Two-Player Strict War

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0 Introduction

A children's card game perhaps not known to some is the game 'War.' It is typically played with a standard deck of 52 cards split between two players. Players then take turns flipping their respective top cards and the winner is whomever has the higher valued card. The winner then places both cards at the bottom of their pile. In the instance where both cards are the same, players then flip the top four cards and compare between their fourth flipped card. Again, the winner is whoever has the higher valued card and they then take all of the flipped cards and place them in the bottom of their pile.

This paper analyzes a simplified card game inspired by 'War.' Namely, suppose two players have a deck of $2n$ cards labeled accordingly by $1, \dots, 2n$. Note already that there will never be a tie between players. Consequently, players take turns as in 'War,' flipping their top cards and comparing values, and we stipulate that when the winner places the two cards at the bottom of their pile, the losers' card must go last. This consistent placement of cards will greatly simplify later proofs. In fact, it will be prominent enough that this restriction is titled the 'winning restriction;' the reason for this name will be made clear later.

Given that there are never any ties, this variant of 'War' will be called 'Strict War.' The aspects of 'Strict War' which we concern ourselves with are the following properties:

1. Number of terminating and periodic games.
2. Expected duration of a terminating game and the expected period of periodic game play.
3. Which hands can actually occur during game play.

Before proceeding to address these questions and more, allow us to establish some terminology and notation.

First, it is assumed that Strict War is played with a deck of cards of size $2n$, hence the game begins with each having n cards. A **round of game play** consists of both players comparing their current card, to which a winner is determined and then places both cards at the bottom of their pile with the lower card placed at the bottom. A **game** then consists of several rounds of game play until no more rounds can be played or the game becomes periodic. To the former, a game is said to **terminate** if one player has all $2n$ cards in some permutation. The initial cards the players begin with are called their **starting hands**, while in general during any round, if both players have the same amount of cards, then they are said to have **a pair of n-hands**. The pairs with unequal sizes that occur during game play are then just called **a pair of hands**. Lastly, a pair of hands is said to be **playable** if there exists another pair of hands such that the outcome of that round is the initial pair of hands; in this case, the initial pair of hands is also said to have **been played to** by the previous pair of hands.

With the terminology established, we introduce some notation. First, denote the total set of all ordered subsets of $\{1, \dots, 2n\}$ by T_{2n}. Then, denote the i-th player by X_i and their starting hand of n cards by $H_i(0)$. After l rounds of game play, denote the hand of X_i by $H_i(l)$, and define **the state of the game after l moves** to be the ordered pair $(H_1(l), H_2(l))$. Consequently, for each l, the state $(H_1(l), H_2(l))$ can be identified as an element of T_{2n}; but note that not every element of T_{2n} can actually occur (see the first Lemma).

1 The Graph

The primary tool used in studying Strict War is the following graph associated to it. Let G_{2n} be the directed graph with set of vertices $V_{2n} = T_{2n}$ and edge set consisting of all ordered pairs

$$E_{2n} := \{[(x_1, y_1), (x_2, y_2)] \mid (x_1, y_1) \text{ plays to } (x_2, y_2)\}$$

Then the graph $G_{2n} := (V_{2n}, E_{2n})$ precisely demonstrates all possible games of Strict War played on $2n$ cards. Questions about game play can be posed as questions about the graph. For instance, a terminating game is then a directed path as a subdiagram of G_{2n}. Moreover, periodic games are not necessarily cycles. Instead, they may be cycles or cycles with ray pointing inward to a node on a cycle; this observation leads to the distinction of periodic games and quasi-periodic games, respectively.

The graph also shows which hands may actually appear as game play. These are clearly the nodes which have a directed edge pointing to it. Moreover, define the **directed degree** of a node v to be a tuple (p, q) where p is the number of edges pointing to v and q is the number of edges pointing out of v. Denote this by

$$\text{ddeg}(v) := (p, q)$$

Then

Lemma 1. *For all $v \in V$*

$$\textit{ddeg}(v) \in \{(0,0), (1,0), (2,0), (0,1), (1,1), (2,1)\}$$

That is, a node can have at most 2 edges pointing to it and at most 1 edge pointing out of it.

Proof. Let $v_0 \in V$. By definition of the directed edges, v_0 can only be pointing to another node v_1 if one can play from v_0 to v_1. Since each pair of hands has only one outcome each round, either player 1 wins or player 2 wins, then either v_0 is connected to another node or it is the final pair of hands i.e. the game has terminated at v_0. Hence, indeed $0 \le q \le 1$.

Similarly, consider the possible nodes that may point to v_0. Again by definition, any such connected node then plays to v_0. However, as v_0 is a pair of hands of the two players, if a node is connected to v_0, then either player 1 won or player 2 won the game before. However, both events are possible, so indeed there are at most two nodes pointing to v_0.

Of particular importance is when there are no nodes pointing to v_0. In which case, either the pair of hands represented by v_0 are of the same size or they are not. If they are, then these nodes represent starting hands for Strict War which cannot be obtained by playing from a different pair of starting hands; or they are of different lengths, which is not a valid pair of starting hands and hence cannot point to any other node. □

Corollary 1. *The connected components of G_{2n} are either isolated points, trees, cycles, or cycles with directed rays pointing to a node of a cycle. Here, a path is considered a tree.*

Clearly the points of T_{2n} which cannot occur as game play are precisely the isolated points of G_{2n}. However, the graph also reduces the analysis of Strict War by considering the proof of the following lemma, which in fact follows from an easier lemma.

Lemma 2. *There are an even number of connected components in G_{2n}. Moreover, each connected component of G_{2n} is isomorphic to another distinct connected component of G_{2n}.*

Lemma 3. *If for some game a pair of hands $(H_1(l), H_2(l))$ plays to $(H_1(l+1), H_2(l+1))$, then $(H_2(l), H_1(l))$ plays to $(H_2(l+1), H_1(l+1))$.*

Proof. Obvious. □

Proof. Returning to the proof of Lemma 2. Suppose H is a connected component of G_{2n}. Then consider the image of H obtained by the map defined by swapping hands i.e. $(H_1(l), H_2(l)) \mapsto (H_2(l), H_1(l))$. This is clearly a directed graph isomorphism. □

In words, the previous two lemmas simply say that game play is symmetric with respect to the player i.e. it does not matter which player is assigned $H_i(l)$. Thus, we can effectively reduce the numbers of cases to consider by half. In order to make a consistent choice among all pairs of connected components, note the following trivial lemma.

Lemma 4. *If a game terminates, then the player with the highest card must win.*

Proof. Suppose otherwise i.e. the player who does not have the highest card wins the game. This then implies that the winning player must have beat the highest card, which is impossible. □

Thus, without loss of generality, we need only analyze the cases in which player 1 has the highest card.

Now, in order to analyze the remaining game play of Strict War, it would appear natural to first look at the game play that ensues from trying each possible choice of $H_1(0)$ and $H_2(0)$ where $H_1(0)$ contains $2n$. However, a more informative algorithm would be to study the game in reverse. That is, for each permutation of $\{1, \ldots, 2n\}$, consider the moves that lead to it and proceed recursively, determining the moves that could lead to those moves and so on.

In doing so, one effectively determines which permutation can a game terminate on, how long a game lasts, how many starting hands end on the same permutation, and consequently, which starting hands lead to periodic or quasi-periodic game play. Indeed, any starting hand will either terminate or become periodic. The posed algorithm determines which starting hands terminate. Hence, the complement of the remaining set of starting hands must lead to periodic or quasi-periodic game play. Moreover, the posed algorithm further allows one to determine the (quasi)period of a starting hand that experiences (quasi)periodic game play.

Thus, while the posed algorithm allows one to determine all starting hands which have the same ending hand, it in fact allows us to reduce the number of cases even further. Indeed, lemmas 2, 3, and 4 were useful in reducing the analysis by half if it was conducted by looking at starting hands. However, in starting from reverse, a priori we are looking at $2(2n)!$ since there are $(2n)!$ permutations of $\{1, \ldots, 2n\}$, but then either player 1 wins or player 2 wins. However, by lemma 2, it still suffices to consider only the cases in which player 1 wins due to the aforementioned symmetry. Consequently, we are again looking at at most $(2n)!$ cases for the final hand of player 1.

Lemma 5. *There are at most $\frac{(2n)!}{4}$ possible hands player 1 can end on.*

Proof. Suppose a gamed terminated with the state of the game being

$$([i_1, \ldots, i_{2n}], [])$$

where $[i_1, \ldots, i_{2n}]$ is some ordering of the cards. As this is not a valid starting hand, there must have necessarily been a round that plays to it. However, since the rules stipulate that the lower of the two compared cards gets stacked, necessarily $i_{2n-1} > i_{2n}$ and the previous pair of hands is given by

$$([i_{2n-1}, i_1, \ldots, i_{2n-2}], [i_{2n}])$$

However, in the event that $n > 1$, this again is not a valid starting pair of hands. Hence, by the same logic there must have been a round that plays to it, so necessarily $i_{2n-3} > i_{2n-2}$ and the previous turn had the pair of hands

$$([i_{2n-3}, i_{2n-1}, i_1, \ldots, i_{2n-4}], [i_{2n-2}, i_{2n}])$$

From here, it follows that the endings that may actually occur must be such that the last two pairs of cards are decreasing i.e. the only endings allowed are permutations $[i_1, \ldots, i_{2n}]$ with $i_{2n-1} > i_{2n}$ and $i_{2n-3} > i_{2n-2}$. There are precisely $\frac{(2n)!}{4}$ such permutations, hence completing the proof. □

Note that the proof does provide a procedure to determine tighter bounds for games played with more cards. In particular,

Corollary 2. *If $n \geq 3$, then there are at most $\frac{3(2n)!}{16}$ possible hands player 1 can end on. If $n \geq 4$, then there are at most $\frac{3(2n)!}{32}$ possible hands player 1 can end on.*

These two bounds can be readily derived by continuing the process outlined in the proof of the previous lemma.

We now turn our attention and prove two lemmas pertaining to periodic game play. Before doing so, there are two types of periodic game play that have been mentioned but we now make more concrete. If a game does not terminate, then it is called **quasi-periodic** if in the repetition of game play the players do not return to their starting hands; otherwise it is called **periodic**. For the former, what is meant is it is possible to play a game such that the repetition begins after l moves. Thus, both periodic and quasi-periodic game plays have a notion of period: the length of the cycle experienced in game play. However, quasi-periodic game play also has a quasi-period given by adding to the period the amount of moves needed to get from the starting hand to the periodic game.

Lemma 6. *Any periodic game has an even period.*

Proof. This follows from the trivial observation that the length of the period is also equal to the number of times each player has won. However, if the game is to be periodic, then after a cycle, a player must have lost as much as they won i.e. both players won the same amount of times. Hence, the period is indeed even. □

We now prove a lemma pertaining to the minimum size of periodic game play, but first prove an even easier lemma, where recall we assume that player 1 has the highest card.

Lemma 7. *The highest card cannot occur as the last or third to last card in player 1's hand during any round of game play.*

Proof. The fact that the highest card cannot be the last card is obvious. If it was, since the corresponding pair of hands is part of a periodic game play, there is a pair of hands that plays to it. If in the previous round player 1 won, then as the lowest card gets placed last, this means the second to last card is higher than its last, which is a contradiction. Thus, player 2 won the round before. But by the same logic, player 1 could not have the won the round before that one, so necessarily player 2 did. Consequently, player 2 won every round prior, eventually leading to a pair of hands where player 2 has only one card. At this state, clearly player 2 could not have won the round prior, so player 1 must have won, which again is a contradiction to the last card being the highest card.

Similarly, suppose the highest card was in the third to last spot. Suppose player 1 won the round before. In this previous round, player 1 thus had their highest card as their last, which was shown before to be impossible. ☐

Corollary 3. *In a periodic game, the highest card cannot be in a position with the same parity as that of n.*

We are now able to prove a lower bound on the length of periodic game play.

Lemma 8. *Any periodic game must have a period of at least $n+2$.*

Proof. Suppose $([i_1,\ldots,i_k],[i_{k+1},\ldots,i_{2n}])$ is a pair of hands in the periodic game play. Now, one of the players has at least n cards. Without loss of generality, we may suppose it is the first player i.e. $k \geq n$. As the game is periodic, player 1 must play through each i_j. In particular, there will be at least $k \geq n$ moves. However, note that once the first card of player 1's hand is i_k, player 2 will have some i_j as their first.

If player 1 wins, then their last two cards are ordered as i_k, i_j. As the game is periodic, player 1 must return i_j to player 2, hence play at least 2 more rounds. If player 1 loses, then player 2's last two cards are ordered as i_j, i_k. As the game is periodic, player 1 must retrieve i_k, hence there are at least two more rounds; once through i_j and once through i_k.

Hence, the period is indeed at least $k+2 \geq n+2$. ☐

We now return to terminating games and give two lemmas pertaining to their lengths.

Lemma 9. *The shortest terminating game which occurs has length n.*

Proof. This is the simple observation that player 1 can win every single round, which is precisely n rounds. ☐

Lemma 10. *The length of any game that terminates has the same parity as n.*

Proof. Denote the length of a terminating game by N. Suppose after these N total rounds, player 1 won n_1 rounds, and player 2 won n_2 rounds. Hence $N = n_1 + n_2$. Now, as it is assumed player 1 wins, we readily obtain that $n_1 = n + n_2$. Indeed, for player 1 to win they must obtain every card they lost to player 2, which is n_2, and then obtain the other n cards player 2 started with. With this system one readily deduces that

$$n_1 = \frac{N+n}{2}$$
$$n_2 = \frac{N-n}{2}$$

In particular, as either left hand side is an integer, 2 must divide their difference (and sum) and indeed N and n are either both even or both odd; N is thus the same parity as n as claimed. ☐

2 Generating Functions

The following are the generating functions $L_n(t)$for the duration of game play with each player starting with n cards which terminate.

$$L_2(t) = 4t^6 + 8t^4 + 12t^2$$
$$L_3(t) = 12t^{15} + 30t^{13} + 54t^{11} + 102t^9 + 162t^7 + 180t^5 + 180t^3$$
$$L_4(t) = 64t^{26} + 324t^{24} + 636t^{22} + 1300t^{20} + 2196t^{18} + 3280t^{16} + 4188t^{14} + 4920t^{12} + 5828t^{10} + 5824t^8 + 6720t^6 + 5040t^4$$
$$\begin{aligned}L_5(t) = {} & 40t^{49} + 150t^{47} + 390t^{45} + 565t^{43} + 820t^{41} + 1850t^{39} + 3615t^{37} \\ & + 6955t^{35} + 11580t^{33} + 18280t^{31} + 30960t^{29} + 44275t^{27} + 58665t^{25} + 70585t^{23} \\ & + 87430t^{21} + 110595t^{19} + 128660t^{17} + 149045t^{15} + 190120t^{13} + 204150t^{11} + 195300t^9 + 189000t^7 + 113400t^5\end{aligned}$$

Interestingly enough, the exponents are all in arithmetic progression with difference 2 starting from n and ending on...who knows! Moreover, note that the coefficients are all even, which should be the case since as noted in the preliminary discussion, there are twice as many hands with the same outcome by swapping the players' hands.

Now, $L_n(1)$ is precisely the amount of starting hands which have a terminating game play. Interestingly enough, note that

$$\begin{aligned} L_2(1) &= 4! = 24 \\ L_3(1) &= 6! = 720 \\ L_4(1) &= 8! = 40320 \\ L_5(1) &= 3232860 \end{aligned}$$

Thus, for a deck of size 4, 6, and 8, there are no games which experience periodic game play. However, this can be rigorously proven, rather trivially, for each of these cases. For a deck of size 5 there are indeed starting hands that experience periodic game play.

Moreover, one may also readily compute the expectations from $L_n(t)$ as $\frac{L'_n(1)}{L_n(1)}$. In particular,

$$\begin{aligned} \frac{L'_2(1)}{L_2(1)} &= \frac{10}{3} \approx 3.\bar{3} \\ \frac{L'_3(1)}{L_3(1)} &= \frac{97}{15} \approx 6.4\bar{6} \\ \frac{L'_4(1)}{L_4(1)} &= \frac{8843}{840} \approx 10.52738095\ldots \\ \frac{L'_5(1)}{L_5(1)} &= \frac{2339111}{161643} \approx 14.47084625\ldots \end{aligned}$$

Interestingly, as mentioned before, the coefficients are even because half occur for player 1 winning and the other occur for player 2 winning. Thus, the generating function for the length of game play where player 1 wins is $\frac{1}{2}L_n(t)$ and similarly for player 2. Consequently, when we compute the expected length of game play where player 1 wins, it is precisely the same values as above! And of course the same holds for player 2 winning! This makes contextual since as game play is symmetric with respect to the players.