HW 15

Mike Murr
Due April 2, 2020

1. a) $\dfrac{e^{z^3}-1}{z^2}$ Isolated singularity at $z=0$.

L'Hôpital's rule

$$\lim_{z\to 0}\frac{e^{z^3}-1}{z^2} = \lim_{z\to 0}\frac{(3z^2)(e^{z^3})}{2z} = \lim_{z\to 0}\frac{3z\,e^{z^3}}{2} = \frac{3}{2}\cdot 0\cdot e^0 = 0$$

So it is a removable singularity.

b) $\cos\left(\frac{1}{z^2}\right)$ Isolated singularity at $z=0$.

Let $w=\frac{1}{z^2}$ $\quad \cos w = \sum_{k=0}^{\infty}(-1)^k\frac{w^{2k}}{(2k)!}$

$$\cos\left(\frac{1}{z^2}\right) = \sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{z^2}\right)^{2k}\frac{1}{(2k)!} = \sum_{k=0}^{\infty}(-1)^k\frac{z^{-4k}}{(2k)!}$$

There is an infinite number of negative powers of z, so essential singularity.

c) Poles at $z=1$ and $z=2$ and $z=3$

d) $$\frac{1-3z+3z^2-z^3}{1-2z+z^2} = \frac{(1-z)^3}{(1-z)^2} = 1-z, \text{ provided } z\neq 1.$$

Removable singularity at $z=1$.

e) $$\frac{e^z+1}{e^{2z}-1} = \frac{e^z+1}{(e^z+1)(e^z-1)}$$ isolated singularity at $z=0$

$$f(z) = \frac{1}{e^z-1} \qquad e^z-1 = \left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}\cdots\right)-1$$

$$= z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}\cdots$$

$$= z\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\frac{z^4}{5!}\cdots\right)$$

$$= z\left(1+\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\frac{z^4}{120}\cdots\right)$$

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1, e) (continued)

$$= z\left(1 + \frac{z}{2} \cdots\right)$$

very approximately $$= z\left(\frac{1}{1 - \frac{z}{2} \cdots}\right)$$

So $$f(z) = \frac{1}{e^z - 1} = \frac{1}{z}\left(1 - \frac{z}{2} \cdots\right)$$

from above

One term with a negative power of z, so it is a pole.

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1. f) $\csc z = \dfrac{1}{\sin z}$ isolated singularities at $z = \pi k$, for k an integer

Consider $z=0$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} \cdots\right)$$

very approximately $$= z \frac{1}{\left(1 + \frac{z^2}{3!} \cdots\right)}$$

So $\csc z = \dfrac{1}{z}\left(1 + \dfrac{z^2}{3!} \cdots\right)$ So it is a pole at $z=0$.

Pole at each isolated singularity $z = \pi k$, for k an integer, since each zero of $\sin z$ at $z = \pi k$ is similar.

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2. a) $$\frac{e^{2z}-1-2z}{z^4} = \frac{\left(1+2z+\frac{(2z)^2}{2!}+\frac{(2z)^3}{3!}+\frac{(2z)^4}{4!}+\cdots\right)-1-2z}{z^4}$$

$$= \frac{\frac{(2z)^2}{2!}+\frac{(2z)^3}{3!}+\frac{(2z)^4}{4!}+\cdots}{z^4}$$

$$= \frac{\frac{2^2z^2}{2}+\frac{2^3z^3}{3!}+\frac{2^4z^4}{4!}+\frac{2^5z^5}{5!}\cdots}{z^4}$$

$$= 2z^{-2}+\frac{4}{3}z^{-1}+\frac{2}{3}+\frac{2^5 z}{5!}+\sum_{k=2}^{\infty}\frac{2^{k+4}z^k}{(k+4)!}$$

Residue is $\frac{4}{3}$

b) $\frac{z}{z^2-4}=f(z)$ at $z_0=-2$

$$\frac{z}{z^2-4}=\frac{z}{(z+2)(z-2)}=\frac{A}{z+2}+\frac{B}{z-2}$$ partial fractions

$A(z-2)+B(z+2)=z$ $\quad 4B=2 \quad B=\frac{1}{2}$

$-4A=-2 \quad A=\frac{1}{2}$

$$f(z)=\frac{1}{2(z+2)}+\frac{1}{2(z-2)}$$ we want powers of $(z+2)$

$$\frac{1}{2(z-2)}=\frac{1}{2z-4}=\frac{1}{2(z+2)-8}=\frac{-\frac{1}{2}}{4-(z+2)}=\frac{-\frac{1}{8}}{1-\frac{(z+2)}{4}}$$

$$= -\frac{1}{8}\sum_{k=0}^{\infty}\left(\frac{z+2}{4}\right)^k$$

So $f(z)=\frac{1}{2(z+2)}-\frac{1}{8}\sum_{k=0}^{\infty}\left(\frac{z+2}{4}\right)^k$ $\quad \text{Res}(f(z);-2)=\frac{1}{2}$

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2.c) $\frac{z}{(\sin z)^2} = g(z)$ Method of book, page 145.

Numerator has zero of order 1. Denominator has zero of order 2.

$z = g(z)[\sin z]^2$

Laurent series for $g(z)$ will have $\frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 \ldots$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots$

$(\sin z)^2 = z^2 - \frac{z^4}{6} - \frac{z^4}{6} + \frac{z^6}{6 \cdot 6} + \frac{z^6}{120} + \frac{z^6}{120}$

$(\sin z)^2 = z^2 - \frac{z^4}{3} + \frac{2 \cdot z^6}{45}$

$g(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 \ldots$

$g(z) \cdot (\sin z)^2 = (a_{-1}) z + a_0 z^2 - \frac{a_{-1} z^3}{3} + a_1 z^3 + a_2 z^4 - \frac{a_0 z^4}{3} \ldots$

This needs to equal z.

So $a_{-1} = 1$ $a_0 = 0$

$-\frac{a_{-1}}{3} + a_1 = 0$

$-\frac{1}{3} + a_1 = 0$ $a_1 = \frac{1}{3}$

$a_2 - \frac{a_0}{3} = 0$ $a_2 = 0$

$g(z) = \frac{1}{z} + 0 + \frac{z}{3} + 0 \cdot z^2 \ldots$

$\mathrm{Res}(g(z); 0) = 1$

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2.d) $\dfrac{1}{1-\cos(2z)} = g(z)$, first four terms

$\cos(2z) = 1 - \dfrac{z^2}{2} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} \cdots \quad \cos(2z) = 1 - \dfrac{(2z)^2}{2} + \dfrac{(2z)^4}{4!} - \dfrac{(2z)^6}{6!} \cdots$

$\cos(2z) = 1 - 2z^2 + \dfrac{2z^4}{3} - \dfrac{4z^6}{45} \cdots$

$1 - \cos(2z) = 2z^2 - \dfrac{2z^4}{3} + \dfrac{4z^6}{45} \cdots$

$g(z) = \dfrac{a_{-2}}{z^2} + \dfrac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 \cdots$

$\boxed{1-\cos(2z)}\left(g(z)\right) = 1$

const. term $\quad 2a_{-2} = 1$

z term $\quad 2a_{-1} = 0$

z^2 term $\quad -\frac{2}{3}a_{-2} + 2a_0 = 0$

z^3 term $\quad -\frac{2}{3}a_{-1} + 2a_1 = 0$

z^4 term $\quad \frac{4}{45}a_{-2} - \frac{2}{3}a_0 + 2a_2 = 0$

$a_{-2} = \frac{1}{2} \quad a_{-1} = 0 \quad -\frac{1}{3} + 2a_0 = 0 \quad a_0 = \frac{1}{6} \quad a_1 = 0$

$\dfrac{2}{45} - \dfrac{1}{9} + 2a_2 = 0 \quad 2a_2 = \frac{1}{15} \quad a_2 = \frac{1}{30}$

$g(z) = \dfrac{1}{2z^2} + 0 \cdot \dfrac{1}{z} + \dfrac{1}{6} + 0 \cdot z + \dfrac{z^2}{30} \cdots$

$\mathrm{Res}(g(z); 0) = 0$