Newman's Short Proof of the Prime Number Theorem

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Dedicated to the Prime Number Theorem on the occasion of its 100th birthday

The prime number theorem, that the number of primes $\leq x$ is asymptotic to $x/\log x$, was proved (independently) by Hadamard and de la Vallée Poussin in 1896. Their proof had two elements: showing that Riemann's zeta function $\zeta(s)$ has no zeros with $\Re(s) = 1$, and deducing the prime number theorem from this. An ingenious short proof of the first assertion was found soon afterwards by the same authors and by Mertens and is reproduced here, but the deduction of the prime number theorem continued to involve difficult analysis. A proof that was elementary in a technical sense—it avoided the use of complex analysis—was found in 1949 by Selberg and Erdös, but this proof is very intricate and much less clearly motivated than the analytic one. A few years ago, however, D. J. Newman found a very simple version of the Tauberian argument needed for an analytic proof of the prime number theorem. We describe the resulting proof, which has a beautifully simple structure and uses hardly anything beyond Cauchy's theorem.

Recall that the notation $f(x) \sim g(x)$ ("f and g are asymptotically equal") means that $\lim_{x\to\infty} f(x)/g(x) = 1$, and that $O(f)$ denotes a quantity bounded in absolute value by a fixed multiple of f. We denote by $\pi(x)$ the number of primes $\leq x$.

Prime Number Theorem. $\pi(x) \sim \dfrac{x}{\log x}$ as $x \to \infty$.

We present the argument in a series of steps. Specifically, we prove a sequence of properties of the three functions

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Phi(s) = \sum_p \frac{\log p}{p^s}, \quad \vartheta(x) = \sum_{p \leq x} \log p \quad (s \in \mathbb{C},\ x \in \mathbb{R});$$

we always use p to denote a prime. The series defining $\zeta(s)$ (the Riemann zeta-function) and $\Phi(s)$ are easily seen to be absolutely and locally uniformly convergent for $\Re(s) > 1$, so they define holomorphic functions in that domain.

(I). $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ *for* $\Re(s) > 1$.

Proof: From unique factorization and the absolute convergence of $\zeta(s)$ we have

$$\zeta(s) = \sum_{r_2, r_3, \cdots \geq 0} (2^{r_2} 3^{r_3} \cdots)^{-s} = \prod_p \Big(\sum_{r \geq 0} p^{-rs} \Big) = \prod_p \frac{1}{1 - p^{-s}} \quad (\Re(s) > 1).$$

(II). $\zeta(s) - \dfrac{1}{s-1}$ *extends holomorphically to* $\Re(s) > 0$.

Proof: For $\Re(s) > 1$ we have

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s}\, dx = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx.$$

The series on the right converges absolutely for $\Re(s) > 0$ because

$$\left| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \right| = \left| s \int_n^{n+1} \int_n^x \frac{du}{u^{s+1}}\, dx \right| \leq \max_{n \leq u \leq n+1} \left| \frac{s}{u^{s+1}} \right| = \frac{|s|}{n^{\Re(s)+1}}$$

by the mean value theorem.

(III). $\vartheta(x) = \mathrm{O}(x)$.

Proof: For $n \in \mathbb{N}$ we have

$$2^{2n} = (1+1)^{2n} = \binom{2n}{0} + \cdots + \binom{2n}{2n} \geq \binom{2n}{n} \geq \prod_{n<p\leq 2n} p = e^{\vartheta(2n)-\vartheta(n)}$$

and hence, since $\vartheta(x)$ changes by $\mathrm{O}(\log x)$ if x changes by $\mathrm{O}(1)$, $\vartheta(x) - \vartheta(x/2) \leq Cx$ for any $C > \log 2$ and all $x \geq x_0 = x_0(C)$. Summing this over $x, x/2, \dots, x/2^r$, where $x/2^r \geq x_0 > x/2^{r+1}$, we obtain $\vartheta(x) \leq 2Cx + \mathrm{O}(1)$.

(IV). $\zeta(s) \neq 0$ *and* $\Phi(s) - 1/(s-1)$ *is holomorphic for* $\Re(s) \geq 1$.

Proof: For $\Re(s) > 1$, the convergent product in (I) implies that $\zeta(s) \neq 0$ and that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1} = \Phi(s) + \sum_p \frac{\log p}{p^s(p^s - 1)}.$$

The final sum converges for $\Re(s) > \frac{1}{2}$, so this and (II) imply that $\Phi(s)$ extends meromorphically to $\Re(s) > \frac{1}{2}$, with poles only at $s = 1$ and at the zeros of $\zeta(s)$, and that, if $\zeta(s)$ has a zero of order μ at $s = 1 + i\alpha$ ($\alpha \in \mathbb{R}$, $\alpha \neq 0$) and a zero of order ν at $1 + 2i\alpha$(so $\mu, \nu \geq 0$ by (II)), then

$$\lim_{\epsilon \searrow 0} \epsilon \Phi(1+\epsilon) = 1,\ \lim_{\epsilon \searrow 0} \epsilon \Phi(1+\epsilon \pm i\alpha) = -\mu, \text{ and } \lim_{\epsilon \searrow 0} \epsilon \Phi(1+\epsilon \pm 2i\alpha) = -\nu.$$

The inequality

$$\sum_{r=-2}^{2} \binom{4}{2+r} \Phi(1+\epsilon+ir\alpha) = \sum_p \frac{\log p}{p^{1+\epsilon}} \left(p^{i\alpha/2} + p^{-i\alpha/2}\right)^4 \geq 0$$

then implies that $6 - 8\mu - 2\nu \geq 0$, so $\mu = 0$, i.e., $\zeta(1+i\alpha) \neq 0$.

(V). $\int_1^{\infty} \frac{\vartheta(x) - x}{x^2}\, dx$ *is a convergent integral.*

Proof: For $\Re(s) > 1$ we have

$$\Phi(s) = \sum_p \frac{\log p}{p^s} = \int_1^{\infty} \frac{d\vartheta(x)}{x^s} = s\int_1^{\infty} \frac{\vartheta(x)}{x^{s+1}}\, dx = s\int_0^{\infty} e^{-st}\vartheta(e^t)\, dt.$$

Therefore (V) is obtained by applying the following theorem to the two functions $f(t) = \vartheta(e^t)e^{-t} - 1$ and $g(z) = \Phi(z+1)/(z+1) - 1/z$, which satisfy its hypotheses by (III) and (IV).

Analytic Theorem. *Let $f(t)$ $(t \geq 0)$ be a bounded and locally integrable function and suppose that the function $g(z) = \int_0^\infty f(t)e^{-zt}\,dt$ $(\Re(z) > 0)$ extends holomorphically to $\Re(z) \geq 0$. Then $\int_0^\infty f(t)\,dt$ exists (and equals $g(0)$).*

(VI). $\vartheta(x) \sim x$.

Proof: Assume that for some $\lambda > 1$ there are arbitrarily large x with $\vartheta(x) \geq \lambda x$. Since ϑ is non-decreasing, we have

$$\int_x^{\lambda x} \frac{\vartheta(t) - t}{t^2}\,dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2}\,dt = \int_1^{\lambda} \frac{\lambda - t}{t^2}\,dt > 0$$

for such x, contradicting (V). Similarly, the inequality $\vartheta(x) \leq \lambda x$ with $\lambda < 1$ would imply

$$\int_{\lambda x}^{x} \frac{\vartheta(t) - t}{t^2}\,dt \leq \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2}\,dt = \int_{\lambda}^{1} \frac{\lambda - t}{t^2}\,dt < 0,$$

again a contradiction for λ fixed and x big enough.

The prime number theorem follows easily from (VI), since for any $\epsilon > 0$

$$\vartheta(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x = \pi(x) \log x,$$

$$\vartheta(x) \geq \sum_{x^{1-\epsilon} \leq p \leq xx} \log p \geq \sum_{x^{1-\epsilon} \leq p \leq x} (1 - \epsilon) \log x$$

$$= (1 - \epsilon) \log x \left[\pi(x) + O(x^{1-\epsilon})\right].$$

Proof of the Analytic Theorem. For $T > 0$ set $g_T(z) = \int_0^T f(t)e^{-zt}\,dt$. This is clearly holomorphic for all z. We must show that $\lim_{T \to \infty} g_T(0) = g(0)$.

Let R be large and let C be the boundary of the region $\{z \in \mathbb{C} \mid |z| \leq R, \Re(z) \geq -\delta\}$, where $\delta > 0$ is small enough (depending on R) so that $g(z)$ is holomorphic in and on C. Then

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C (g(z) - g_T(z)) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{dz}{z}$$

by Cauchy's theorem. On the semicircle $C_+ = C \cap \{\Re(z) > 0\}$ the integrand is bounded by $2B/R^2$, where $B = \max_{t \geq 0} |f(t)|$, because

$$|g(z) - g_T(z)| = \left| \int_T^\infty f(t) e^{-zt}\,dt \right| \leq B \int_T^\infty |e^{-zt}|\,dt = \frac{Be^{-\Re(z)T}}{\Re(z)} \quad (\Re(z) > 0)$$

and

$$\left| e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{1}{z} \right| = e^{\Re(z)T} \cdot \frac{2\Re(z)}{R^2}.$$

Hence the contribution to $g(0) - g_T(0)$ from the integral over C_+ is bounded in absolute value by B/R. For the integral over $C_- = C \cap \{\Re(z) < 0\}$ we look at $g(z)$ and $g_T(z)$ separately. Since g_T is entire, the path of integration for the integral involving g_T can be replaced by the semicircle $C'_- = \{z \in \mathbb{C} \mid |z| = R, \Re(z) < 0\}$, and the integral over C'_- is then bounded in absolute value by $2\pi B/R$

by exactly the same estimate as before since

$$|g_T(z)| = \left|\int_0^T f(t)e^{-zt}\,dt\right| \le B\int_{-\infty}^T |e^{-zt}|\,dt = \frac{Be^{-\Re(z)T}}{|\Re(z)|} \quad (\Re(z) < 0).$$

Finally, the remaining integral over C_- tends to 0 as $T \to \infty$ because the integrand is the product of the function $g(z)(1 + z^2/R^2)/z$, which is independent of T, and the function e^{zT}, which goes to 0 rapidly and uniformly on compact sets as $T \to \infty$ in the half-plane $\Re(z) < 0$. Hence $\limsup_{T\to\infty} |g(0) - g_T(0)| \le 2B/R$. Since R is arbitrary this proves the theorem.

Historical remarks. The "Riemann" zeta function $\zeta(s)$ was first introduced and studied by Euler, and the product representation given in (I) is his. The connection with the prime number theorem was found by Riemann, who made a deep study of the analytic properties of $\zeta(s)$. However, for our purposes the nearly trivial analytic continuation property (II) is sufficient. The extremely ingenious proof in (III) is in essence due to Chebyshev, who used more refined versions of such arguments to prove that the ratio of $\vartheta(x)$ to x (and hence also of $\pi(x)$ to $x/\log x$) lies between 0.92 and 1.11 for x sufficiently large. This remained the best result until the prime number theorem was proved in 1896 by de la Vallée Poussin and Hadamard. Their proofs were long and intricate. (A simplified modern presentation is given on pages 41–47 of Titchmarsh's book on the Riemann zeta function [T].) The very simple proof reproduced in (IV) of the non-vanishing of $\zeta(s)$ on the line $\Re(s) = 1$ was given in essence by Hadamard (the proof of this fact in de la Vallée Poussin's first paper had been about 25 pages long) and then refined by de la Vallée Poussin and by Mertens, the version given by the former being particularly elegant. The Analytic Theorem and its use to prove the prime number theorem as explained in steps (V) and (VI) above are due to D. J. Newman. Apart from a few minor simplifications, the exposition here follows that in Newman's original paper [N] and in the expository paper [K] by J. Korevaar.

We refer the reader to P. Bateman and H. Diamond's survey article [B] for a beautiful historical perspective on the prime number theorem.

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