Generalized Gambler's Ruin Problem

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Abstract

We consider the classic Gambler's Ruin problem with our experimental mathematics approach and extend it to higher dimensions and even other graphs, e.g., ring graphs. Both the probability and expected duration are studies numerically and symbolically. The problem of the number of visits to the origin of a random walk is explored as well.

1 Introduction

Gambler's Ruin problem is a classic in probability theory. In this article, we apply the experimental mathematics approach to the Gambler's Ruin problem.

At first, we review the classic Gambler's Ruin problem in 1-dimension. We mainly consider its probability to reach the exit state, the expected duration and the probability generating function of the duration. Then we extend our method to 2D situation where the game become much more complicated and can be defined in different ways. Moreover, we generalize the problem to even higher dimension and also look at its analogue on ring or torus graph.

In strategic gambling, the strategy at each state could be different. So even in 1-dimensional situation, it is interesting to explore the result when a gambler have a set of moves at each state. We also discuss the problem of the number of visits to the origin of a random walk in a high dimension space.

This article is accompanied by a Maple package, `GamblersRuin.txt`, which can be found on the front of the article
`https://sites.math.rutgers.edu/ zeilberg/EM20/projs.html`.

Readers are encouraged to download the package and use the procedures to experiment.

2 The classic Gambler's Ruin problem

In the classic Gambler's Ruin problem, a gambler starts with an initial fortune of i dollars and on each game, the gambler wins \$1 with probability p or loses \$1 with probability $q = 1 - p$, where $0 \leq p \leq 1$. The gambler will stop playing if either N dollars are accumulated or all money has been lost. Then the natural question is that what is the probability that the gambler will end up with N dollars.

At first we'd like to use probability theory to solve this problem. Then we will introduce our experimental mathematics approach and highlight the advantage of our methodology. If $p = 0$ or $p = 1$, then it is trivial. Let's begin with the fair game, i.e., $p = \frac{1}{2}$. Define $X(t)$ to be the random variable of the amount of dollars the gambler possesses after t rounds of games, assuming the gambling is ongoing. Then $X(t)$ is a martingale because by conditional probability

$$E[X(t+1)] = \frac{1}{2}(E[X(t)] - 1) + \frac{1}{2}(E[X(t)] + 1).$$

Then by the property of martingale, at the stopping time T we have

$$E[X(T)] = X(0) = i,$$

which leads to

$$i = P(N) \times N + P(0) \times 0.$$

Hence when it is a fair game, the probability that the gambler walks out of the casino with N dollars is $\frac{i}{N}$. But in reality, the games in casino are usually unfair. So for general $0 < p < 1$ and for any initial state $0 \leq i \leq N$, let P_i be the probability that the gambler's fortune will reach N instead of 0. By conditional probabitlity we have

$$P_i = pP_{i+1} + qP_{i-1}$$

with the boundary condition $P_0 = 0$ and $P_1 = 1$. Then

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}) = \frac{q^2}{p^2}(P_{i-1} - P_{i-2}) = \cdots = \frac{q^i}{p^i}(P_1 - P_0).$$

By successively expressing P_i as a formula of P_i, we have

$$P_i = \sum_{k=0}^{k=i-1} (\frac{q}{p})^k P_1.$$

If $\frac{q}{p} = 1$, i.e. $p = \frac{1}{2}$, we can see that $P_N = NP_1$ and $P_i = i/N$, which is liner. This result is consistent with the previous one obtained with martingale argument.

When $\frac{q}{p} \neq 1$, by sum of geometric series, we have

$$P_N = \frac{1 - (q/p)^N}{1 - q/p} P_1.$$

Then

$$P_1 = \frac{1 - q/p}{1 - (q/p)^N}$$

and

$$P_i = \frac{1 - (q/p)^i}{1 - (q/p)^N}.$$

Another random variable we are interested in is the duration, which is the number of rounds of games played until the gambler has N dollars or lose all money. When $p = \frac{1}{2}$, $X(t)^2 - t$ is a martingale because

$$E[X(t+1)^2 - (t+1)] = \frac{1}{2}(X(t) - 1)^2 + \frac{1}{2}(X(t) + 1)^2 - (t+1) = X(t)^2 - t.$$

Consider the stopping time T, and shift each the number of each state by $-i$ for convenience, we have

$$E(X(T)^2 - T] = P_N(N-i)^2 + P_0 i^2 - T = X(0)^2 - 0.$$

Since we shift each state, $X(0) = 0$, hence the right hand side is 0.

$$T = \frac{i}{N}(N-i)^2 + \frac{N-i}{N} i^2 = i(N-i).$$

However, when $p \neq 1/2$, $X(t)^2 - t$ is no longer a martingale and the problem becomes much more complicated. By using `ExpDuration(N, p)` procedure in `GamblersRuin.txt`, we can immediately get the expected duration for numerical N and symbolic p for each state $1 \leq i \leq N-1$. For example, `ExpDuration(5, p)` returns

$$[\frac{2\,pt^3 + 2\,pt^2 - pt + 1}{pt^4 - 2\,pt^3 + 4\,pt^2 - 3\,pt + 1}, -\frac{pt^3 - 4\,pt^2 + 2\,pt - 2}{pt^4 - 2\,pt^3 + 4\,pt^2 - 3\,pt + 1},$$
$$\frac{pt^3 + pt^2 - 3\,pt + 3}{pt^4 - 2\,pt^3 + 4\,pt^2 - 3\,pt + 1}, -\frac{2\,pt^3 - 8\,pt^2 + 9\,pt - 4}{pt^4 - 2\,pt^3 + 4\,pt^2 - 3\,pt + 1}].$$

To get the closed-form formula for symbolic N and p, we just need to call `ExpDurationCF(N, i, p)`, which uses recurrence relation and boundary condition to find the explicit formula. We have the following theorem.

Theorem 1. *The expected duration of a Gambler's Ruin problem with the initial state i dollars, exit condition N dollars and probability $p \neq \frac{1}{2}$ to win each game is*

$$\frac{1}{2\,p-1}\left(\left(-\frac{p-1}{p}\right)^i N - \left(-\frac{p-1}{p}\right)^N i - N + i\right)\left(\left(-\frac{p-1}{p}\right)^N - 1\right)^{-1}$$

Note that when p goes to $\frac{1}{2}$, the above formula goes to $i(N-i)$. But we can reach much further than expectation. How about the probability generating function of the random variable T, the number of duration? At first we can use the procedure `PGF(N, p, t)` to explore for numerical N and sympolic p. t here is the symbol for the generating function. For instance, `PGF(5, p)` returns

$$[\frac{t\,(p^4t^3 - 2\,p^3t^2 + 4\,p^2t^2 - 2\,pt^2 - p + 1)}{p^4t^4 - 2\,p^3t^4 + p^2t^4 + 3\,p^2t^2 - 3\,pt^2 + 1},$$

$$\frac{\left(p^4t^2 - 3\,p^3t^2 + p^3t + 3\,p^2t^2 - pt^2 + p^2 - 2\,p + 1\right)t^2}{p^4t^4 - 2\,p^3t^4 + p^2t^4 + 3\,p^2t^2 - 3\,pt^2 + 1},$$
$$\frac{t^2\left(p^4t^2 - p^3t^2 - p^3t + 3\,p^2t + p^2 - 3\,pt + t\right)}{p^4t^4 - 2\,p^3t^4 + p^2t^4 + 3\,p^2t^2 - 3\,pt^2 + 1},$$
$$\frac{t\left(p^4t^3 - 4\,p^3t^3 + 2\,p^3t^2 + 6\,p^2t^3 - 2\,p^2t^2 - 4\,pt^3 + t^3 + p\right)}{p^4t^4 - 2\,p^3t^4 + p^2t^4 + 3\,p^2t^2 - 3\,pt^2 + 1}].$$

For more general situation, we obtain the following theorem by using the procedure `PGFcf`.

Theorem 2. *The probability generating function of the random variable T, the duration of Gambler's Ruin is*

$$\frac{1}{-2^N\left(\frac{t\,(p-1)}{-1+\sqrt{4\,p^2t^2-4\,p\,t^2+1}}\right)^N+\left(-\frac{2\,t\,(p-1)}{1+\sqrt{4\,p^2t^2-4\,p\,t^2+1}}\right)^N}\left(\left(-\frac{2\,t\,(p-1)}{1+\sqrt{4\,p^2t^2-4\,p\,t^2+1}}\right)^N 2^i\left(\frac{t\,(p-1)}{-1+\sqrt{4\,p^2t^2-4\,p\,t^2+1}}\right)^i\right.$$
$$-2^N\left(\frac{t\,(p-1)}{-1+\sqrt{4\,p^2t^2-4\,p\,t^2+1}}\right)^N\left(-\frac{2\,t\,(p-1)}{1+\sqrt{4\,p^2t^2-4\,p\,t^2+1}}\right)^i-2^i\left(\frac{t\,(p-1)}{-1+\sqrt{4\,p^2t^2-4\,p\,t^2+1}}\right)^i+\left(\right.$$
$$\left.\left.-\frac{2\,t\,(p-1)}{1+\sqrt{4\,p^2t^2-4\,p\,t^2+1}}\right)^i\right)$$

Since we can easily get the probability generating function for any given numerical N and i, it will be much easier to get the variance and higher moments of T, which might be extremely difficult for human approach or probabilistic argument.

As an example, when $N = 10$ and $i = 5$, from `PGF(10, p, t)[5]` we obtain the probability generating function

$$F := \frac{\left(5\,p^4 - 10\,p^3 + 10\,p^2 - 5\,p + 1\right)t^5}{5\,p^4t^4 - 10\,p^3t^4 + 5\,p^2t^4 + 5\,p^2t^2 - 5\,pt^2 + 1}.$$

Then by `StatAnal(F, t, K)` we can easily find out the scaled moment about the mean (for m-th moment where $m > 2$, the first two being expectation and variance). So we have the following theorems.

Theorem 3. *The expectation of the random variable T, the duration of Gambler's Ruin when $N = 10$ and $i = 5$ is*

$$5\,\frac{p^4 - 2\,p^3 + 4\,p^2 - 3\,p + 1}{5\,p^4 - 10\,p^3 + 10\,p^2 - 5\,p + 1}.$$

Theorem 4. *The variance of the random variable T, the duration of Gambler's Ruin when $N = 10$ and $i = 5$ is*

$$\frac{-100\,p^6 + 300\,p^5 - 380\,p^4 + 260\,p^3 - 100\,p^2 + 20\,p}{25\,p^8 - 100\,p^7 + 200\,p^6 - 250\,p^5 + 210\,p^4 - 120\,p^3 + 45\,p^2 - 10\,p + 1}$$

Theorem 5. *The skewness of the random variable T, the duration of Gambler's Ruin when $N = 10$ and $i = 5$ is*

$$\left(1000\,p^{10} - 5000\,p^9 + 10600\,p^8 - 12400\,p^7 + 8600\,p^6 - 3400\,p^5 + 480\,p^4 + 240\,p^3 - 160\,p^2 + 40\,p\right) \Big/ \Bigg(\left(125\,p^{12} - 750\,p^{11} + 2250\,p^{10} - 4375\,p^9 + 6075\,p^8 - 6300\,p^7 + 4975\,p^6 - 3000\,p^5 + 1365\,p^4 - 455\,p^3 + 105\,p^2 - 15\,p + 1\right) \left(\frac{-100\,p^6 + 300\,p^5 - 380\,p^4 + 260\,p^3 - 100\,p^2 + 20\,p}{25\,p^8 - 100\,p^7 + 200\,p^6 - 250\,p^5 + 210\,p^4 - 120\,p^3 + 45\,p^2 - 10\,p + 1}\right)^{3/2}\Bigg)$$

Theorem 6. *The kurtosis of the random variable T, the duration of Gambler's Ruin when $N = 10$ and $i = 5$ is*

$$\Big(\left(-10000\,p^{14} + 70000\,p^{13} - 172000\,p^{12} + 122000\,p^{11} + 284000\,p^{10} - 870000\,p^9 + 1180800\,p^8 - 1021200\,p^7 + 615600\,p^6 - 265600\,p^5 + 80720\,p^4 - 15840\,p^3 + 1440\,p^2 + 80\,p\right) \left(25\,p^8 - 100\,p^7 + 200\,p^6 - 250\,p^5 + 210\,p^4 - 120\,p^3 + 45\,p^2 - 10\,p + 1\right)^2\Big) \Big/ \Big(\left(625\,p^{16} - 5000\,p^{15} + 20000\,p^{14} - 52500\,p^{13} + 100500\,p^{12} - 148000\,p^{11} + 172750\,p^{10} - 162500\,p^9 + 124150\,p^8 - 77100\,p^7 + 38700\,p^6 - 15500\,p^5 + 4845\,p^4 - 1140\,p^3 + 190\,p^2 - 20\,p + 1\right) \left(-100\,p^6 + 300\,p^5 - 380\,p^4 + 260\,p^3 - 100\,p^2 + 20\,p\right)^2\Big)$$

Theorem 7. *The fifth scaled moment about the mean of the random variable T, the duration of Gambler's Ruin when $N = 10$ and $i = 5$ is*

$$\left(100000\,p^{18} - 900000\,p^{17} + 2140000\,p^{16} + 3280000\,p^{15} - 30140000\,p^{14} + 82180000\,p^{13} - 133668000\,p^{12} + 148108000\,p^{11} - 117200000\,p^{10} + 66800000\,p^9 - 26438400\,p^8 + 6061600\,p^7 + 236000\,p^6 - 869600\,p^5 + 396480\,p^4 - 97760\,p^3 + 11520\,p^2 + 160\,p\right) \Big/ \Bigg(\left(3125\,p^{20} - 31250\,p^{19} + 156250\,p^{18} - 515625\,p^{17} + 1253125\,p^{16} - 2375000\,p^{15} + 3631250\,p^{14} - 4575000\,p^{13} + 4813750\,p^{12} - 4263750\,p^{11} + 3191250\,p^{10} - 2018750\,p^9 + 1075875\,p^8 - 479750\,p^7 + 177000\,p^6 - 53125\,p^5 + 12650\,p^4 - 2300\,p^3 + 300\,p^2 - 25\,p + 1\right) \left(\frac{-100\,p^6 + 300\,p^5 - 380\,p^4 + 260\,p^3 - 100\,p^2 + 20\,p}{25\,p^8 - 100\,p^7 + 200\,p^6 - 250\,p^5 + 210\,p^4 - 120\,p^3 + 45\,p^2 - 10\,p + 1}\right)^{5/2}\Bigg)$$

As long as there are sufficient time and computing resources, we can continue this process to find as high moment as we want. The aforementioned example fully demonstrates the advantage and power of our experimental mathematics approach as no or little humen research has been conducted on the higher moments of the duration of Gambler's Ruin.

Additionally, if we fix i and let N goes to infinity and consider the any fixed higher moment, we may consider this as a generating function and hence possibly we can get a finite closed formula which stores "infinite information". For the experimental mathematics approach to discrete probability problems, its methodology and beyond, we refer interested readers to [2], [3], [4], [5], [7], [8], [9] and [10]. In the remainder of this article, we consider various generalization of the classic Gambler's Ruin problem.

3 2D Gambler's Ruin

3.1 The Rule of 2D Gambler's Ruin

After the thorough 1D Gambler's Ruin problem, we explore and run a number of numeric and symbolic tests for a generalization of the Gambler's Ruin problem to two dimensions. In the 2D Gambler's Ruin game, there are two simultaneous non-interacting instances of Gambler's Ruin being played (referred to from hereon out as the 'vertical game' and the 'horizontal game'), and there are four possible ending states: winning both games, winning the vertical game and losing the horizontal game, winning the horizontal game and losing the vertical game, or losing both games. There are two possible ways of modelling this game. Suppose the player has probability p of winning \$1 in the vertical game, and probability q of winning \$1 in the horizontal game. In 2D Gambler's Ruin Type 1, every round, the player will either gain or lose a dollar in each game with the associated respective probabilities: so if the player has bank $[\$i, \$j]$, the four possibilities in the next round are $[\$i+1, \$j+1]$, $[\$i+1, \$j-1]$, $[\$i-1, \$j+1]$, and $[\$i-1, \$j-1]$; unless the vertical or horizontal game has already been won or lost, and then only the other game will be played. In 2D Gambler's Ruin Type 2, we have a third probability g, which is the probability of playing the vertical game. The player first uses g to determine which game they are playing, then plays either the vertical or the horizontal game with the associated probability: so if the player has bank $[\$i, \$j]$, the four possibilities in the next round are $[\$i+1, \$j]$, $[\$i-1, \$j]$, $[\$i, \$j+1]$, and $[\$i, \$j-1]$; unless the vertical or horizontal game has already been won or lost, and then only the other game will be played.

We will use an $M \times N$ matrix to represent a vertical game with win condition $\$M-1$ and a horizontal win condition $\$N-1$, and position $[i, j]$ in the matrix represents having $\$i-1$ in the vertical game and $\$j-1$ in the horizontal game. Note that this means that in all analysis of the 2D Gambler's Ruin game, the \$0 states are included.

As will become immediately apparent, the probabilities of winning the Type 1 game and Type 2 game from any given starting position are the same; the Gambler's Ruin Type 2 game has horizontal and vertical probabilities as the Type 1 game has, it just takes two moves to get to positions where the Type 1 game would only take one move to get to. Therefore, the difference lies in the fact that the expected duration of the Type 2 game is longer so long as the starting position is not in an edge row or column. The probability g of playing the vertical or horizontal game affects the most likely path that the gambler takes to their end node, but does not affect the probability of ending in each various node or the expected duration of the game.

The next subsection gives examples and density plots for small square ending conditions, large square ending conditions, and large narrow ending conditions at various balanced or unbalanced probabilities. Here we will see the effects of g on the path taken. And then we use patterns noted in our numerical study to calculate the symbolic probability of winning a game from a symbolic starting position.

3.2 Numerical Gambler's Ruin Games

In this section, we look at the probability of ending in each node for various numerically banked games with numeric or symbolic probabilities, their expected duration, and look at probability densities of paths from an internal starting point. As Game Type 1 and Game Type 2 have the same probabilities of ending in each node, they will not be both calculated for node probabilities.

We will present this section almost entirely in graphs, as the probability matrices very quickly become unweildy for displaying in a paper. However, we give an example here of an equal win condition of \$3 for both games and symbolic p, q:

$$P(1,1) = \begin{bmatrix} 1 & -\frac{q-1}{q^2-q+1} & \frac{q^2-2\,q+1}{q^2-q+1} & 0 \\ -\frac{p-1}{p^2-p+1} & \frac{pq-p-q+1}{p^2q^2-p^2q-pq^2+p^2+pq+q^2-p-q+1} & -\frac{pq^2-2\,pq-q^2+p+2\,q-1}{(q^2-q+1)(p^2-p+1)} & 0 \\ \frac{p^2-2\,p+1}{p^2-p+1} & -\frac{p^2q-p^2-2\,pq+2\,p+q-1}{p^2q^2-p^2q-pq^2+p^2+pq+q^2-p-q+1} & \frac{(pq-p-q+1)^2}{p^2q^2-p^2q-pq^2+p^2+pq+q^2-p-q+1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P(M,1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{p^2}{p^2-p+1} & -\frac{(q-1)p^2}{(q^2-q+1)(p^2-p+1)} & \frac{p^2(q-1)^2}{p^2q^2-p^2q-pq^2+p^2+pq+q^2-p-q+1} & 0 \\ \frac{p}{p^2-p+1} & -\frac{p(q-1)}{p^2q^2-p^2q-pq^2+p^2+pq+q^2-p-q+1} & \frac{\left(q^2-2\,q+1\right)p}{p^2q^2-p^2q-pq^2+p^2+pq+q^2-p-q+1} & 0 \\ 1 & -\frac{q-1}{q^2-q+1} & \frac{q^2-2\,q+1}{q^2-q+1} & 0 \end{bmatrix}$$

$$P(1,N) = \begin{bmatrix} 0 & \frac{q^2}{q^2-q+1} & \frac{q}{q^2-q+1} & 1 \\ 0 & -\frac{(p-1)q^2}{(q^2-q+1)(p^2-p+1)} & -\frac{q(p-1)}{p^2q^2-p^2q-pq^2+p^2+pq+q^2-p-q+1} & -\frac{p-1}{p^2-p+1} \\ 0 & \frac{q^2(p-1)^2}{p^2q^2-p^2q-pq^2+p^2+pq+q^2-p-q+1} & \frac{\left(p^2-2\,p+1\right)q}{p^2q^2-p^2q-pq^2+p^2+pq+q^2-p-q+1} & \frac{p^2-2\,p+1}{p^2-p+1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P(M,N) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{p^2q^2}{p^2q^2-p^2q-pq^2+p^2+pq+q^2-p-q+1} & \frac{p^2q}{p^2q^2-p^2q-pq^2+p^2+pq+q^2-p-q+1} & \frac{p^2}{p^2-p+1} \\ 0 & \frac{pq^2}{(q^2-q+1)(p^2-p+1)} & \frac{pq}{p^2q^2-p^2q-pq^2+p^2+pq+q^2-p-q+1} & \frac{p}{p^2-p+1} \\ 0 & \frac{q^2}{q^2-q+1} & \frac{q}{q^2-q+1} & 1 \end{bmatrix}$$

The procedures in our Maple package `GamblersRuin.txt` will automatically produce these matrices with numeric entries along with the figures shown in the following subsections.

3.2.1 Small Square Matrix

We first wish to examine a small square matrix, representing a total vertical and horizontal bank of \$4.

First, we see the probability of landing in each node from each starting point for different p, q:

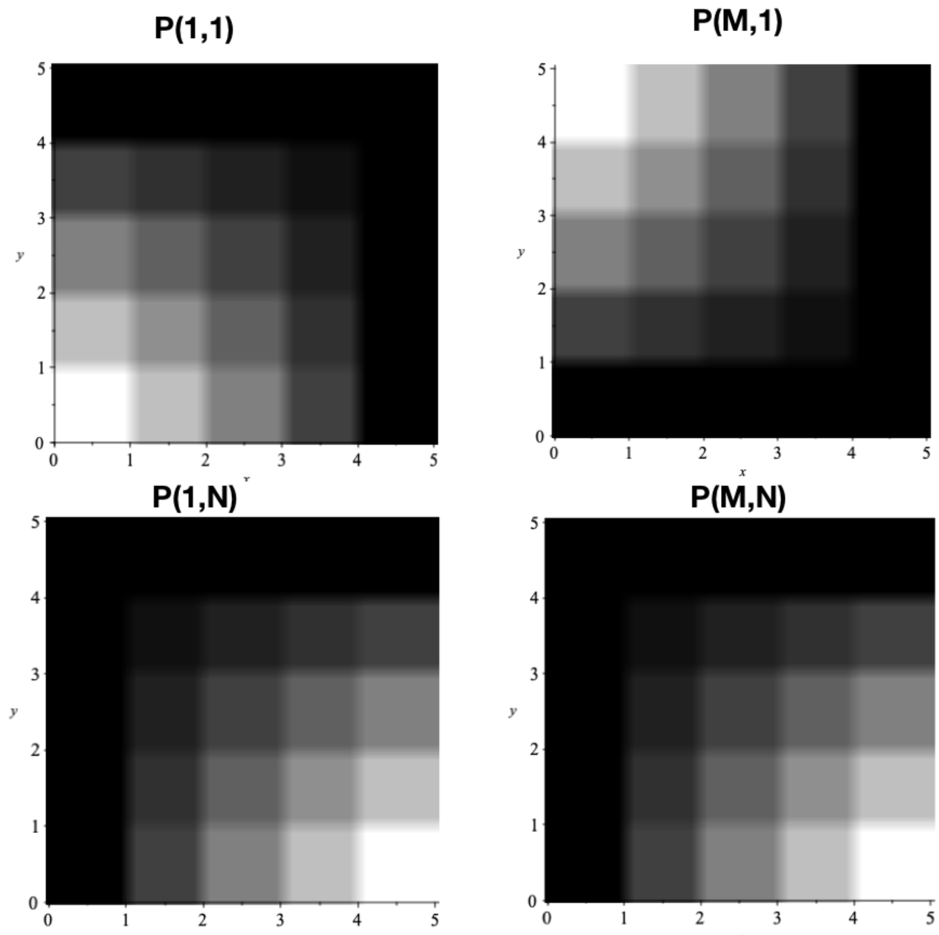

Figure 1: p=1/2, q=1/2

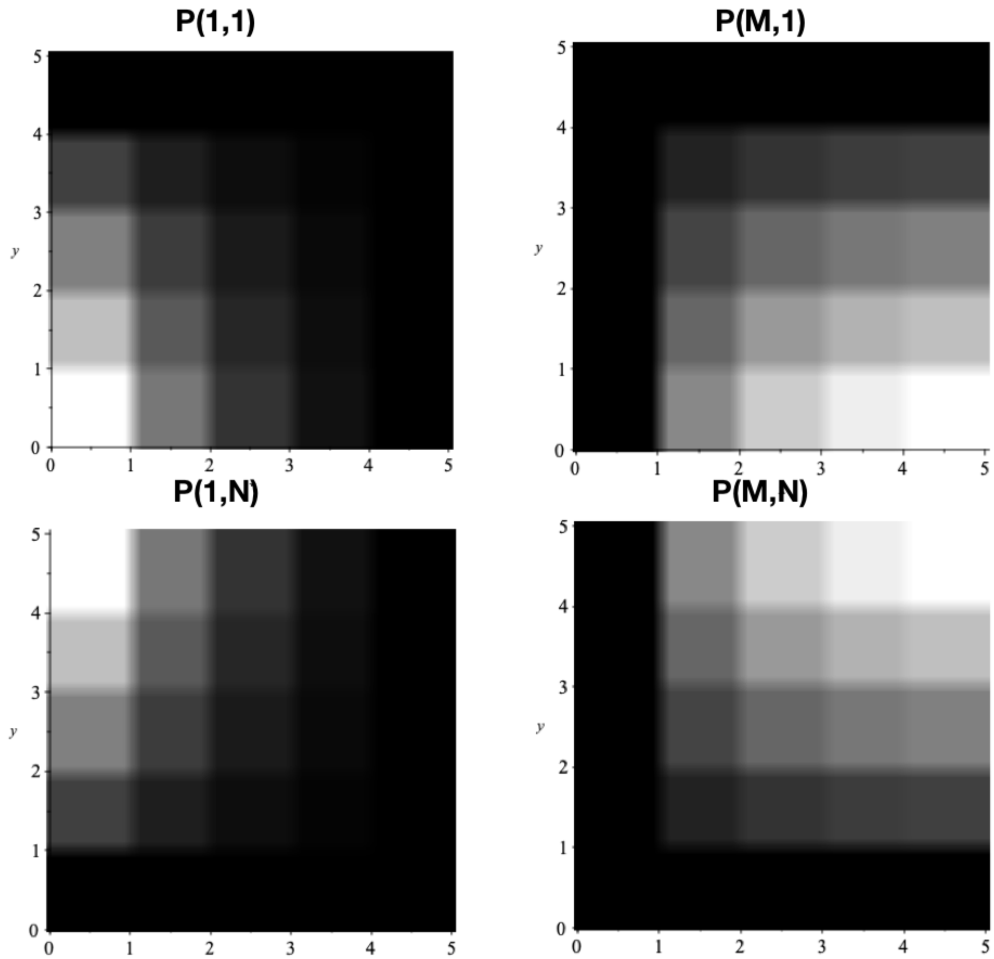

Figure 2: p=1/2, q=2/3

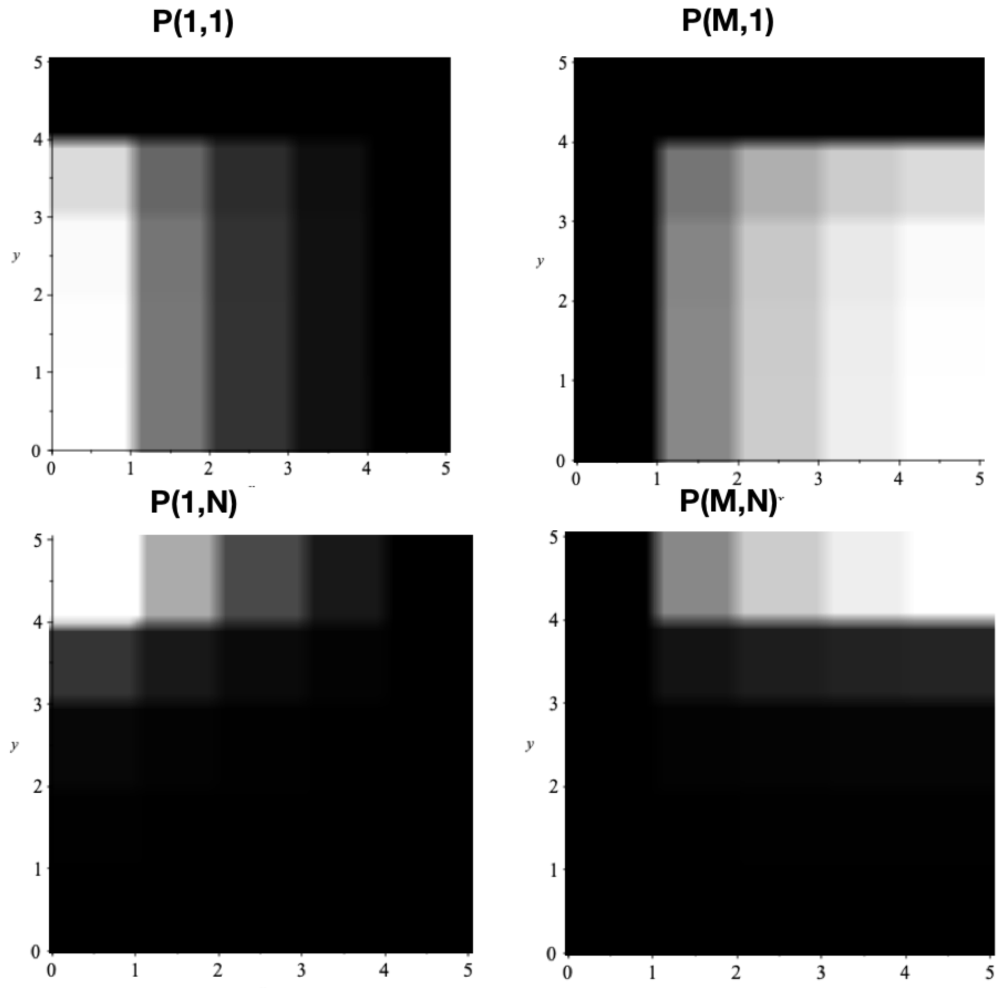

Figure 3: p=1/8, q=2/3

Next we see the expected duration of Game Type 1, Game Type 2, and their differences from each starting point for different p, q. Note that the density plot function will always scale the largest value to white, and smallest values to black, so the difference matrix of Game 2 ED-Game 1ED is also given.

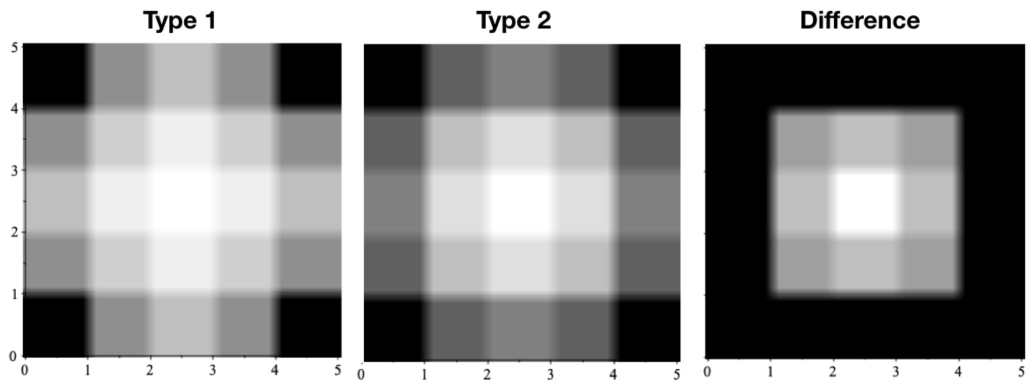

Figure 4: p=1/2, q=1/2

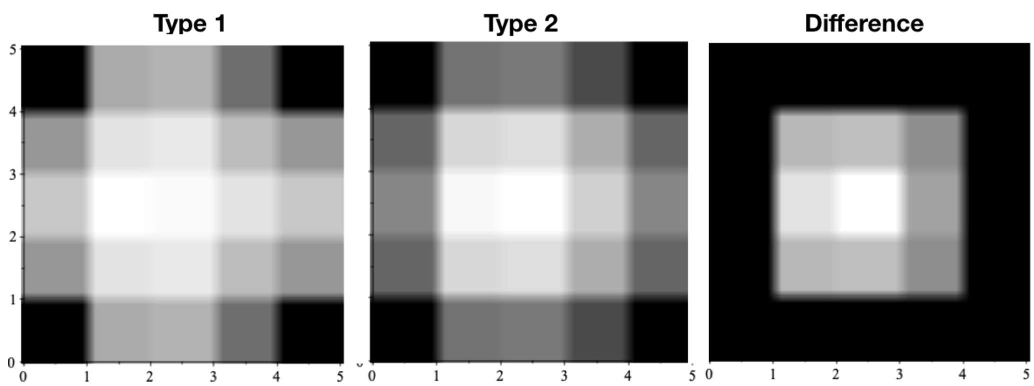

Figure 5: p=1/2, q=2/3

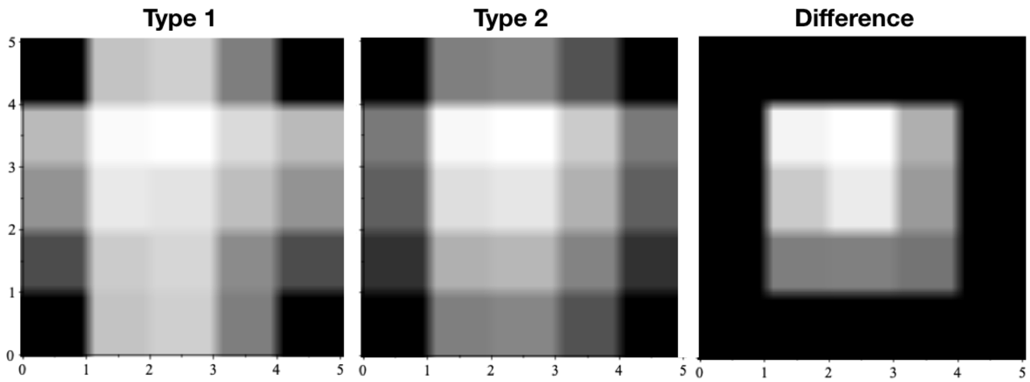

Figure 6: p=1/8, q=2/3

$$D(p=1/2, q=1/2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 5/3 & 2 & 5/3 & 0 \\ 0 & 2 & 8/3 & 2 & 0 \\ 0 & 5/3 & 2 & 5/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D(p=1/2, q=2/3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{13}{7} & \frac{27}{14} & \frac{10}{7} & 0 \\ 0 & \frac{16}{7} & \frac{18}{7} & \frac{23}{14} & 0 \\ 0 & \frac{13}{7} & \frac{27}{14} & \frac{10}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D(p=1/8, q=2/3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{77}{65} & 6/5 & \frac{71}{65} & 0 \\ 0 & \frac{19}{10} & \frac{144}{65} & \frac{29}{20} & 0 \\ 0 & \frac{149}{65} & \frac{12}{5} & \frac{107}{65} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally, we can trace the path at from starting point $[2,3]$, here only using $p = 1/2, q = 1/2$ for Game Type 1 and Game Type 2 with $g = 1/2$, $g = 4/5$.

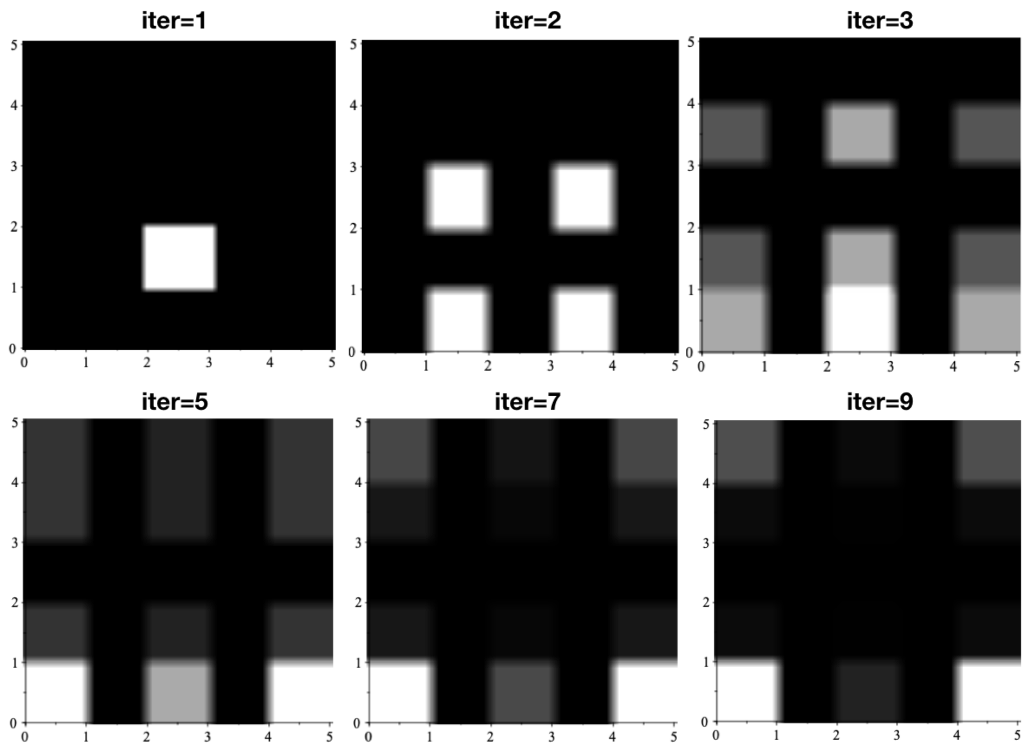

Figure 7: Type 1 Game path, p=1/2, q=1/2

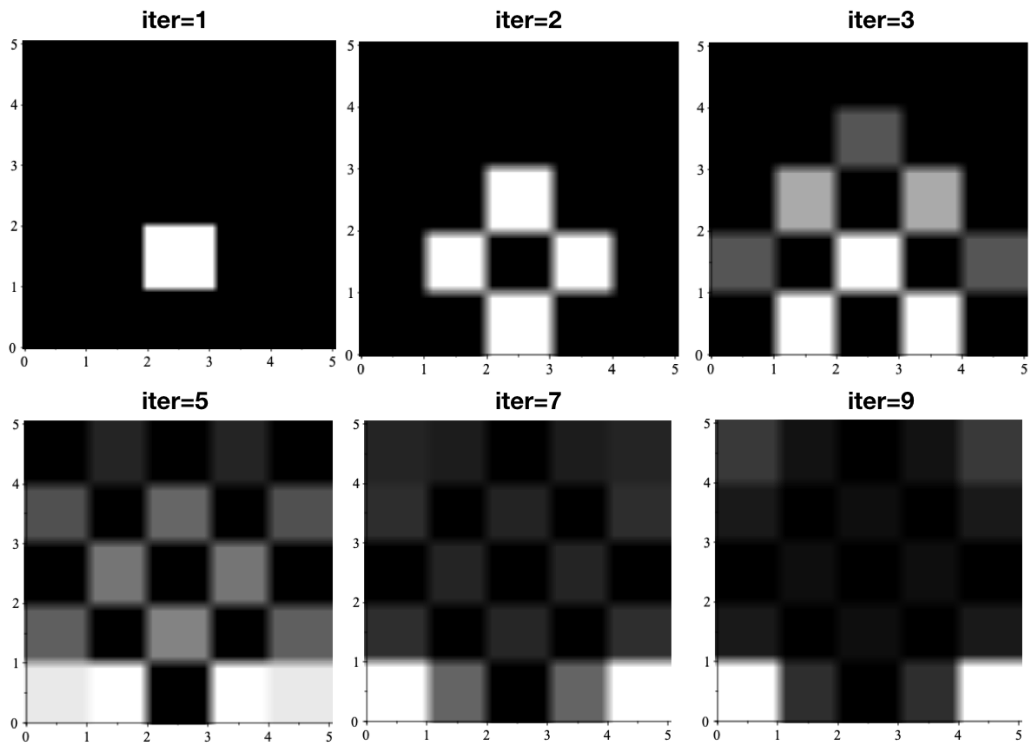

Figure 8: Type 2 Game path, p=1/2, q=1/2, g=1/2

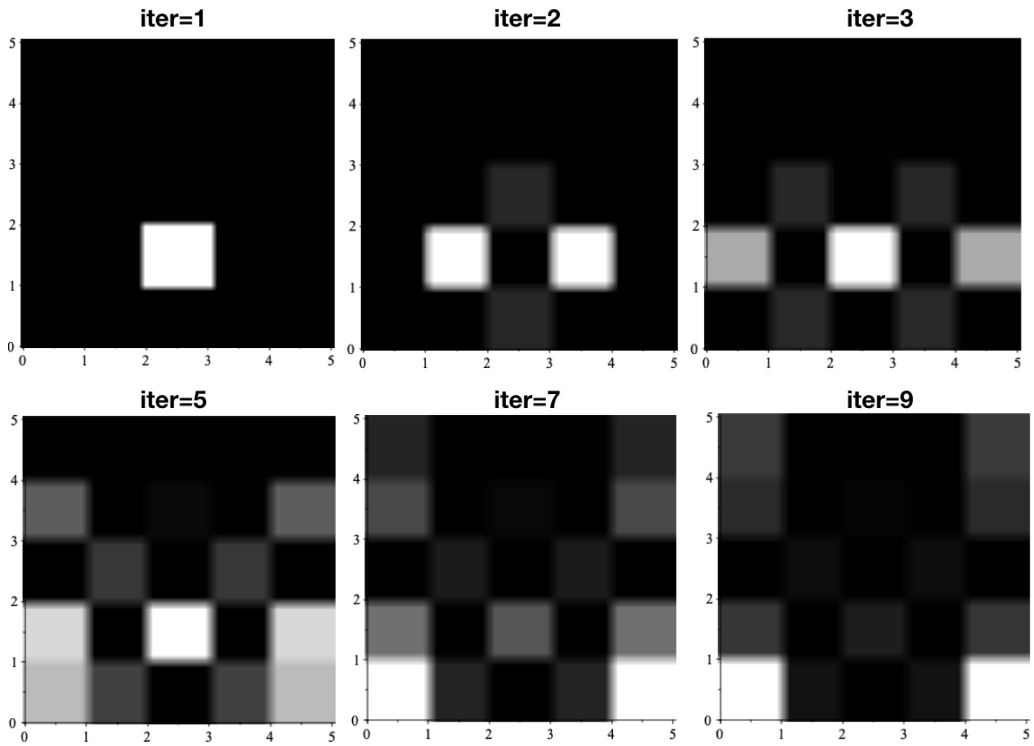

Figure 9: Type 2 Game path, p=1/2, q=1/2, g=4/5

3.3 Large Square Matrix

For the 20×20 matrix representing an equal bank of \$19 for each game, we will look at probabilities for ending in each node for $p = 1/2, q = 2/3$:

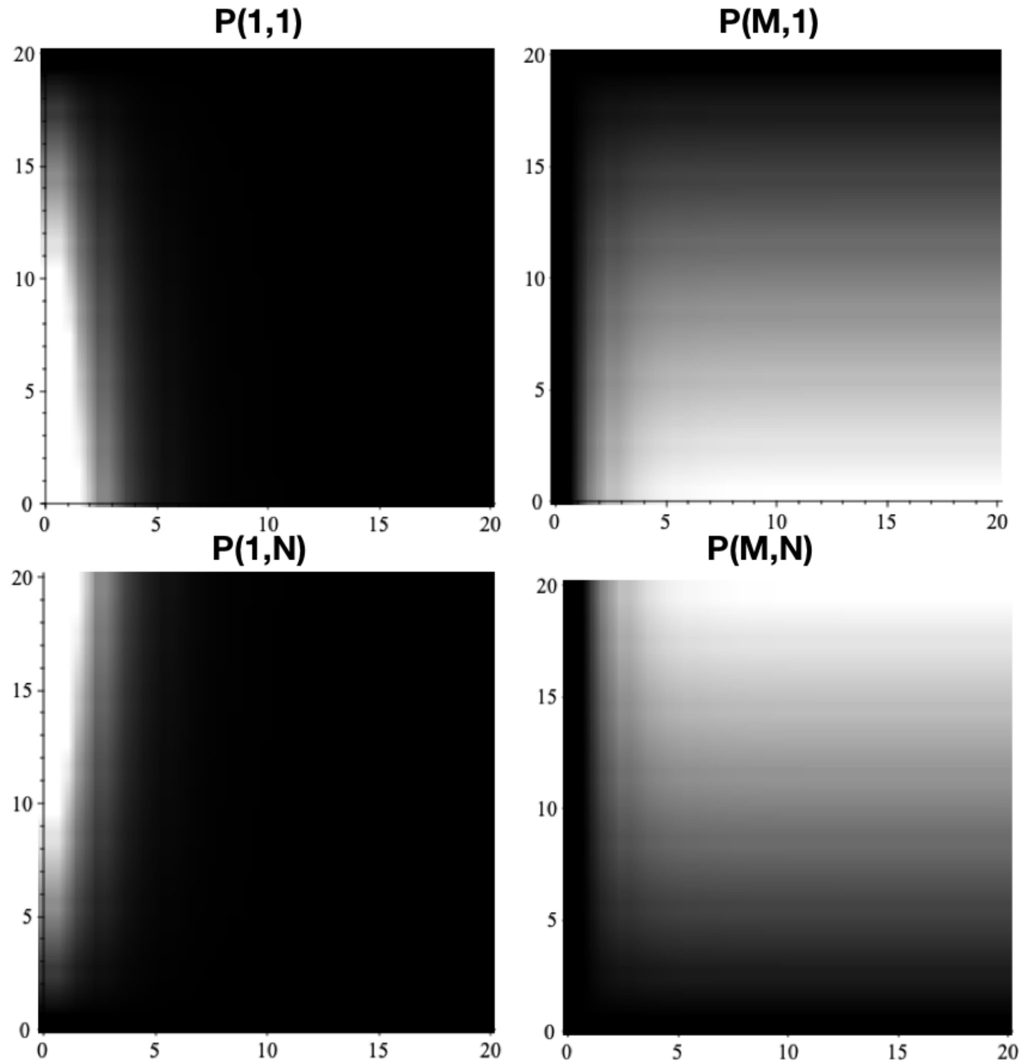

Figure 10: p=1/2, q=2/3

And the expected durations and their differences. Here the matrix is large and unweildy enough that we will not report it.

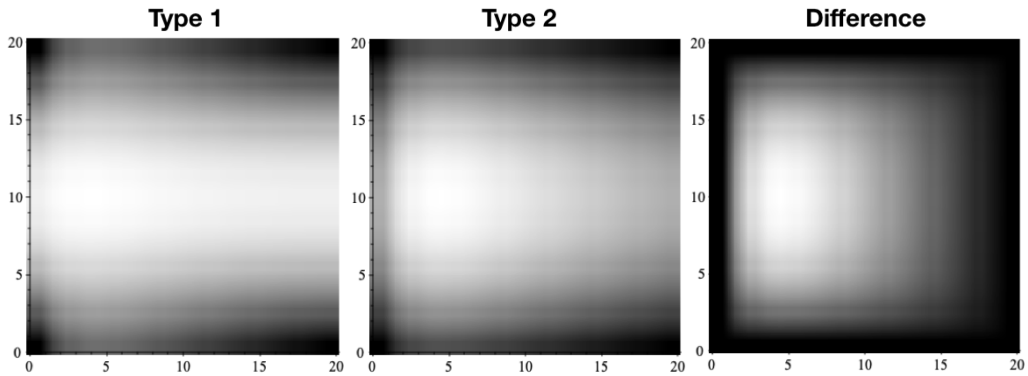

Figure 11: p=1/2, q=2/3

3.3.1 Large Narrow Matrix

For the 20×5 matrix representing a bank of \$19 for the vertical game and \$4 for the horizontal game, we will look at probabilities for ending in each node for $p = 1/2, q = 2/3$:

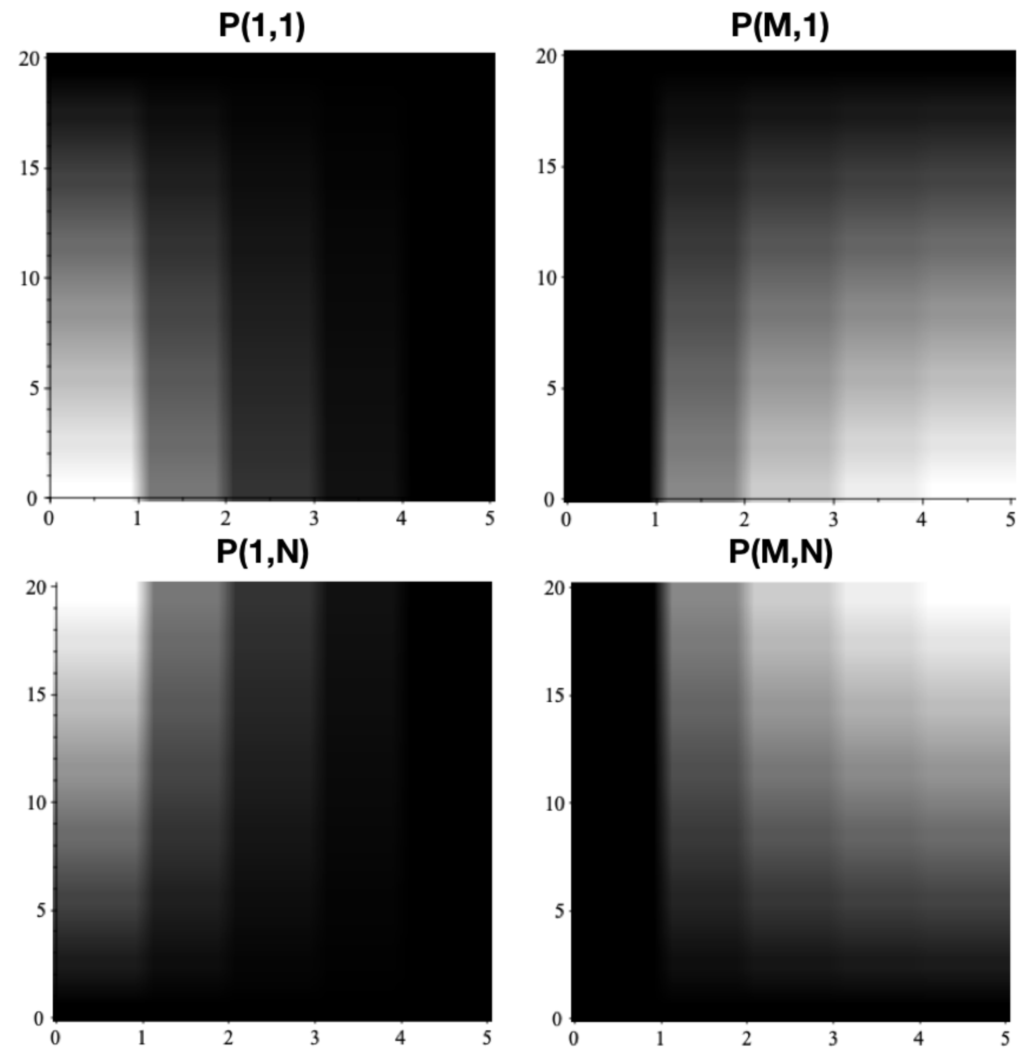

Figure 12: p=1/2, q=2/3

And the expected durations and their differences:

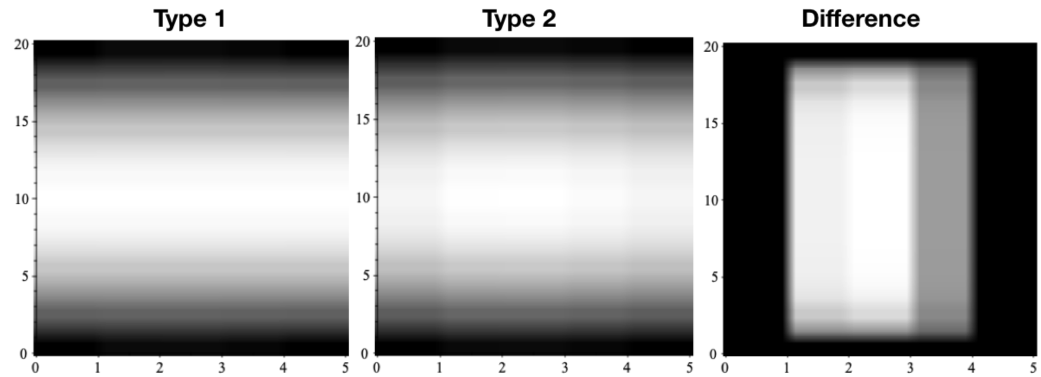

Figure 13: p=1/2, q=2/3

3.4 Symbolic Gambler's Ruin Games

From testing (and as matches the intuition that the two games are independent), it is immediately evident that the probability matrix of winning both games is the outer product of the probability vector of winning the vertical game with the probability vector of winning the horizontal game. This makes the symbolic probability incredibly simple to calculate: we simply calculate separately the symbolic probability V of winning a vertical game of length M with starting bank \$$m$, and H of winning the horizontal game of length N with starting bank \$$n$, then the probability of winning both is VH, of winning the vertical but losing the horizontal is $V(1-H)$, etc. As such, we can fully symbolically calculate the probability of ending in each node:

$$P(1,1) = \frac{\left(\left(\frac{1-p}{p}\right)^M - \left(\frac{1-p}{p}\right)^m\right)\left(\left(\frac{1-q}{q}\right)^N - \left(\frac{1-q}{q}\right)^n\right)}{\left(\left(\frac{1-p}{p}\right)^M - 1\right)\left(\left(\frac{1-q}{q}\right)^N - 1\right)} \tag{3.1}$$

$$P(M,1) = \frac{\left(1 - \left(\frac{1-p}{p}\right)^m\right)\left(\left(\frac{1-q}{q}\right)^n - \left(\frac{1-q}{q}\right)^N\right)}{\left(\left(\frac{1-p}{p}\right)^M - 1\right)\left(\left(\frac{1-q}{q}\right)^N - 1\right)} \tag{3.2}$$

$$P(1,N) = \frac{\left(\left(\frac{1-p}{p}\right)^M - \left(\frac{1-p}{p}\right)^m\right)\left(\left(\frac{1-q}{q}\right)^n - 1\right)}{\left(\left(\frac{1-p}{p}\right)^M - 1\right)\left(\left(\frac{1-q}{q}\right)^N - 1\right)} \tag{3.3}$$

$$P(M,N) = \frac{\left(\left(\frac{1-p}{p}\right)^m - 1\right)\left(\left(\frac{1-q}{q}\right)^n - 1\right)}{\left(\left(\frac{1-p}{p}\right)^M - 1\right)\left(\left(\frac{1-q}{q}\right)^N - 1\right)} \tag{3.4}$$

The expected duration does not nearly map as easily from the 1D to the 2D case. Techniques for solving a multivariate recursion relationship seem to always default to finding the multivariate probability generating function, in which having symbolic boundary conditions and being limited to Maple's single variable rsolve make the problem non-trivial. The boundary row and columns do share the expected values for the 1D game, so clearly the setup should take advantage of the fact that the boundary probability generating function is already calculable in Section 2.

4 High-Dimensional Gambler's Ruin

In this section, we experiment with generalizing Gambler's Ruin for higher dimensions. We begin with the formulation, and then give some results.

4.1 Formulation

The original formulation of Gambler's ruin has a single player A starting with a capital of n units of currency. A repeatedly plays a game by flipping a coin and either earning 1 unit of currency with probability p or loses 1 unit of currency with probability $1 - p$. The game stops when either A wins by gaining N units of currency or loses by having 0 units of currency at the end. We can re-formulate this 1-dimensional game equivalently by treating A as a single point on an interval, whose starting position is n and moves left with probability p or right with probability $1 - p$. The analysis for probability of winning when starting with n units of currency as well as expected duration of the game has been analyzed comprehensively []. A natural extension is: Can we analyze what happens when a single player can choose from different games? Formally, let player A start with a capital of n_k units of currency for the k-th subgame. A repeatedly plays a game by choosing a subgame and flips a coin, earning 1 unit of currency with probability p_k or loses 1 unit of currency with probability $1 - p_k$. A "wins" the k-th subgame when A earns N_k units of currency for that particular sub-game. We focus primarily on what happens when $k = 3$. There are three vectors: the starting capital for each subgame $n = \{n_1, n_2, n_3\}$, the goal capital for each subgame $N = \{N_1, N_2, N_3\}$, and the probability vector for winning 1 unit of currency for each subgame $p = \{p_1, p_2, p_3\}$.

There are a couple of concerns:

- What constitutes a "win" or "loss" for the whole game for A?
- How would A go about choosing a subgame to play?

To answer the first question, consider that A could "win" the whole game either by winning 1 of the games, or by winning every game. We subsequently consider both policies of winning. To answer the second question, suppose A has not won any of the subgames, i.e. the current capital for the k-th subgame is not 0 or N_k. Then, A could randomly choose by picking any of the 3 subgames to play. However, if A has won at least one subgame, say the k-th subgame, then A would not want to risk their current position in the k-th subgame by gambling again. In this case, A would only randomly choose a subgame from $S = \{1, 2, 3\} - \{k\}$, where each subgame is equally likely to be chosen. Likewise, if A had 0 units of currency for a particular subgame, then A could not choose to play that subgame; A can only randomly choose from the other subgames to play. Therefore, how A chooses a subgame depends on the current configuration of A's capital.

4.2 "Win All Games" Policy

If we consider A to "win" only if A wins every subgame, then the only winning configuration of capital for A is (N_1, N_2, N_3). The configuration of capital that makes A lose is $(0, 0, 0)$. Every other configuration of capital (C_1, C_2, C_3) with $0 \leq C_i \leq N_i$ is valid and can move to another configuration of capital. How the configuration moves depends on how A chooses the subgame. In this case, if A wins at least one of the subgames (or if A has zero capital for that particular subgame), then A should not play that subgame anymore.

- Consider the case (C_1, C_2, C_3) such that $0 < C_i < N_i$. Then, the movements to other configurations of capital can be characterized as follows: $(C_1, C_2, C_3) \rightarrow (C_1 + 1, C_2, C_3)$ with probability $\frac{1}{3}p_1$ if the first subgame was chosen (with probability 1/3). $(C_1, C_2, C_3) \rightarrow (C_1 - 1, C_2, C_3)$ with probability $\frac{1}{3}(1 - p_1)$ if the first subgame was chosen. Similar definitions hold if second or third subgame was chosen.

- Consider the case where one of the C_i's is either 0 or N_i. Without loss of generality, let C_1 be either 0 or N_1. Then, A can only choose between two subgames, since the first subgame should not be played (or cannot be played for the case of 0) anymore. So, $(C_1, C_2, C_3) \rightarrow (C_1, C_2 + 1, C_3)$ with probability $\frac{1}{2}p_2$ if the second subgame was chosen (with probability 1/2). $(C_1, C_2, C_3) \rightarrow (C_1, C_2 - 1, C_3)$ with probability $\frac{1}{2}(1 - p_2)$ if the second subgame was chosen. Similar definitions hold if third subgame was chosen.

- Consider the case where two C_i's are either 0 or N_i. Without loss of generality, suppose C_1 be neither 0 nor N_1. Then, $(C_1, C_2, C_3) \rightarrow (C_1 + 1, C_2, C_3)$ with probability p_1. $(C_1, C_2, C_3) \rightarrow (C_1 - 1, C_2, C_3)$ with probability $(1 - p_1)$.

Now, we analyze the probability of winning such a scenario given a starting capital of (n_1, n_2, n_3) and where $(p_1, p_2, p_3) = (p, q, r)$. The recurrence relation for the probability of winning for a particular configuration of capital $P(i, j, k)$ can be given as follows, where U_i represents the boundary value, i.e. either $U_i = 0$ or $U_i = N_i$:

$$\begin{aligned}
P(i,j,k) &= \frac{1}{3}(pP(i+1,j,k) + (1-p)P(i-1,j,k)) \\
&+ \frac{1}{3}(qP(i,j+1,k) + (1-q)P(i,j-1,k)) \\
&+ \frac{1}{3}(rP(i,j,k+1) + (1-r)P(i,j,k-1)) \\
P(i,j,U_3) &= \frac{1}{2}(pP(i+1,j,U_3) + (1-p)P(i-1,j,U_3)) \\
&+ \frac{1}{2}(qP(i,j+1,U_3) + (1-q)P(i,j-1,U_3)) \\
P(i,U_2,k) &= \frac{1}{2}(pP(i+1,U_2,k) + (1-p)P(i-1,U_2,k)) \\
&+ \frac{1}{2}(rP(i,U_2,k+1) + (1-r)P(i,U_2,k-1)) \\
P(U_1,j,k) &= \frac{1}{2}(qP(U_1,j+1,k) + (1-q)P(U_1,j-1,k)) \\
&+ \frac{1}{2}(rP(U_1,j,k+1) + (1-r)P(U_1,j,k-1)) \\
P(i,U_2,U_3) &= (pP(i+1,U_2,U_3) + (1-p)P(i-1,U_2,U_3)) \\
P(U_1,j,U_3) &= (qP(U_1,j+1,U_3) + (1-q)P(U_1,j-1,U_3)) \\
P(U_1,U_2,k) &= (rP(U_1,U_2,k+1) + (1-r)P(U_1,U_2,k-1)) \\
P(0,0,0) &= 0, \quad P(N_1,N_2,N_3) = 1
\end{aligned}$$

For the case where $(n_1, n_2, n_3) = (4, 4, 4)$, and $(p_1, p_2, p_3) = (.5, .5, .5)$, here are some values for the probability of winning:

$$\begin{aligned}
P(1,2,3) &= 0.09375 \\
P(2,2,3) &= 0.1875 \\
P(2,3,3) &= 0.28125
\end{aligned}$$

Now, we analyze the expected duration E for such a scenario given a starting capital of (n_1, n_2, n_3) and where $(p_1, p_2, p_3) = (p, q, r)$. The recurrence relation $E(i, j, k)$ for the expected duration with current capital (i, j, k) is given as follows, again where either $U_i = 0$ or $U_i = N_i$:

$$\begin{aligned}
E(i,j,k) &= \frac{1}{3}(pE(i+1,j,k) + (1-p)E(i-1,j,k)) \\
&+ \frac{1}{3}(qE(i,j+1,k) + (1-q)E(i,j-1,k)) \\
&+ \frac{1}{3}(rE(i,j,k+1) + (1-r)E(i,j,k-1)) + 1 \\
E(i,j,U_3) &= \frac{1}{2}(pE(i+1,j,U_3) + (1-p)E(i-1,j,U_3)) \\
&+ \frac{1}{2}(qE(i,j+1,U_3) + (1-q)E(i,j-1,U_3)) + 1 \\
E(i,U_2,k) &= \frac{1}{2}(pE(i+1,U_2,k) + (1-p)E(i-1,U_2,k)) \\
&+ \frac{1}{2}(rE(i,U_2,k+1) + (1-r)E(i,U_2,k-1)) + 1 \\
E(U_1,j,k) &= \frac{1}{2}(qE(U_1,j+1,k) + (1-q)E(U_1,j-1,k)) \\
&+ \frac{1}{2}(rE(U_1,j,k+1) + (1-r)E(U_1,j,k-1)) + 1 \\
E(i,U_2,U_3) &= (pE(i+1,U_2,U_3) + (1-p)E(i-1,U_2,U_3)) + 1 \\
E(U_1,j,U_3) &= (qE(U_1,j+1,U_3) + (1-q)E(U_1,j-1,U_3)) + 1 \\
E(U_1,U_2,k) &= (rE(U_1,U_2,k+1) + (1-r)E(U_1,U_2,k-1)) + 1 \\
E(0,0,0) &= 0, \quad E(N_1,N_2,N_3) = 0
\end{aligned}$$

For the case where $(n_1, n_2, n_3) = (4, 4, 4)$, and $(p_1, p_2, p_3) = (.5, .5, .5)$, here are some values for the probability of winning:

$$\begin{aligned}
E(1,2,3) &= 10 \\
E(2,2,3) &= 11 \\
E(2,3,3) &= 10
\end{aligned}$$

The Maple code for these programs are given as GWinProbAll, and GExpDurAll. There is also a maple code for providing the probability generating function for the expected duration, given as GPGFExpDurAll. For all three maple programs, the probabilities can either be given as a number, or a symbol.

4.3 "Win One Game" Policy

In this case, we consider A to "win" if A wins one of the subgames. The configuration of capital that makes A lose is $(0,0,0)$. However, now the configuration of capital that makes A win is any triple containing either N_1, N_2, or N_3. Every other configuration of capital (C_1, C_2, C_3) with $0 \leq C_i < N_i$ is valid and can move to another configuration of capital. How the configuration moves depends on how A chooses the subgame, similar to the "win all games" policy, except now there is more restriction, since winning at least one subgame means A exits the whole game.

- Consider the case (C_1, C_2, C_3) such that $0 < C_i < N_i$. Then, the movements to other configurations of capital can be characterized as follows: $(C_1, C_2, C_3) \to (C_1 + 1, C_2, C_3)$ with probability $\frac{1}{3}p_1$ if the first subgame was chosen (with probability 1/3). $(C_1, C_2, C_3) \to (C_1 - 1, C_2, C_3)$ with probability $\frac{1}{3}(1 - p_1)$ if the first subgame was chosen. Similar definitions hold if second or third subgame was chosen.
- Consider the case where one or two of the C_i's is 0. Then, A can only choose either two or one subgame, respectively. The update is similar to "win all games" policy.
- Consider the case where one or two of the C_i's are N_i. In this case, A has already won, so there is no need to move to another configuration of capital. Instead, A would want to exit.

Now, we analyze the probability of winning in the "win one game" policy given a starting capital of (n_1, n_2, n_3) and where $(p_1, p_2, p_3) = (p, q, r)$. If any configuration includes any of N_i, then the probability of winning is 1 since A has already won the game. This is reflected in the new recurrence relation, and it makes A's game much easier to play. The recurrence relation for the probability of winning for a particular configuration of capital $P(i, j, k)$ can be given as follows:

$$
\begin{aligned}
P(i,j,k) &= \frac{1}{3}(pP(i+1,j,k) + (1-p)P(i-1,j,k)) \\
&+ \frac{1}{3}(qP(i,j+1,k) + (1-q)P(i,j-1,k)) \\
&+ \frac{1}{3}(rP(i,j,k+1) + (1-r)P(i,j,k-1)) \\
P(i,j,0) &= \frac{1}{2}(pP(i+1,j,0) + (1-p)P(i-1,j,0)) \\
&+ \frac{1}{2}(qP(i,j+1,0) + (1-q)P(i,j-1,0)) \\
P(i,j,N_3) &= 1 \\
P(i,0,k) &= \frac{1}{2}(pP(i+1,0,k) + (1-p)P(i-1,0,k)) \\
&+ \frac{1}{2}(rP(i,0,k+1) + (1-r)P(i,0,k-1)) \\
P(i,N_2,k) &= 1 \\
P(0,j,k) &= \frac{1}{2}(qP(0,j+1,k) + (1-q)P(0,j-1,k)) \\
&+ \frac{1}{2}(rP(0,j,k+1) + (1-r)P(0,j,k-1)) \\
P(N_1,j,k) &= 1 \\
P(i,0,0) &= (pP(i+1,0,0) + (1-p)P(i-1,0,0)) \\
P(i,N_2,N_3) &= 1 \\
P(0,j,0) &= (qP(0,j+1,0) + (1-q)P(0,j-1,0)) \\
P(N_1,j,N_3) &= 1 \\
P(0,0,k) &= (rP(0,0,k+1) + (1-r)P(0,0,k-1)) \\
P(N_1,N_2,k) &= 1 \\
P(0,0,0) &= 0, \quad P(N_1,N_2,N_3) = 1
\end{aligned}
$$

For the case where $(n_1, n_2, n_3) = (4, 4, 4)$, and $(p_1, p_2, p_3) = (.5, .5, .5)$, here are some values for the probability of winning:

$$
\begin{aligned}
P(1,2,3) &= 0.90625 \\
P(2,2,3) &= 0.9375 \\
P(2,3,3) &= 0.96875
\end{aligned}
$$

Now, we analyze the expected duration E for such a scenario given a starting capital of (n_1, n_2, n_3) and where $(p_1, p_2, p_3) = (p, q, r)$. The recurrence relation $E(i, j, k)$ for the expected duration with current capital (i, j, k) is given as follows:

$$\begin{aligned}
E(i,j,k) &= \frac{1}{3}(pE(i+1,j,k) + (1-p)E(i-1,j,k)) \\
&\quad + \frac{1}{3}(qE(i,j+1,k) + (1-q)E(i,j-1,k)) \\
&\quad + \frac{1}{3}(rE(i,j,k+1) + (1-r)E(i,j,k-1)) + 1 \\
E(i,j,0) &= \frac{1}{2}(pE(i+1,j,0) + (1-p)E(i-1,j,0)) \\
&\quad + \frac{1}{2}(qE(i,j+1,0) + (1-q)E(i,j-1,0)) + 1 \\
E(i,j,N_3) &= 0 \\
E(i,0,k) &= \frac{1}{2}(pE(i+1,0,k) + (1-p)E(i-1,0,k)) \\
&\quad + \frac{1}{2}(rE(i,0,k+1) + (1-r)E(i,0,k-1)) + 1 \\
E(i,N_2,k) &= 0 \\
E(0,j,k) &= \frac{1}{2}(qE(0,j+1,k) + (1-q)E(0,j-1,k)) \\
&\quad + \frac{1}{2}(rE(0,j,k+1) + (1-r)E(0,j,k-1)) + 1 \\
E(N_1,j,k) &= 0 \\
E(i,0,0) &= (pE(i+1,0,0) + (1-p)E(i-1,0,0)) + 1 \\
E(i,N_2,N_3) &= 0 \\
E(0,j,0) &= (qE(0,j+1,0) + (1-q)E(0,j-1,0)) + 1 \\
E(N_1,j,N_3) &= 0 \\
E(0,0,k) &= (rE(0,0,k+1) + (1-r)E(0,0,k-1)) + 1 \\
E(N_1,N_2,k) &= 0 \\
E(0,0,0) &= 0, \quad E(N_1,N_2,N_3) = 0
\end{aligned}$$

For the case where $(n_1, n_2, n_3) = (4, 4, 4)$, and $(p_1, p_2, p_3) = (.5, .5, .5)$, here are some values for the probability of winning:

$$E(1,2,3) = 5.5593$$
$$E(2,2,3) = 5.4526$$
$$E(2,3,3) = 3.9932$$

The Maple code for these programs are given as GWinProbOne, and GExpDurOne. There is also a maple code for providing the probability generating function for the expected duration, given as GPGFExpDurOne. For all three maple programs, the probabilities can either be given as a number, or a symbol.

4.4 Extensions

A first extension would be to get symbolic results for symbolic N_1, N_2, ...etc. It is clear that if player A follows the "win one game" policy, then not only is A's chances of winning much higher, but A should also need to play fewer subgames to win. However, one extension could be figuring out what happens with "win any two games" policy. How close would the probability of winning be to "win one game" policy? Lastly, I would like to explore whether or not there is a reduction from k subgames to any $v < k$ subgames. If there is, then generalizing Gambler's Ruin in this way to k subgames can be done with v subgames.

5 Gambler's Ruin on Periodic Graph

5.1 Ring Graph with Numerical Parameters

From our Maple package, we have plenty of results for gambler's ruin on single path graph. There, if you currently have i dollars, then with probability p next round you would have $i+1$ dollars and with probability 1-p next time you would have $i-1$ dollars. The absorbing state is just the two ends, 0 dollar and N dollars. Now, consider a cycle graph with N vertices $v_0, v_1, v_2, \ldots, v_{N-1}$. If we play the similar game here, which means on state v_i, you have probability p to arrive v_{i+1} or v_0 if $i+1 = N$ and probability $1-p$ to arrive v_{i-1} , also the game ends for state v_0. Then nothing changes, because we still move around from 0 to N.

However, things become different if we can move farther to get across absorbing points. For convenience, we use $\mathbb{Z}_N$ to label our states or vertices. Here is the idea. You win j dollars with probability $p[j]$. On the graph, it means on state v_i, you arrive v_{i+j} with probability $p[j]$. Also, you leave if you arrive v_0. This game is both interesting and realistic. For example, if you have some casino tokens, and you can either gamble with them or exchange 10 tokens with cash. Then, you do not want to leave if you do not have exact $10k$ token, because 5 tokens are useless to exchange for anything unless you keep gambling.

For coding, this is just a special case for random walk on a complete graph. Just give a transition matrix A such that $A[i][j]$ is the probability of going from i to j. Then by using the solve function in Maple, with respect of starting point, we can give the probability of wining, the expectation of duration and probability generating functions for the duration. The code is attached.

For example, if $N = 6$ and we have probability $1/4$ to move $\pm 1, \pm 2$, then the matrix A is

$$\begin{pmatrix} 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \end{pmatrix}$$

Using the whole transition matrix can always solve random walk on any graphs. However, the shortage of this code is we have to give a specific transition matrix A. Then we can only use numerical N instead symbolic N.

5.2 Ring Graph with Symbolic parameters

For symbolic N, here is an interesting problem that we can experiment with coding. Again, we are working on a cycle with $\mathbb{Z}_N$ labeling. If we start at n, for each round, you get $+2$ with probability p_1, get $+1$ with probability p_2, get -1 with probability p_3 and get -2 with probability p_4. The goal is arriving at exactly 0. Of course, $p_1 + p_2 + p_3 + p_4 = 1$. Here is the problem, what is the expectation of the duration of this game.

By generalization of the code for gambler's ruin, we can give the closed form of this expectation. `ExpDurationR(N,n,p1,p2,p3,p4)` means the expectation of duration, with symbolic inputs N, n. Then we can plug in different parameters to find patterns. For example, we let $N = 40$, and n goes from 0 to 40, then we can give the line chart with three probability sets.

- Data Set A: $p_1 = \frac{1}{4}, p_2 = \frac{1}{4}, p_3 = \frac{1}{4}, p_4 = \frac{1}{4}$
- Data Set B: $p_1 = \frac{1}{6}, p_2 = \frac{1}{3}, p_3 = \frac{1}{3}, p_4 = \frac{1}{6}$
- Data Set C: $p_1 = \frac{1}{3}, p_2 = \frac{1}{6}, p_3 = \frac{1}{6}, p_4 = \frac{1}{3}$

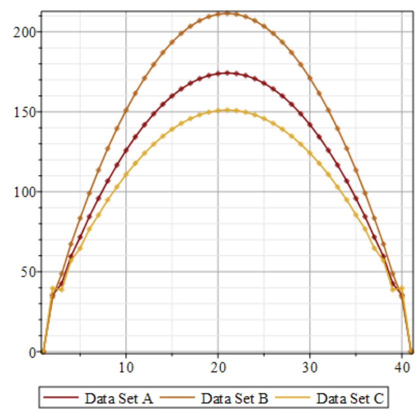

Figure 14: ExpDurationR with N=40

From the data above, we can find relation between expectation of duration and probability sets. To clarify this, we can let $N = 40$, $n = 20$. Then use the probability.

- Data Set i: $p_1 = \frac{i}{80}, p_2 = \frac{40-i}{80}, p_3 = \frac{40-i}{80}, p_4 = \frac{i}{80}$

The surprising right end of figure above is really interesting. Also, if we just focus on this situation, it gives the following beautiful figure.

5.3 Torus Graph

A natural generalization of ring graphs is torus lattice. The game here is similar with above. Consider a torus lattice with C_M vertically times C_N

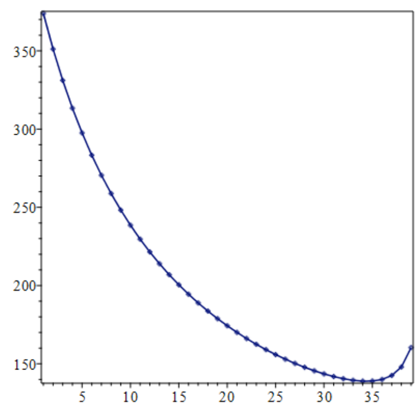

Figure 15: Relation between i and ExpDurationR(40,20,Data Set i)

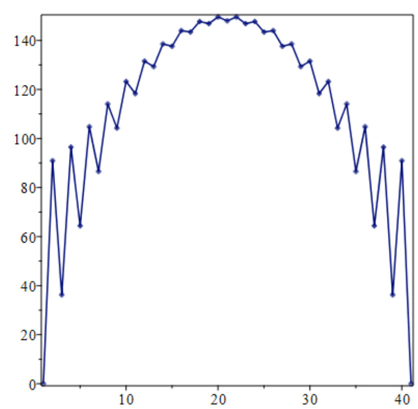

Figure 16: ExpDurationR(40,i,19/40, 1/40, 1/40, 19/40)

horizontally. For each step it has probability p to go right, probability q to go up, probability r to go left, probability s to go down. Also, we may have set of winning sites and set of losing sites. Then, we are interested in probability of winning, expectation of duration and probability generating function of duration.

For symbolic inputs of M, N and initial position, the previous idea of giving closed form of result does not work here. However, we can still use the numerical parameters to give probability of winning, expectation of duration and probability generating function of duration.

The functions `ProbWinningTR`, `ExpDurationTR` and `PGFTR` work for questions above. The (i, j) entry is the result if we start at (i,j) position. If we plot an example ExpDurationTR(20, 20, 1/4, 1/4, 1/4, 1/4, [[1, 1]], [[10, 10]]). We can get

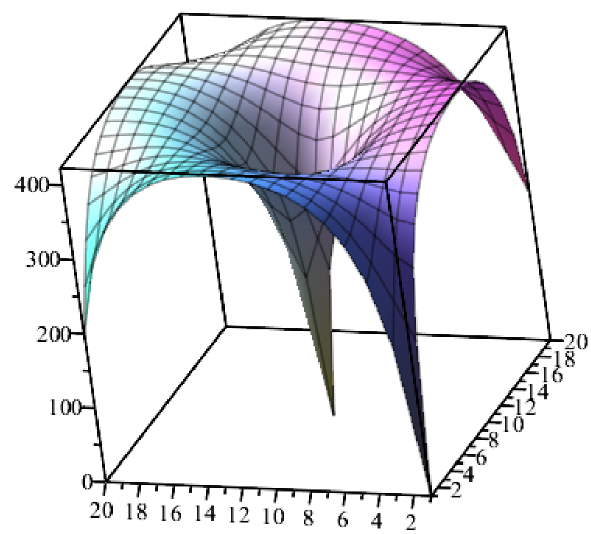

Figure 17: ExpDurationTR(20, 20, 1/4, 1/4, 1/4, 1/4, [[1, 1]], [[10, 10]])

and `ProbWinningTR(20, 20, 1/4, 1/4, 1/4, 1/4, [[1, 1]], [[10, 10]])`.

The idea here can also work on toroidal rectangle and higher dimensional lattice. But the key point is the same: give the transition matrix according to the problem, then use the solve function to solve the equation from transition.

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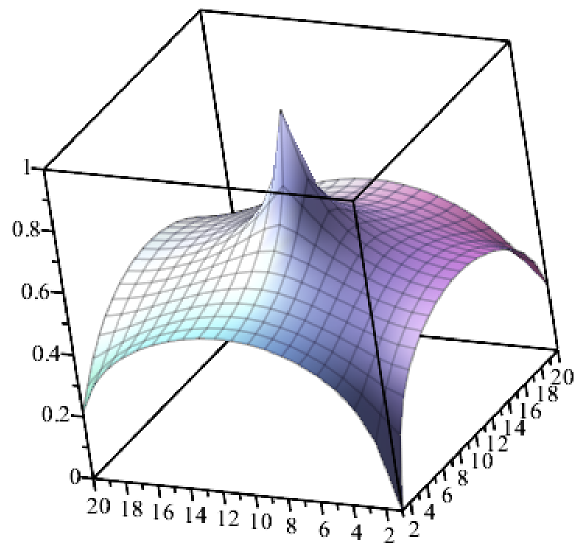

Figure 18: ProbWinningTR(20, 20, 1/4, 1/4, 1/4, 1/4, [[1, 1]], [[10, 10]])

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