

Computing the Generating Function of a C-finite sequence

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Input: A C -finite sequence $a(n)$ that is a solution of a (homogeneous) **linear recurrence equation with constnt coefficient**

$$a(n) = R_1 a(n-1) + \dots + R_d a(n-d) \quad ,$$

subject to the initial conditions

$$a(0) = a_0, \dots, a(d-1) = a_{d-1} \quad .$$

Note: this ‘infinite’ sequence is a **finite** object, and we represent it in class as a pair of lists [INI, REC] where

$$INI = [a_0, \dots, a_{d-1}] \quad , \quad REC = [R_1, \dots, R_d] \quad .$$

The (ordinary) generating function of a sequence $a(n), 0 \leq n < \infty$ is, by definition

$$f(x) = \sum_{n=0}^{\infty} a(n)x^n \quad .$$

The **output** is the generating function that is always **rational** as explained below.

How to go from the C -finite description to the generating function?

For $n \geq d$ we have

$$a(n) = R_1 a(n-1) + \dots + R_d a(n-d) \quad ,$$

Hence

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a(n)x^n = \sum_{n=0}^{d-1} a(n)x^n + \sum_{n=d}^{\infty} a(n)x^n \\ &= \sum_{n=0}^{d-1} a(n)x^n + \sum_{n=d}^{\infty} (R_1 a(n-1) + R_2 a(n-2) + \dots + R_d a(n-d)) x^n \quad . \end{aligned}$$

Distributing, we have

$$\begin{aligned} f(x) &= \sum_{n=0}^{d-1} a(n)x^n \\ &+ \sum_{n=d}^{\infty} R_1 a(n-1)x^n + \sum_{n=d}^{\infty} R_2 a(n-2)x^n + \dots + \sum_{n=d}^{\infty} R_d a(n-d)x^n \quad . \end{aligned}$$

Rewritng: we get

$$f(x) = \sum_{n=0}^{d-1} a(n)x^n + R_1x \sum_{n=d}^{\infty} a(n-1)x^{n-1} + R_2x^2 \sum_{n=d}^{\infty} a(n-2)x^{n-2} + \dots + R_dx^d \sum_{n=d}^{\infty} a(n-d)x^{n-d} .$$

Changing the index of summation we get

$$f(x) = \sum_{n=0}^{d-1} a(n)x^n + R_1x \sum_{n=d-1}^{\infty} a(n)x^n + R_2x^2 \sum_{n=d-2}^{\infty} a(n)x^n + \dots + R_dx^d \sum_{n=0}^{\infty} a(n)x^n .$$

It follows that

$$f(x) = POLYNOMIAL_{d-1}(x) + (R_1x + \dots + R_dx^d)f(x)$$

where $POLYNOMIAL_{d-1}(x)$ is **some** polynomial of degree $d-1$. Hence

$$f(x)(1 - (R_1x + \dots + R_dx^d)) = POLYNOMIAL_{d-1}(x) ,$$

and hence

$$f(x) = \frac{POLYNOMIAL_{d-1}(x)}{(1 - R_1x - \dots - R_dx^d)} .$$

This explains why $f(x)$ is **always** a rational function, and why its denominator is what it is. As for the numerator, the best way is to find it empirically as we did in class.

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