Exp Math S2021: Homework 25, Problem 4

Exercise 1. Let $L = [a_1, ..., a_k]$, and n be the number of cells in L.

$$\frac{n!}{\prod_{\text{cells } [i,j] \text{ in } L} h_{i,j}} = \frac{\Delta(a_1 + k - 1, a_2 + k - 2, \ldots, a_k)(a1 + \cdots + a_k)!}{(a_1 + k - 1)!(a_2 + k - 1)! \ldots a_k!},$$

where $h_{i,j}$ is the hook length of the cell $[i, j]$ and $\Delta(x_1, \ldots, x_k) = \prod_{1 \le i < j \le k} (x_i - x_j)$.

Proof. Since $n! = (a_1 + \cdots + a_k)!$ for any $L = [a_1, \ldots, a_k]$, it suffices to prove

$$\prod_{\text{cells } [i,j] \text{ in } L} h_{i,j} = \frac{(a_1 + k - 1)!(a_2 + k - 1)! \ldots a_k!}{\Delta(a_1 + k - 1, a_2 + k - 2, \ldots, a_k)}.$$

For the sake of notation, I am going to write the left and right hand sides as $LHS([a_1, \ldots, a_k])$ and $RHS([a_1, \ldots, a_k])$, respectively. Base case: when $n = 1$, we have

$$LHS([a_1]) = \prod_{\text{cells } [i,j] \text{ in } [a_1]} h_{i,j} = a_1! = RHS([a_1]).$$

Now suppose $LHS(L_{k-1}) = RHS(L_{k-1})$ holds for arbitrary $L_{k-1} = [a_1, \ldots, a_{k-1}]$. Then, adding a row of length a_k gives us $L_k := [a_1, \ldots, a_k]$.

$$\begin{aligned}
RHS(L_k) &= \frac{(a_1 + k - 1)!(a_2 + k - 2)! \ldots a_k!}{\Delta(a_1 + k - 1, a_2 + k - 2, \ldots, a_k)} \\
&= \frac{(a_1 + k - 2)!(a_2 + k - 3)! \ldots a_{k-1}!}{\Delta(a_1 + k - 2, a_2 + k - 3, \ldots, a_{k-1})} \frac{(a_1 + k - 1)(a_2 + k - 2) \ldots (a_{k-1} + 1)a_k!}{(a_1 + k - 1 - a_k)(a_2 + k - 2 - a_k) \ldots (a_{k-1} + 1 - a_k)} \\
&= RHS(L_{k-1}) \prod_{i=1}^{k-1} \frac{a_i + k - i}{a_i + k - i - a_k} a_k!.
\end{aligned}$$

Now, looking at the left-hand side, let $H_{i,j}$ denote the hook length in L_{k-1} of the cell $[i, j]$.

$$\begin{aligned}
LHS(L_k) &= \prod_{\text{cells } [i,j] \text{ in } L_k} h_{i,j} \\
&= \prod_{\text{cells } [i,j] \text{ in } L_k, j \le a_k} h_{i,j} \prod_{\text{cells } [i,j] \text{ in } L_k, j \le a_k} h_{i,j} \\
&= \prod_{1 \le j \le a_k} h_{k,j} \left(\prod_{\text{cells } [i,j] \text{ in } L_{k-1}, j \le a_k} H_{i,j} + 1 \right) \left(\prod_{\text{cells } [i,j] \text{ in } L_{k-1}, j > a_k} H_{i,j} \right),
\end{aligned}$$

since when going from L_{k-1} to L_k, add one entry below the first a_k elements of the previous rows, increasing their hook length by one, and do not change the hook number for the remaining columns.

The hook length of the elements of the $k-$th row is obviously

$$\prod_{1 \le j \le a_k} h_{k,j} = a_k!$$

Since the cell $[i, j]$ and $[i, j+1]$ have the same number of elements below them for $1 \leq j \leq a_k - 1$, we have $H_{i,j} = H_{i,j+1} + 1$ for such j. Thus,

$$\prod_{\text{cells } [i,j] \text{ in } L_{k-1}, j \leq a_k} H_{i,j} + 1 = \prod_{i=1}^{k-1} (H_{i,1} + 1) \prod_{\text{cells } [i,j] \text{ in } L_{k-1}, j < a_k} H_{i,j}$$
$$= \prod_{i=1}^{k-1} a_i + k - i \prod_{\text{cells } [i,j] \text{ in } L_{k-1}, j < a_k} H_{i,j},$$

where $H_{i,1} = a_i + k - i - 1$ since there are a_i elements in its row and $k - i - 1$ elements below it. So, we have

$$LHS(L_k) = a_k! \prod_{i=1}^{k-1} a_i + k - i \prod_{\text{cells } [i,j] \text{ in } L_{k-1}, j \neq a_k} H_{i,j}$$
$$= \frac{a_k! \prod_{i=1}^{k-1} a_i + k - i \prod_{\text{cells } [i,j] \text{ in } L_{k-1}} H_{i,j}}{\prod_{i=1}^{k-1} H_{i,a_k}}$$
$$= \frac{a_k! \prod_{i=1}^{k-1} a_i + k - i}{\prod_{i=1}^{k-1} a_i + k - i - a_k} LHS(L_{k-1}),$$

since $[i, a_k]$ has $a_i - a_k$ entries to the right of it and $k - i - 1$ entries below it in L_{k-1}. Finally, we have

$$LHS(L_k) = LHS(L_{k-1}) \prod_{i=1}^{k-1} \frac{a_i + k - i}{a_i + k - i - a_k} a_k!$$
$$= RHS(L_{k-1}) \prod_{i=1}^{k-1} \frac{a_i + k - i}{a_i + k - i - a_k} a_k!$$
$$= RHS(L_k),$$

concluding our proof. □