$$A = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \binom{a_1 + a_2 + \cdots + a_n}{\pi(1)-1+a_1 \quad \pi(2)-2+a_2 \quad \cdots \quad \pi(n)-n+a_n}$$

$$B = \prod_{1 \le i < j \le n} (a_i - i - a_j + j) \cdot \binom{a_1 + a_2 + \cdots + a_n}{a_1 \; a_2 \cdots a_n} \Big/ \prod_{1 \le i < j \le n} (a_i + j - i)$$

Proof of $A = B$ using determinant.

$$A = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \frac{(a_1 + a_2 + \cdots + a_n)!}{(\pi(1)-1+a_1)!\,(\pi(2)-2+a_2)! \cdots (\pi(n)-n+a_n)!}$$

$$= (a_1 + a_2 + \cdots + a_n)! \cdot \left[\frac{1}{(j-i+a_i)!}\right]_{i,j}$$

Multiple $(a_i + n - i)!$ on ith row

$$= (a_1 + a_2 + \cdots + a_n)! \left[\frac{(a_i+n-i)!}{(j-i+a_i)!}\right]_{i,j} \cdot \frac{1}{\prod_{i=1}^{n} (a_i+n-i)!}$$

$$= \left[\frac{(a_i+n-i)!}{(j-i+a_i)!}\right]_{i,j} \cdot \binom{a_1 + a_2 + \cdots + a_n}{a_1 \; a_2 \; \cdots \; a_n} \cdot \frac{1}{\prod_{i=1}^{n} \prod_{j=i+1}^{n} (a_i+j-i)}$$

Comparing this to B, it suffices to prove

$$\left[\frac{(a_i+n-i)!}{(j-i+a_i)!}\right]_{i,j} = \prod_{1 \le i < j \le n} (a_i - i - a_j + j)$$

Suppose $b_i = a_i - i$, it's equivalent to prove

$$\left[\frac{(b_i+n)!}{(b_i+j)!}\right]_{i,j} = \prod_{1 \le i < j \le n} (b_i - b_j)$$

It's not hard to prove this using Vandermonde matrix. Either, we can use the proof of Vandermonde determinant.

Pf: $D = \left[\frac{(b_i+n)!}{(b_i+j)!}\right]_{i,j}$ is a polynomial of b_i's with degree $\sum_{j=1}^{n} (n-j) = \frac{1}{2}n(n-1)$.

If, for $i \neq j$, one substitudes b_i for b_j. one can get a matrix with two equal rows, which has thus a zero determinant. Thus, $b_i - b_j$ is a factor of D. We have

$D = Q \cdot \prod_{1 \le i < j \le n} (b_i - b_j)$ where Q is a polynomial. D has the same degree as $\prod_{1 \le i < j \le n} (b_i - b_j)$, then Q is a constant. The constant is one by considering the coefficient of monomial $b_1^{n-1} b_2^{n-2} \cdots b_{n-1}$

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