generating function

$$x^4/(x^6 - x^4 - x^2 + 1).$$

3 Number of parts in partitions with restricted largest part

Given a partition p, let $L(p)$ denote the number of parts of p. We are interested in the behavior of the random variable $L(P_{nk})$ where P_{nk} is sampled from partitions of n with largest part exactly k. For instance, the average number of parts over such partitions is exactly

$$E[L(P_{nk})] = \frac{1}{p(n,k)} \sum_{p \in P(n,k)} L(p)),$$

where $P(n,k)$ is the set of partitions of n with largest part k, and $p(n,k)$ is the *number* of such partitions. It makes sense to study the sum on its own terms and introduce the normalizing factor $p(n,k)$ later. To that end, let

$$S(n,k) = \sum_{p \in P(n,k)} L(p).$$

The cases $k = 1$ and $k = 2$ are delightful:

$$S(n, 1) = n$$
$$S(n, 2) = \frac{3n^2}{8} + \frac{(-1)^n - 3}{8} n - \frac{(-1)^n - 1}{16}.$$

The sequence $S(n, 2)$ has a more striking representation in terms of the *pentagonal numbers* q_n:

$$S(n, 2) = q_{(-1)^n \lfloor n/2 \rfloor}; \qquad q_n = \frac{n(3n-1)}{2}.$$

Proposition 1.

$$S(n, 2) = \begin{cases} \frac{k(3k-1)}{2}, & \text{if } n = 2k \\ \frac{k(3k+1)}{2}, & \text{if } n = 2k+1 \end{cases}$$

Proof. Note there is an obvious one-to-one correspondence between partitions of n with largest part 2, and pairs $[a_1, a_2]$ such that $a_2 \geqslant 1$ and

$$a_1 + 2a_2 = n.$$

The partition that corresponds to the pair $[a_1, a_2]$ has $a_1 + a_2 = n - a_2$ parts.

When $n = 2k$, clearly $1 \leqslant a_2 \leqslant k$. Thus, the total number of parts in all partitions of n with largest part 2 is

$$\begin{aligned} S(n, 2) &= \sum_{i=1}^{k} n - i \\ &= 2k^2 - \frac{k(k+1)}{2} \\ &= \frac{k(3k-1)}{2} \end{aligned}$$

Similarly, when $n = 2k + 1$, we have

$$\begin{aligned} S(n, 2) &= \sum_{i=1}^{k} n - i \\ &= k(2k+1) - \frac{k(k+1)}{2} \\ &= \frac{k(3k+1)}{2} \end{aligned}$$

□

Corollary 1. *The average number of parts in a partition of* n *with largest part 2 is*

$$E(P_{n,2}) = \begin{cases} \frac{3k-1}{2}, & \text{if } n = 2k \\ \frac{3k+1}{2}, & \text{if } n = 2k+1 \end{cases}$$

Proof. Since a partition of n with largest part 2 is defined by how many parts are equal to 2, there are clearly k such partitions. Thus, the follows directly from the previous proposition. $\square$

Corollary 2. *The total number of parts in a partition with largest part at most 2 is*

$$Z(n,2) = \begin{cases} \frac{3k(k+1)}{2}, & \text{if } n = 2k, \\ \frac{k(3k+5)}{2}, & \text{if } n = 2k+1. \end{cases}$$

Corollary 3. *The average number of parts in a partition with largest part at most 2 is*

$$A_{\leqslant}(n,2) = \begin{cases} \frac{3k}{2}, & \text{if } n = 2k, \\ \frac{k(3k+5)}{2(k+1)}, & \text{if } n = 2k+1. \end{cases}$$

Proof. Since there are exactly k partitions with largest part 2 and 1 partition with largest part 1, this follows from the previous corollary. $\square$

Proposition 2. *The total number of parts in partitions of $n \geqslant 3$ with largest part 3 are*

$$S(n,3) = \begin{cases} \frac{k(22k^2-3k-1)}{2}, & \text{if } n = 6k \\ k^2(11k+4), & \text{if } n = 6k+1 \\ \frac{k(22k^2+19k+3)}{2}, & \text{if } n = 6k+2 \\ 11k^3+15k^2+7k+1, & \text{if } n = 6k+3 \\ \frac{(11k+4)(2k+1)(k+1)}{2} & \text{if } n = 6k+4 \\ (k+1)(11k^2+15k+5), & \text{if } n = 6k+5 \end{cases}$$

Proof. A partition of n with largest part 3 can be expressed as $a_1 + 2a_2 + 3a_3 = n$, where a_i denotes the number of times i appears in the partition. Thus, the total number of parts in the partition given by $a_1 + 2a_2 + 3a_3$ is $a_1 + a_2 + a_3 = n - a_2 - 2a_3$.

Note that if $n = 6k + r$, where $0 \leqslant r < 3$, then $1 \leqslant a_3 \leqslant 2k$ and $0 \leqslant a_2 \leqslant \lfloor \frac{n-3a_3}{2} \rfloor$, so

$$S(6k+r,3) = \sum_{a_3=1}^{2k} \sum_{a_2=0}^{\lfloor \frac{6k+r-3a_3}{2} \rfloor} 6k + r - a_2 - 2a_3.$$

For each r, we want to consider the odd and even values for a_3 separately. Note that when $a_3 = 2i$ for some i, we have $a_2 \leqslant 3k - 3i + \lfloor r/2 \rfloor$. When $a_3 = 2i - 1$ for some i, we have $a_2 \leqslant 3k - 3i + 1 + \lfloor (r+1)/2 \rfloor$.

$$S(6k,3) = \sum_{i=1}^{k}\left(\sum_{j=0}^{3k-3i}(6k-j-4i) + \sum_{j=0}^{3k-3i+1}(6k-j-4i+2)\right)$$
$$= \frac{k(22k^2-3k-1)}{2}.$$
$$S(6k+1,3) = \sum_{i=1}^{k}\left(\sum_{j=0}^{3k-3i}(6k+1-j-4i) + \sum_{j=0}^{3k-3i+2}(6k+1-j-4i+2)\right)$$
$$= k^2(11k+4).$$
$$S(6k+2,3) = \sum_{i=1}^{k}\left(\sum_{j=0}^{3k+1-3i}(6k+2-j-4i) + \sum_{j=0}^{3k-3i+2}(6k+2-j-4i+2)\right)$$
$$= \frac{k(22k^2+19k+3)}{2}.$$

When $3 \leqslant r < 6$, we can now also have $a_3 = 2k+1$. Note that when $r = 3$, we have

$$a_1 + 2a_2 + 3(2k+1) := 6k+3 \implies [a_1, a_2, a_3] = [0, 0, 2k+1].$$

Thus, there is one partition, and it has $2k+1$ parts. When $r = 4$,

$$a_1 + 2a_2 + 3(2k+1) := 6k+4 \implies [a_1, a_2, a_3] = [1, 0, 2k+1].$$

So, there is one partition, and it has $2k+2$ parts. Finally, when $r = 5$,

$$a_1 + 2a_2 + 3(2k+1) := 6k+5 \implies [a_1, a_2, a_3] = [2, 0, 2k+1] \text{ or } [0, 1, 2k+1].$$

Hence, there is one partition with $2k+2$ parts and one partition with $2k+3$ parts.

$$S(6k+3,3) = (2k+1) + \sum_{i=1}^{k}\left(\sum_{j=0}^{3k+1-3i}(6k+3-j-4i) + \sum_{j=0}^{3k+3i+3}(6k+3-j-4i+2)\right)$$
$$= 11k^3 + 15k^2 + 7k + 1$$
$$S(6k+4,3) = (2k+2) + \sum_{i=1}^{k}\left(\sum_{j=0}^{3k+2-3i}(6k+4-j-4i) + \sum_{j=0}^{3k+3i+3}(6k+4-j-4i+2)\right)$$
$$= \frac{(11k+4)(2k+1)(k+1)}{2}$$
$$S(6k+5,3) = (2k+3+2k+2) + \sum_{i=1}^{k}\left(\sum_{j=0}^{3k+2-3i}(6k+4-j-4i) + \sum_{j=0}^{3k+3i+4}(6k+4-j-4i+2)\right)$$
$$= (k+1)(11k^2+15k+5)$$

□

These examples show that $S(n,1)$ and $S(n,2)$ are both C-finite. The sequence $S(n,3)$ is A308265 in the OEIS, and there Colin Barker conjectured in 2019 that it is C-finite as well. *Our* conjecture is that $S(n,k)$ is C-finite for every k, and we now have a constructive proof of this fact.

Theorem 1. *The generating function* $S_k(t) = \sum_{n\geqslant 0} S(n,k)t^n$ *equals* $Z_k(t) - Z_{k-1}(t)$*, where*

$$Z_k(t) = \sum_{i=1}^{k} \frac{t^i}{(1-t^i)^2} \prod_{\substack{1\leqslant j\leqslant k\\ j\neq i}} \frac{1}{1-t^j}.$$

The function $Z_k(t)$ is the generating function of the sum

$$Z(n,k) = \sum_{p\in P'(n,k)} L(p),$$

where $P'(n,k)$ is the set of partitions of n with largest part *at most* k. This makes the relation $S_k(t) = Z_k(t) - Z_{k-1}(t)$ obvious, and, as we shall see, this sum is more natural from a generating function perspective.

Proof. Every partition in $P'(n,k)$ is uniquely specified by the length of its "runs." There are so many k's, so many $(k-1)$'s, and so on down to so many 1's. Therefore every partition in $P'(n,k)$ is uniquely specified by a sequence of nonnegative integers $n(1), n(2), \ldots n(k)$ such that $n = kn(k) + (k-1)n(k-1) + \ldots + 1\cdot n(1)$. Written this way, the length of a partition is $n(1)+n(2)+\ldots+n(k)$. This gives us an equivalent formulation for $Z(n,k)$:

$$Z(n,k) = \sum_{\substack{n(i)\geqslant 0\\ n(1)+2n(2)+\cdots+kn(k)=n}} (n(1)+\ldots+n(k)).$$

This expression of $Z(n,k)$ is almost a convolution, and we can make it completely so by breaking it into k terms:

$$Z(n,k) = \sum_{i=1}^{k} Z(n,k,i), \tag{1}$$

where

$$Z(n,k,i) = \sum_{\substack{n(1),n(2),\ldots,n(k)\geqslant 0\\ n(1)+2n(2)+\cdots+kn(k)=n}} n(i).$$

The sums $Z(n,k,i)$ are special convolutions. Here's the lemma that we need:

$$\sum_{\substack{n(i)\geqslant 0\\ n(1)+2n(2)+\cdots+kn(k)=n}} b_1(n(1))b_2(n(2))\cdots b_k(n(k)) = [t^n]f_1(t)f_2(t^2)f_3(t^3)\ldots f_k(t^k), \tag{2}$$

where

$$f_i(t) = \sum_{n \geqslant 0} b_i(n) t^n.$$

We can recognize $Z(n, k, i)$ by taking every $b_j(n)$ to be identically 1 except for the case $j = i$, where we take $b_j(n) = n$. The generating function of the identically-1 sequence is $(1 - t)^{-1}$, and the generating function of the identity sequence is $t/(1 - t)^2$. Therefore our lemma gives

$$Z(n, k, i) = [t^n] \frac{t^i}{(1 - t^i)^2} \prod_{\substack{1 \leqslant j \leqslant k \\ j \neq i}} \frac{1}{1 - t^j}, \tag{3}$$

and summing over i yields the result for $Z_k(t)$. □

This theorem is both a theoretical and a computational "win." Theoretically, the generating functions $S_k(t)$ and $Z_k(t)$ are rational, so the sums $S(n, k)$ and $Z(n, k)$ are C-finite with respect to n. This resolves at least one conjecture in the OEIS. Computationally, everything is expressible in terms of finite operations on "known" generating functions, so a computer can easily manipulate these generating functions to generate interesting identities.

Here are the first few cases:

$$\begin{aligned} S_1(t) &= \frac{t}{(1 - t)^2} \\ S_2(t) &= \frac{t^2(t^2 + t + 1)}{(1 - t)^3 (1 + t)^2} \\ S_3(t) &= \frac{t^3 \left(2\,t^4 + 3\,t^3 + 3\,t^2 + 2\,t + 1\right)}{(1 - t)^4 \left(t^2 + t + 1\right)^2 \left(t + 1\right)^2}. \end{aligned}$$

Maple can translate these generating functions into closed forms by using `convert` with `FormalPowerSeries`. Inspecting these closed forms reveals the following asymptotics:

$$\begin{aligned} S(n, 1) &\sim n \\ S(n, 2) &\sim \frac{3}{8} n^2 \\ S(n, 3) &\sim \frac{11}{216} n^3. \end{aligned}$$

It seems as though $S(n, k) \sim C_k n^k$ for some rationals C_k. Not only is this true, but we even know C_k.

Theorem 2. *As* $n \to \infty$,

$$S(n, k) \sim \frac{H_k}{k!^2} n^k$$

where $H_k = \sum_{j=1}^{k} \frac{1}{j}$ *is the* k*th* harmonic number.

FIRST STEP: PARTIAL FRACTIONS By abstract nonsense, we know the generating function $S_k(t)$ has a *partial fraction decomposition.* That is, after writing it as a rational function and factoring the denominator into linear terms, we may express it as a linear combination of reciprocal powers of these linear terms. For instance:

$$S_2(t) = \frac{13}{16(1-t)} + \frac{1}{8\ (1+t)^2} - \frac{3}{16(1+t)} + \frac{3}{4\ (1-t)^3} - \frac{3}{2\ (1-t)^2}. \qquad (4)$$

This is particularly interesting for asymptotics, because each term of the expansion has a simple closed-form expression:

$$\begin{aligned}[t^n]\frac{C}{(1-at)^{m+1}} &= C\binom{n+m}{m}a^n \\ &\sim \frac{C}{m!}a^n n^k.\end{aligned}$$

Using this idea in (4) shows that

$$\begin{aligned}S(n,2) &\sim \frac{13}{16}n^0 + \frac{1}{8}(-1)^n n - \frac{3}{16}(-1)^n + \frac{3}{8}n^2 - \frac{3}{2}n \\ &\sim \frac{3}{8}n^2.\end{aligned}$$

This is our general strategy: If there is a term of the form $C/(1-t)^{m+1}$, and $m+1$ is *strictly larger* than any other power that appears, then the sequence is asymptotically $\frac{C}{m!}n^m$.

This *does* happen with $Z_k(t)$:

$$Z_k(t) = \sum_{i=1}^{k} \frac{t^i}{(1-t^i)^2} \prod_{\substack{1\leqslant j\leqslant k \\ j\neq i}} \frac{1}{1-t^j}.$$

The denominators of each term split into roots of unity. In particular, the term $(1-t)$ will appear exactly $k+1$ times, and every other linear factor will appear no more than k times. (Because 1 is the only root of unity that is an nth root for *every* n.) Thus, the partial fraction expansion of $Z_k(t)$, and therefore that of $S_k(t)$, will contain a term of the form $C/(1-t)^{k+1}$, and this is the largest term as far as asymptotics are concerned. It suffices to compute C, the coefficient on this term.

SECOND STEP: RESIDUES Elementary complex analysis gives us a nice way to compute the coefficients of a partial fraction expansion into linear terms. If

$$f(t) = \sum_j \sum_\nu \frac{c_{j\nu}}{(r_j - t)^\nu},$$

then

$$c_{j\nu} = \mathrm{Res}_{r_j}(r_j - t)^{\nu-1} f(t),$$

where $\mathrm{Res}_z f(t)$, the *residue* of $f(t)$ at $t = z$, is the coefficient on $(t - z)^{-1}$ in the series expansion of $f(t)$ about $t = z$. In particular, the coefficient that we seek is

$$C = \mathrm{Res}_1 (1-t)^k Z_k(t)$$

Fortunately, this residue is simple to compute. Let

$$P_m(t) = \frac{1-t^m}{1-t} = 1 + t + \cdots + t^{m-1}.$$

Then we can write

$$(1-t)^k Z_k(t) = \frac{1}{1-t} \sum_{i=1}^{k} \frac{t^i}{P_i(t)} \prod_{j=1}^{k} \frac{1}{P_j(t)},$$

which implies

$$\begin{aligned} \mathrm{Res}_{t=1} (1-t)^k Z_k(t) &= \sum_{i=1}^{k} \frac{1}{P_i(1)} \prod_{j=1}^{k} \frac{1}{P_j(1)} \\ &= \sum_{i=1}^{k} \frac{1}{i} \frac{1}{k!} \\ &= \frac{H_k}{k!}. \end{aligned}$$

It follows that the highest order term in the partial fraction expansion of $Z_k(t)$ is

$$\frac{H_k}{k!} \frac{1}{(1-t)^{k+1}},$$

and this contributes $\frac{H_k}{k!^2} n^k$ to the asymptotics of the coefficients of $Z_k(t)$. Since $S_k(t) = Z_k(t) - Z_{k-1}(t)$, we have

$$\begin{aligned} S(n,k) &\sim \frac{H_k}{k!^2} n^k - \frac{H_{k-1}}{(k-1)!^2} n^{k-1} \\ &\sim \frac{H_k}{k!^2} n^k. \end{aligned}$$

AVERAGES Now that we have an asymptotic formula for the sum $S(n,k)$, it's time to get asymptotics for the averages $S(n,k)/p(n,k)$. Fortunately the partition function is very well understood:

$$p(n,k) \sim \frac{n^{k-1}}{(k-1)!k!}.$$

Therefore

$$\frac{S(n,k)}{p(n,k)} \sim \frac{H_k}{k} n.$$

NEXT STEPS Our technique here gave us a lot of information. We should take stock of what *else* it might give us.

Here's an example: Rather than number of parts, why not study $2^{\text{number of parts}}$? Then the sum which defines $S(n, k)$ is *directly* a convolution. Who knows what we'll get out!

Define a new sequence of functions:

$$Q(n, k) = \sum_{p \in P(n,k)} 2^{L(p)}.$$

This is equivalent to

$$Q(n, k) = \sum_{\substack{n(k) \geqslant 1 \\ n(1), n(2), \ldots, n(k-1) \geqslant 0 \\ n(1)+2n(2)+\cdots+kn(k)=n}} 2^{n(1)+\ldots+n(k)},$$

and this is nearly a convolution! In fact, if we let $E(n, k)$ be the same sum but we write $n(k) \geqslant 0$ rather than $n(k) \geqslant 1$, then $Q(n, k) = E(n, k) - E(n, k-1)$ and most of the important properties will be preserved.

What's the generating function of $E(n, k)$ with respect to n? It is

$$f_1(t) f_2(t^2) \cdots f_k(t^k),$$

where

$$f_i(t) = \sum_{n \geqslant 0} 2^n t^n = \frac{1}{1 - 2t}.$$

Therefore the whole thing has generating function

$$\prod_{i=1}^{k} \frac{1}{1 - 2t^i}.$$

This generating function is *much* nicer than the first one we considered. Its smallest pole is at $t = 1/2$, and the next one is $t = 1/\sqrt{2}$. The residue at $t = 1/2$ is

$$R_k = \prod_{i=2}^{k} \frac{1}{1 - 2^{1-i}}.$$

Therefore

$$\sum_{n \geqslant 0} E(n, k) t^n - \frac{R_k}{1/2 - t}$$

has radius of convergence $1/\sqrt{2}$, which implies $E(n, k) = R_k 2^n + O(\sqrt{2} + \epsilon)^n$ for every $\epsilon > 0$. Picking a small enough ϵ will show that

$$E(n, k) \sim R_k 2^n.$$

(Or something like that. Maybe I'm off by a constant factor.)

We can do another example. Let

$$\mathsf{T}(n,k) = \sum_{\substack{n(i)\geqslant 0 \\ n(1)+2n(2)+\cdots+kn(k)=n}} 1^{n(1)}2^{n(2)}\cdots k^{n(k)}.$$

For example, the partition $(3,3,2)$ of 8 would contribute $3^2 \cdot 2^1 = 18$ to the sum.

It is clear that

$$\mathsf{T}(n,k) = [t^n]f_1(t)f_2(t^2)\cdots f_k(t^k),$$

where

$$f_i(t) = \sum_{n\geqslant 0}(it)^n = \frac{1}{1-it}.$$

Therefore the generating function of $\mathsf{T}(n,k)$ with respect to n is

$$\prod_{i=1}^{k}\frac{1}{1-it^i}.$$

The asymptotics here are more difficult! The smallest poles are no longer simple, meaning that we have to work harder to get a nice formula.

IMPORTANT REMARK It seems more natural to consider partitions with largest part *at most* k rather than *exactly* k. That's what this shift to $\mathsf{E}(n,k)$ is, and what the shift to $\mathsf{Z}(n,k)$ was before.