for general $k$. $syt(a_1, a_2, \cdots, a_k)$ ?

Try to find some formula. $syt(a_1, a_2, \dots, a_k)$

$$= \sum_{b=(b_1, b_2, \cdots, b_k) \in \mathbb{R}^k} c_b \binom{n}{a_1+b_1 \quad a_2+b_2 \quad \cdots \quad a_k+b_k}.$$

where $c_b$ is a coefficient subject to $b$.

Then using the recursive property of multinomial coefficients, we can prove the formula by induction.

The next thing is about boundaries.

Similar to $k=3$ situation, we need

$syt(a_1, a_2, \cdots a_k) = 0$ when

$a_2 = a_1+1$. or $a_3 = a_2+1$ or $\cdots$ or $a_k = a_{k-1}+1$

when $a_2 = a_1+1$.

$$\binom{n}{a_1+b_1 \quad a_2+b_2 \quad \cdots \quad a_k+b_k} = \binom{n}{a_2-1+b_1 \quad a_1+1+b_2 \quad a_3+b_3 \quad \cdots}$$

$$= \binom{n}{a_1+1+b_2 \quad a_2-1+b_1 \quad a_3+b_3 \quad \cdots}$$

In other words, we just want

$$C_{(b_1, b_2, \cdots, b_k)} = -C_{(b_2+1, b_1-1, \cdots b_k)}$$

For other boundaries, we want

$$C(b_1, \dots, b_i, b_{i+1}, \dots, b_k) = -C(b_1, \dots, b_{i+1}+1, b_i - 1, \dots, b_k) \quad (*)$$

This is an interesting property. When a number is exchanged with a neighbor, we add $(+1, -1)$ on them. After trying a 1000000000t of examples, I find a good idea: using the permutation to record the "exchange". For any permutation $\pi$ of $[1, 2, \dots, k]$, according to $(*)$.

$$C(b_1, \dots, b_k) = \mathrm{sgn}(\pi) \cdot C((\pi[1]-1)+b_{\pi[1]}, (\pi[2]-2)+b_{\pi[2]}, \dots, (\pi[k]-k)+b_{\pi[k]})$$

Remark, ① $\mathrm{sgn}(\pi)$ is the parity of permutation; it comes from that $\pi$ is composition of "exchange"

② $b_{\pi[i]}$ moves from position $\pi[i]$ to $i$, and adding $(+1, -1)$ during exchange is recorded, moving left $\pi[i]-i$ gives $(\pi[i]-i)$.

In fact, you can forget all the things above, and check the following conclusion by induction.

$syt(a_1, a_2, \cdots, a_k) =$

$$\sum_{\pi \in permute(k)} sgn(\pi) \binom{n}{\pi[1]-1+a_1 \quad \pi[2]-2+a_2 \quad \cdots \quad \pi[k]-k+a_k}$$

where $n = a_1 + a_2 + a_3 + \cdots + a_k$.

Proof: checking each situation of recursion.

① If $n=1$ reture $1$ ✓

② $$\binom{n}{\pi[1]-1+a_1 \quad \cdots \quad \pi[k]-k+a_k} = \sum_i \binom{n}{\cdots \quad \pi[i]-i+a_i-1 \quad \cdots}$$

for $a_1 > a_2 > \cdots > a_k > 1$ ✓

③ If $a_k = 1$, we have $\pi[k] \le k$, then

$\pi[k] - k + a_k - 1 \le 0$, $=0$ only when $\pi[k] = k$.

$$\binom{n}{\pi[1]-1+a_1 \quad \cdots \quad \pi[k]-k+a_k-1}$$

$$= \begin{cases} 0 & \text{if } \pi[k] \neq k \\ \binom{n}{\pi[1]-1+a_1 \quad \cdots \quad \pi[k-1]-(k-1)+a_{k-1}} & \end{cases}$$ ✓

④ If $a_i = a_{i+1}$, recursion holds because of ⊛

✓

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