10.

On the zeros of Riemann's zeta-function on the critical line

INTRODUCTION.

We denote by $N_0(T)$ the number of zeros of $\zeta(s)=\zeta(\sigma+it)$ for which $\sigma=\frac{1}{2}$, $0<t<T$. A theorem due to Hardy and Littlewood[1]), then says that there exist positive constants K and T_0 such that

$$N_0(2T) - N_0(T) > KT,$$

for $T>T_0$. In this paper we shall prove the slightly better

Theorem. There exist positive constants K and T_0 such that

$$N_0(2T) - N_0(T) > KT \log\log\log T,$$

for $T>T_0$.

A_1, A_2, ... denote positive absolute constants, the constants implied by the 0's are also absolute.

§ 1.
Proof of the Theorem.[2])

Let

(1) $$Z(t) = -\frac{1}{2}\pi^{-\frac{1}{4}-\frac{it}{2}} e^{\frac{\pi}{4}t} \Gamma\left(\frac{1}{4}+\frac{it}{2}\right) \zeta\left(\frac{1}{2}+it\right),$$

[1]) G. H. Hardy and J. E. Littlewood, The Zeros of Riemanns Zeta-Function on the Critical Line, Math. Zeitschrift 10 (1921), 283-317.

[2]) This proof follows the same lines as the proof of Hardy and Littlewood's theorem given by E. C. Titchmarsh, The Zeta-Function of Riemann, Cambridge Tracts No. 26 (1930) § 3.4.

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then it is known[1]) that $Z(t)$ is real for real t. Further let

(2) $$\eta(s) = \sum_{p\le\xi}(1-\tfrac{1}{2}p^{-s}-\tfrac{1}{8}p^{-2s}) = \sum_{\nu} a_\nu \cdot \nu^{-s},$$

where p runs through the prime numbers, and ξ is a positive number to be fixed later. If we put

(3) $$\Phi(x) = 2\sum_1^\infty e^{-n^2\pi x} - \frac{1}{\sqrt{x}},$$

it is known that the functions $Z(t)e^{-\frac{\pi}{4}t}$ and

$$-\sqrt{\frac{\pi}{2}}\, e^{\frac{1}{2}x}\, \Phi(e^{2x})$$

are Fourier transforms of each-other. We easily see that this is also the case with the functions $Z(t)e^{-\frac{1}{2}\delta t}\left(\frac{\nu_2}{\nu_1}\right)^{it}$ and

$$-\sqrt{\frac{\pi}{2}}\, e^{\frac{1}{2}x+\frac{i}{4}\left(\frac{\pi}{2}-\delta\right)}\sqrt{\frac{\nu_1}{\nu_2}}\, \Phi\left(e^{i\left(\frac{\pi}{2}-\delta\right)+2x}\left(\frac{\nu_1}{\nu_2}\right)^2\right),$$

where ν_1, ν_2 and δ are positive numbers. Now let

(4) $$Z_1(t) = Z(t)\,|\eta(\tfrac{1}{2}+it)|^2 = \sum \frac{a_{\nu_1} a_{\nu_2}}{\sqrt{\nu_1\nu_2}}\left(\frac{\nu_1}{\nu_2}\right)^{it} Z(t),$$

we then find that the Fourier transform of

(5) $$\int_t^{t+H} Z_1(u)\, e^{-\frac{1}{2}\delta u}\, du$$

is

(6) $$-\sqrt{\frac{\pi}{2}}\, e^{\frac{1}{2}x+\frac{i}{4}\left(\frac{\pi}{2}-\delta\right)}\left(\frac{e^{iHx}-1}{x}\right)\sum \frac{a_{\nu_1} a_{\nu_2}}{\nu_2}\,\Phi\left(e^{i\left(\frac{\pi}{2}-\delta\right)+2x}\left(\frac{\nu_1}{\nu_2}\right)^2\right).$$

In the following we put $\delta = \frac{2}{T}$. Further let $\xi=\sqrt{\log\log T}$ and $\frac{1}{\log\xi}<H<1$, then we write

(7) $$I=\int_t^{t+H} Z_1(u)\, e^{-\frac{u}{T}}\, du\,, \qquad J=\int_t^{t+H} |Z_1(u)|\, e^{-\frac{u}{T}}\, du \quad (T<t<2T).$$

[1]) Titchmarsh, Loc. cit. § 3.31. (Titchmarsh writes $\phi(\pi x)$ where we write $\phi(x)$).

We now prove

(8)
$$\int_T^{2T} I^2\, dt < A_1 \frac{H T^{\frac{1}{2}}}{\log \xi} \quad (T > T_0).$$

It is

$$\int_T^{2T} I^2\, dt < \int_{-\infty}^{\infty} \left| \int_t^{t+H} Z_1(u)\, e^{-\frac{1}{2}\delta u}\, du \right|^2 dt;$$

since (5) and (6) are the Fourier transforms of each-other Parsevals theorem gives:

(9)
$$\int_T^{2T} I^2\, dt < 2\pi \int_{-\infty}^{\infty} e^x \left| \sum \frac{a_{\nu_2} a_{\nu_1}}{\nu_2} \Phi\!\left(e^{i\left(\frac{\pi}{2}-\delta\right)+2x}\left(\frac{\nu_1}{\nu_2}\right)^2\right) \right|^2 \frac{\sin^2\frac{1}{2}Hx}{x^2}\, dx$$

$$= 2\pi \sum \frac{a_{\nu_1} a_{\nu_2} a_{\nu_3} a_{\nu_4}}{\nu_2 \nu_4} \int_{-\infty}^{\infty} \Phi\!\left(e^{i\left(\frac{\pi}{2}-\delta\right)+2x}\left(\frac{\nu_1}{\nu_3}\right)^2\right) \Phi\!\left(e^{i\left(\frac{\pi}{2}-\delta\right)+2x}\left(\frac{\nu_3}{\nu_4}\right)^2\right) e^x \frac{\sin^2\frac{1}{2}Hx}{x^2}\, dx$$

$$= 4\pi \sum \frac{a_{\nu_1} a_{\nu_2} a_{\nu_3} a_{\nu_4}}{\sqrt{\nu_1 \nu_2 \nu_3 \nu_4}} \int_0^{\infty} \Phi\!\left(e^{i\left(\frac{\pi}{2}-\delta\right)}\frac{\nu_1 \nu_4}{\nu_2 \nu_3}y\right) \Phi\!\left(e^{-i\left(\frac{\pi}{2}-\delta\right)}\frac{\nu_2 \nu_3}{\nu_1 \nu_4}y\right) \frac{\sin^2\!\left(\frac{1}{4}H\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)}{\left(\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)^2} \frac{dy}{\sqrt{y}}$$

$$= 8\pi \sum \frac{a_{\nu_1} a_{\nu_2} a_{\nu_3} a_{\nu_4}}{\sqrt{\nu_1 \nu_2 \nu_3 \nu_4}} \int_1^{\infty} \Phi\!\left(e^{i\left(\frac{\pi}{2}-\delta\right)}\frac{\nu_1 \nu_4}{\nu_2 \nu_3}y\right) \Phi\!\left(e^{-i\left(\frac{\pi}{2}-\delta\right)}\frac{\nu_2 \nu_3}{\nu_1 \nu_4}y\right) \frac{\sin^2\!\left(\frac{1}{4}H\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)}{\left(\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)^2} \frac{dy}{\sqrt{y}}$$

the last form being obtained by using the relation [1]):

(10)
$$\Phi\!\left(\frac{1}{z}\right) = z^{-\frac{1}{2}}\, \Phi(z).$$

Now we consider the integral

(11)
$$\int_1^{\infty} \Phi\!\left(i e^{-i\delta}\frac{\nu_1 \nu_4}{\nu_2 \nu_3}y\right) \Phi\!\left(-i e^{i\delta}\frac{\nu_2 \nu_3}{\nu_1 \nu_4}y\right) \frac{\sin^2\!\left(\frac{1}{4}H\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)}{\left(\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)^2} \frac{dy}{\sqrt{y}},$$

where the ν's satisfy the inequality:

$$1 \leqq \nu \leqq \prod_{p \leqq \xi} p^2 < e^{A_1 \xi} < \log T \,(T > T_0).$$

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[1]) This is equivalent to (3) § 3.11 of Titchmarsh. Loc. cit.

It is

$$\Phi\!\left(i e^{-i\delta}\frac{\nu_1 \nu_4}{\nu_2 \nu_3}y\right) \Phi\!\left(-i e^{i\delta}\frac{\nu_2 \nu_3}{\nu_1 \nu_4}y\right) = P + Q_1 + Q_2 + R,$$

where

$$P = 4 \sum_{m,\, n \geqslant 1} e^{-\left(n^2\pi i e^{-i\delta}\frac{\nu_1 \nu_4}{\nu_2 \nu_3} - m^2\pi i e^{i\delta}\frac{\nu_2 \nu_3}{\nu_1 \nu_4}\right)y},$$

and

$$Q_1 = -2\, e^{-i\left(\frac{\pi}{4}-\frac{\delta}{2}\right)} \sqrt{\frac{\nu_1 \nu_4}{\nu_2 \nu_3}}\; y^{-\frac{1}{2}} \sum_{n=1}^{\infty} e^{-n^2\pi i e^{-i\delta}\frac{\nu_1 \nu_4}{\nu_2 \nu_3}y}.$$

Q_2 is an expression of a similar type, while

$$R = \frac{1}{y}.$$

Now

(12)
$$\int_1^{\infty} R\, \frac{\sin^2\!\left(\frac{1}{4}H\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)}{\left(\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)^2} \frac{dy}{\sqrt{y}} < \int_0^{\infty} \frac{\sin^2\!\left(\frac{1}{4}H\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)}{\left(\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)^2} \frac{dy}{y}$$

$$= \int_{-\infty}^{\infty} \frac{\sin^2\frac{1}{4}Hu}{u^2}\, du = O(H) = O(1).$$

For Q_1 we get

$$\int_1^{\infty} Q_1 \frac{\sin^2\!\left(\frac{1}{4}H\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)}{\left(\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)^2} \frac{dy}{\sqrt{y}} =$$

$$= O\!\left(\sqrt{\frac{\nu_1 \nu_4}{\nu_2 \nu_3}} \sum_{n=1}^{\infty} \int_1^{\infty} e^{-n^2\pi e^{i\left(\frac{\pi}{2}-\delta\right)}\frac{\nu_1 \nu_4}{\nu_2 \nu_3}y} \frac{\sin^2\!\left(\frac{1}{4}H\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)}{\left(\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)^2} \frac{dy}{y}\right).$$

Turning here in each integral the line of integration through $-\left(\frac{\pi}{2}-\delta\right)$, putting $y = 1 + e^{-\left(\frac{\pi}{2}-\delta\right)i}r$, and observing that

$$\frac{\sin^2\!\left(\frac{1}{4}H\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)}{\left(\log\frac{\nu_2 \nu_4}{\nu_1 \nu_3}y\right)^2}$$

is regular and $O(H^2)$ in the lower half-plane, we get

$$(13) \quad \int_1^\infty Q_1 \frac{\sin^2\left(\frac{1}{4} H \log \frac{\nu_2 \nu_4}{\nu_1 \nu_3} y\right)}{\left(\log \frac{\nu_2 \nu_4}{\nu_1 \nu_3} y\right)^2} \frac{dy}{\sqrt{y}} = O\left(\sqrt{\frac{\nu_1 \nu_4}{\nu_2 \nu_3}} H^2 \sum_1^\infty \int_0^\infty e^{-n^2 \pi \frac{\nu_1 \nu_4}{\nu_2 \nu_3} r} dr\right)$$

$$= O\left(H^2 \sqrt{\nu_2 \nu_3} \sum_1^\infty \frac{1}{n^2}\right) = O(\log T).$$

Similarly

$$(13') \qquad \int_1^\infty Q_2 \frac{\sin^2\left(\frac{1}{4} H \log \frac{\nu_2 \nu_4}{\nu_1 \nu_3} y\right)}{\left(\log \frac{\nu_2 \nu_4}{\nu_1 \nu_3} y\right)^2} \frac{dy}{\sqrt{y}} = O(\log T).$$

We now write

$$P = 4 \sum_{m, n \geq 1} e^{-\left(n^2 \frac{\nu_1 \nu_4}{\nu_2 \nu_3} + m^2 \frac{\nu_2 \nu_3}{\nu_1 \nu_4}\right) \pi y \sin \delta + i \left(m^2 \frac{\nu_2 \nu_3}{\nu_1 \nu_4} - n^2 \frac{\nu_1 \nu_4}{\nu_2 \nu_3}\right) \pi y \cos \delta} = 2 P_1 + 4 P_2,$$

where

$$P_1 = 2 \sum_{\frac{n}{\nu_2 \nu_3} = \frac{m}{\nu_1 \nu_4} > 0} e^{-\left(n^2 \frac{\nu_1 \nu_4}{\nu_2 \nu_3} + m^2 \frac{\nu_2 \nu_3}{\nu_1 \nu_4}\right) \pi y \sin \delta} = 2 \sum_{\mu=1}^\infty e^{-2 \frac{\nu_1 \nu_2 \nu_3 \nu_4}{(\nu_1 \nu_4, \nu_2 \nu_3)^2} \mu^2 \pi y \sin \delta}.$$

(a, b) here and in the following denotes the greatest common divisor of a and b, and

$$P_2 = \sum_{\frac{n}{\nu_2 \nu_3} \neq \frac{m}{\nu_1 \nu_4} > 0} e^{-\left(n^2 \frac{\nu_1 \nu_4}{\nu_2 \nu_3} + m^2 \frac{\nu_2 \nu_3}{\nu_1 \nu_4}\right) \pi y \sin \delta + i \left(m^2 \frac{\nu_2 \nu_3}{\nu_1 \nu_4} - n^2 \frac{\nu_1 \nu_4}{\nu_2 \nu_3}\right) \pi y \cos \delta}.$$

Hence

$$\int_1^\infty P_2 \frac{\sin^2\left(\frac{1}{4} H \log \frac{\nu_2 \nu_4}{\nu_1 \nu_3} y\right)}{\left(\log \frac{\nu_2 \nu_4}{\nu_1 \nu_3} y\right)^2} \frac{dy}{\sqrt{y}} =$$

$$= \sum_{\frac{n}{\nu_2 \nu_3} \neq \frac{m}{\nu_1 \nu_4}} \int_1^\infty e^{-\left(n^2 \frac{\nu_1 \nu_4}{\nu_2 \nu_3} + m^2 \frac{\nu_2 \nu_3}{\nu_1 \nu_4}\right) \pi y \sin \delta + i \left(m^2 \frac{\nu_2 \nu_3}{\nu_1 \nu_4} - n^2 \frac{\nu_1 \nu_4}{\nu_2 \nu_3}\right) \pi y \cos \delta}$$

$$\frac{\sin^2\left(\frac{1}{4} H \log \frac{\nu_2 \nu_4}{\nu_1 \nu_3} y\right)}{\left(\log \frac{\nu_2 \nu_4}{\nu_1 \nu_3} y\right)^2} \frac{dy}{\sqrt{y}}.$$

Here we may put $y = 1 \pm ir$, according to the sign of $\left(m^2 \frac{\nu_2 \nu_3}{\nu_1 \nu_4} - n^2 \frac{\nu_1 \nu_4}{\nu_2 \nu_3}\right)$, thus we see that it is less than

$$O\left(H^2 \sum_{\frac{n}{\nu_2 \nu_3} \neq \frac{m}{\nu_1 \nu_4}} e^{-\left(n^2 \frac{\nu_1 \nu_4}{\nu_2 \nu_3} + m^2 \frac{\nu_2 \nu_3}{\nu_1 \nu_4}\right) \pi \sin \delta} \int_0^\infty e^{-\left| m^2 \frac{\nu_2 \nu_3}{\nu_1 \nu_4} - n^2 \frac{\nu_1 \nu_4}{\nu_2 \nu_3} \right| \pi r \cos \delta} dr\right)$$

$$= O\left(\nu_1 \nu_2 \nu_3 \nu_4 \sum_{\nu_1 \nu_4 n \neq \nu_2 \nu_3 m} \frac{e^{-\left((\nu_1 \nu_4 n)^2 + (\nu_2 \nu_3 m)^2\right) \frac{\pi \sin \delta}{\nu_1 \nu_2 \nu_3 \nu_4}}}{\left| (m \nu_2 \nu_3)^2 - (n \nu_1 \nu_4)^2 \right|}\right)$$

$$= O\left(\nu_1 \nu_2 \nu_3 \nu_4 \sum_{n \neq m} \frac{e^{-(n^2 + m^2) \frac{\pi \sin \delta}{\nu_1 \nu_2 \nu_3 \nu_4}}}{\left| m^2 - n^2 \right|}\right)$$

$$= O\left(\nu_1 \nu_2 \nu_3 \nu_4 \sum_{m=2}^\infty \frac{e^{-m^2 \frac{\pi \sin \delta}{\nu_1 \nu_2 \nu_3 \nu_4}}}{m} \sum_{n=1}^{m-1} \frac{1}{m-n}\right)$$

$$= O\left(\nu_1 \nu_2 \nu_3 \nu_4 \sum_2^\infty \frac{\log m}{m} e^{-m^2 \frac{\pi \sin \delta}{\nu_1 \nu_2 \nu_3 \nu_4}}\right) = O\left(\nu_1 \nu_2 \nu_3 \nu_4 \sum_2^{\frac{\nu_1 \nu_2 \nu_3 \nu_4}{\delta}} \frac{\log m}{m}\right)$$

$$+ O\left(\nu_1 \nu_2 \nu_3 \nu_4 \sum_{\frac{\nu_1 \nu_2 \nu_3 \nu_4}{\delta}}^\infty e^{-m^2 \frac{\pi \sin \delta}{\nu_1 \nu_2 \nu_3 \nu_4}}\right) = O\left(\nu_1 \nu_2 \nu_3 \nu_4 \log^2 \frac{\nu_1 \nu_2 \nu_3 \nu_4}{\delta}\right) = O(\log^6 T),$$

hence

$$(14) \qquad \int_1^\infty P_2 \frac{\sin^2\left(\frac{1}{4} H \log \frac{\nu_2 \nu_4}{\nu_1 \nu_3} y\right)}{\left(\log \frac{\nu_2 \nu_4}{\nu_1 \nu_3} y\right)^2} \frac{dy}{\sqrt{y}} = O(\log^6 T).$$

Finally we have to discuss what part P_1 contributes to the integral (11), it is

$$2 \sum_{\mu=1}^\infty e^{-\mu^2 \pi x} = \Phi(x) + \frac{1}{\sqrt{x}} = \begin{cases} \dfrac{1}{\sqrt{x}} + O(1), & \text{for } x \leq 1, \\[2ex] \dfrac{1}{\sqrt{x}} + O\left(\dfrac{1}{\sqrt{x}}\right) & \text{for } x \geq 1, \end{cases}$$

this is an immediate consequenze of (10). This gives:

$$\int_1^\infty P_1 \frac{\sin^2\left(\tfrac14 H \log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)}{\left(\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)^2}\frac{dy}{\sqrt{y}} = \frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{2\nu_1\nu_2\nu_3\nu_4\sin\delta}}\int_1^\infty \frac{\sin^2\left(\tfrac14 H\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)}{\left(\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)^2}\frac{dy}{y}$$

$$+ O\left(\int_1^{\frac{(\nu_1\nu_4,\,\nu_2\nu_3)^2}{2\nu_1\nu_2\nu_3\nu_4\sin\delta}} \frac{\sin^2\left(\tfrac14 H\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)}{\left(\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)^2}\frac{dy}{\sqrt{y}}\right)$$

$$+ O\left(\frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{\nu_1\nu_2\nu_3\nu_4\sin\delta}}\int_{\frac{(\nu_1\nu_4,\,\nu_2\nu_3)^2}{2\nu_1\nu_2\nu_3\nu_4\sin\delta}}^\infty \frac{\sin^2\left(\tfrac14 H\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)}{\left(\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)^2}\frac{dy}{y}\right).$$

Here

$$\frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{2\nu_1\nu_2\nu_3\nu_4\sin\delta}}\int_1^\infty \frac{\sin^2\left(\tfrac14 H\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)}{\left(\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)^2}\frac{dy}{y} = \frac{(\nu_1\nu_4,\,\nu_2\nu_3)\tfrac14 H}{\sqrt{2\nu_1\nu_2\nu_3\nu_4\sin\delta}}\int_{\tfrac14 H\log\frac{\nu_2\nu_4}{\nu_1\nu_3}}^\infty \frac{\sin^2 u}{u^2}du =$$

$$= \frac{(\nu_1\nu_4,\,\nu_2\nu_3)\tfrac14 H}{\sqrt{2\nu_1\nu_2\nu_3\nu_4\sin\delta}}\left\{\int_0^\infty \frac{\sin^2 u}{u^2}du + \int_0^{\tfrac14 H\log\frac{\nu_1\nu_3}{\nu_2\nu_4}} \frac{\sin^2 u}{u^2}du\right\} =$$

$$= \tfrac{1}{16}\sqrt{2}\,\pi H \frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{\nu_1\nu_2\nu_3\nu_4\sin\delta}} + \tfrac18\sqrt{2}\,H\frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{\nu_1\nu_2\nu_3\nu_4\sin\delta}}\int_0^{\tfrac14 H\log\frac{\nu_1\nu_3}{\nu_2\nu_4}} \frac{\sin^2 u}{u^2}du,$$

and

$$\int_1^{\frac{(\nu_1\nu_4,\,\nu_2\nu_3)^2}{2\nu_1\nu_2\nu_3\nu_4\sin\delta}} \frac{\sin^2\left(\tfrac14 H\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)}{\left(\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)^2}\frac{dy}{\sqrt{y}} \le \int_1^{\frac{(\nu_1\nu_4,\,\nu_2\nu_3)^2}{\sqrt{2\nu_1\nu_2\nu_3\nu_4\sin\delta}}} \frac{dy}{\sqrt{y}} + \int_{\frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{2\nu_1\nu_2\nu_3\nu_4\sin\delta}}}^\infty \frac{1}{\left(\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)^2}\frac{dy}{\sqrt{y}} =$$

$$= O\left(\frac{(\nu_1\nu_4,\,\nu_2\nu_3)^{\frac12}}{(\nu_1\nu_2\nu_3\nu_4\sin\delta)^{\frac14}}\right) + O\left(\frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{\nu_1\nu_2\nu_3\nu_4\sin\delta}}\cdot\frac{1}{\log^2\left(\frac{\nu_2\nu_4}{\nu_1\nu_3}\frac{1}{\sin\delta}\right)}\right) =$$

$$= O\left(\frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{\nu_1\nu_2\nu_3\nu_4\sin\delta}}\frac{1}{\log T}\right),$$

and

$$\frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{\nu_1\nu_2\nu_3\nu_4\sin\delta}}\int_{\frac{(\nu_1\nu_4,\,\nu_2\nu_3)^2}{2\nu_1\nu_2\nu_3\nu_4\sin\delta}}^\infty \frac{\sin^2\left(\tfrac14 H\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)}{\left(\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)^2}\frac{dy}{y} =$$

$$= \tfrac14 H\frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{\nu_1\nu_2\nu_3\nu_4\sin\delta}}\int_{\tfrac12 H\log\frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\nu_1\nu_3\sqrt{2\sin\delta}}}^\infty \frac{\sin^2 u}{u^2}du =$$

$$= O\left(\frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{\nu_1\nu_4,\,\nu_2\nu_3\sin\delta}}\frac{1}{\log\left(\frac{(\nu_1\nu_4,\,\nu_2\nu_3)^2}{\sin\delta}\right)}\right) = O\left(\frac{(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{\nu_1\nu_2\nu_3\nu_4\sin\delta}}\frac{1}{\log T}\right).$$

Hence

$$(15)\qquad \int_1^\infty P_1 \frac{\sin^2\left(\tfrac14 H\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)}{\left(\log\frac{\nu_2\nu_4}{\nu_1\nu_3}y\right)^2}\frac{dy}{\sqrt{y}} = \frac{\tfrac18\sqrt{2}\,H(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{\nu_1\nu_2\nu_3\nu_4\sin\delta}}$$

$$\left\{\frac{\pi}{2} + \int_0^{\tfrac14 H\log\frac{\nu_1\nu_3}{\nu_2\nu_4}} \frac{\sin^2 u}{u^2}du + O\left(\frac{1}{H\log T}\right)\right\}$$

We now get for the integral (11), from (12), (13), (13′), (14) and (15):

$$\frac{\tfrac14\sqrt{2}\,H(\nu_1\nu_4,\,\nu_2\nu_3)}{\sqrt{\nu_1\nu_2\nu_3\nu_4\sin\delta}}\left\{\frac{\pi}{2} + \int_0^{\tfrac14 H\log\frac{\nu_1\nu_3}{\nu_2\nu_4}} \frac{\sin^2 u}{u^2}du + O\left(\frac{1}{H\log T}\right)\right\}.$$

If this is inserted in (9), we find:

$$(16)\qquad \int_T^{2T} I^2\,dt < \sqrt{2}\,\pi^2 \frac{H}{\sqrt{\sin\delta}}\sum \frac{a_{\nu_1}a_{\nu_2}a_{\nu_3}a_{\nu_4}}{\nu_1\nu_2\nu_3\nu_4}(\nu_1\nu_4,\,\nu_2\nu_3) +$$

$$+ 2\sqrt{2}\,\pi\frac{H}{\sqrt{\sin\delta}}\sum \frac{a_{\nu_1}a_{\nu_2}a_{\nu_4}a_{\nu_3}}{\nu_1\nu_2\nu_3\nu_4}(\nu_1\nu_4,\,\nu_2\nu_3)\int_0^{\tfrac14 H\log\frac{\nu_1\nu_3}{\nu_2\nu_4}} \frac{\sin^2 u}{u^2}du +$$

$$+ O\left(\frac{\sqrt{T}}{\log T}\sum \frac{|a_{\nu_1}a_{\nu_2}a_{\nu_3}a_{\nu_4}|}{\nu_1\nu_2\nu_3\nu_4}(\nu_1\nu_4,\,\nu_2\nu_3)\right).$$

It is easily seen that the second series on the righthand side vanishes identically, since it changes the sign when $\nu_1 \nu_2 \nu_3 \nu_4$ is changed into $\nu_4 \nu_3 \nu_2 \nu_1$.

It remains to discuss the two sums

$$\sum_1 = \sum \frac{a_{\nu_1} a_{\nu_2} a_{\nu_3} a_{\nu_4}}{\nu_1 \nu_2 \nu_3 \nu_4} (\nu_1 \nu_4, \nu_2 \nu_3),$$

and

$$\sum_2 = \sum \frac{|a_{\nu_1}| |a_{\nu_2}| |a_{\nu_3}| |a_{\nu_4}|}{\nu_1 \nu_2 \nu_3 \nu_4} (\nu_1 \nu_4, \nu_2 \nu_3).$$

We first consider $\sum_1$. Remembering (2) we now put

$$\left(\sum a_\nu \cdot \nu^{-s} \right)^2 = \prod_{p \leq \xi} \left(1 - p^{-s} + \frac{1}{8} p^{-3s} + \frac{1}{64} p^{-4s} \right) = \sum \frac{b_\nu}{\nu^s},$$

where $b_\nu = \sum_{\nu_1 \nu_2 = \nu} a_{\nu_1} a_{\nu_2}$. Then we have:

$$(17) \qquad \sum_1 = \sum \frac{b_\nu b_\mu}{\nu \mu} (\nu, \mu) = \sum_\nu \frac{b\nu}{\nu} \sum_\mu \frac{b\mu}{\mu} (\nu, \mu).$$

Now if $(\mu_1, \mu_2) = 1$,

$$\frac{b \mu_1}{\mu_1} (\nu, \mu_1) \cdot \frac{b \mu_2}{\mu_2} (\nu, \mu_2) = \frac{b \mu_1 \mu_2}{\mu_1 \mu_2} (\nu, \mu_1 \mu_2),$$

thus we easily get

$$\sum_\mu \frac{b\mu}{\mu} (\nu, \mu) = \prod_{p \leq \xi} \left(1 - \frac{(\nu, p)}{p} + \frac{(\nu, p^3)}{8p^3} + \frac{(\nu, p^4)}{64 p^4} \right) =$$

$$= \prod_{p \leq \xi} \left(1 - \frac{1}{p} + \frac{1}{8p^3} + \frac{1}{64 p^4} \right) \cdot \prod_{p|\nu} \left(\frac{\frac{(\nu, p^3)}{8p^3} + \frac{(\nu, p^4)}{64 p^4}}{1 - \frac{1}{p} + \frac{1}{8p^3} + \frac{1}{64 p^4}} \right).$$

Inserting in (17), we have

$$\sum_1 = \prod_{p \leq \xi} \left(1 - \frac{1}{p} + \frac{1}{8p^3} + \frac{1}{64 p^3} \right) \sum_\nu \frac{b_\nu}{\nu} \prod_{p|\nu} \left(\frac{\frac{(\nu, p^3)}{8p^3} + \frac{(\nu, p^4)}{64 p^4}}{1 - \frac{1}{p} + \frac{1}{8p^3} + \frac{1}{64 p^4}} \right) +$$

$$= \prod_{p \leq \xi} \left(1 - \frac{1}{p} + O\left(\frac{1}{p^3}\right) \right) \sum_\nu \varrho_\nu.$$

Now if $(\nu_1, \nu_2) = 1$ we see that $\varrho_{\nu_1} \varrho_{\nu_2} = \varrho_{\nu_1 \nu_2}$, hence

$$\sum_\nu \varrho_\nu = \prod_{p \leq \xi} (1 + \varrho_p + \varrho_{p^3} + \varrho_{p^4}) =$$

$$= \prod_{p \leq \xi} \left(1 + \frac{-\frac{1}{p}\left(\frac{1}{8p^2} + \frac{1}{64 p^3}\right) + \frac{1}{8p^3}\left(\frac{1}{8} + \frac{1}{64 p}\right) + \frac{1}{64 p^4} \cdot \frac{9}{64}}{1 - \frac{1}{p} + \frac{1}{8p^3} + \frac{1}{64 p^4}} \right) =$$

$$= \prod_{p \leq \xi} \left(1 + O\left(\frac{1}{p^3}\right) \right).$$

The formula above then gives

$$(18) \qquad \sum_1 = \prod_{p \leq \xi} \left(1 - \frac{1}{p} + O\left(\frac{1}{p^3}\right) \right) \left(1 + O\left(\frac{1}{p^3}\right) \right) = \prod_{p \leq \xi} \left(1 - \frac{1}{p} + O\left(\frac{1}{p^3}\right) \right) =$$

$$= e^{-\sum_{p \leq \xi} \frac{1}{p} + O\left(\sum \frac{1}{p^2}\right)} = e^{-\log\log \xi + O(1)} = O\left(\frac{1}{\log \xi}\right),$$

and similarly we find

$$(18') \qquad \sum_2 = O(\log^3 \xi).$$

(16) now gives

$$\int_T^{2T} I^2 dt = O\left(\frac{H}{\sqrt{\sin \delta \log \xi}} \right) + O\left(\frac{\sqrt{T} \log^3 \xi}{\log T} \right) = O\left(\frac{H \sqrt{T}}{\log \xi} \right),$$

whence (8) follows.

We next prove that

$$(19) \qquad J > A_3 T^{-\frac{1}{2}} (H - \psi),$$

where

$$(20) \qquad \int_T^{2T} |\psi|^2 dt < A_4 \frac{T}{\log^2 \xi} \quad (T > T_0).$$

We have, if $s = \frac{1}{2} + it$, $T \leq t \leq 2T + H$, $T > T_0$,

$$(21) \qquad T^{\frac{1}{4}} |Z_1(t)| e^{-\frac{t}{T}} > A_5 T^{\frac{1}{4}} e^{\frac{\pi}{4}t} \left| \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \right| |\zeta(s) \eta^2(s)|.$$

$$> A_6 \left| \zeta(s) \prod_{p \leq \xi} (1 - p^{-s}) \right| > A_6 R \left\{ \zeta(s) \prod_{p \leq \xi} (1 - p^{-s}) \right\}.$$

Now if $v \leq \sqrt{\log T}$ [1],

$$\zeta(s) = \sum_{n \leq \frac{T\sqrt{\log T}}{v}} n^{-s} - \frac{\left(\frac{T\sqrt{\log T}}{v}\right)^{1-s}}{1-s} + O\left(\frac{\sqrt{v}}{T^{\frac{1}{2}}(\log T)^{\frac{1}{4}}}\right),$$

hence

$$(22) \qquad \zeta(s)\prod_{p \leq \xi}(1-p^{-s}) = \zeta(s)\sum_{\substack{v \mid \Pi p \\ p \leq \xi}} \frac{\mu(v)}{v^s} = \sum_{m \leq T\sqrt{\log T}} \frac{c_m}{m^s} -$$

$$\prod_{p \leq \xi}\left(1-\frac{1}{p}\right)\frac{(T\sqrt{\log T})^{1-s}}{1-s} + O\left(\frac{1}{\sqrt{T}}\right) = \sum_{m \leq T\sqrt{\log T}} \frac{c_m}{m^s} + O(T^{-\frac{1}{2}}),$$

where

$$c_m = \sum_{\substack{d \mid m \\ d \mid \Pi p \\ p \leq \xi}} \mu(d) = \begin{cases} 0 \text{ if } m \text{ is divisible by a prime } \leq \xi, \\ 1 \text{ if } m \text{ is not divisible by a prime } \leq \xi. \end{cases}$$

Inserting (22) in (21) and integrating, we find

$$(23) \qquad T^{\frac{1}{4}}J > A_6 R\left\{\int_t^{t+H}\zeta(s)\prod_{p \leq \xi}(1-p^{-s})\,dt\right\} = A_6 H +$$

$$A_6 R\left[i\sum_{\xi < m \leq T\sqrt{\log T}}\frac{c_m}{m^s\log m}\right]_{s=\frac{1}{2}+it}^{s=\frac{1}{2}+i(t+H)} + O(T^{-\frac{1}{4}}) \geq$$

$$A_6 H - A_6|g(t)| - A_6|g(t+H)| - O(T^{-\frac{1}{4}}) = A_3(H-\psi),$$

where

$$g(t) = \sum_{\xi < m \leq T\sqrt{\log T}}\frac{c_m}{m^s\log m}.$$

We now consider

$$(24) \qquad \int_T^{2T}|g(t)|^2\,dt = T\sum_{\xi < m \leq T\sqrt{\log T}}\frac{|c_m|^2}{m\log^2 m} + O\left(\sum_{\substack{m \neq n \\ 1 < m,\, n \leq T\sqrt{\log T}}}\frac{1}{\sqrt{mn}\log m\log n}\left|\log\frac{m}{n}\right|\right)$$

$$\lesssim T\sum_{m > \xi}\frac{c_m}{m\log^2 m} + O\left(\sum_{\substack{m \neq n \\ 1 < m,\, n \leq T\sqrt{\log T}}}\frac{1}{\sqrt{mn}\log m\log n}\left|\log\frac{m}{n}\right|\right).$$

[1]) Titchmarsh. Loc. cit. Theorem 19. § 2. 12.

But

$$(25) \qquad \sum_{\substack{m \neq n \\ 1 < m \leq T\sqrt{\log T} \\ 1 < n \leq T\sqrt{\log T}}}\frac{1}{\sqrt{mn}\log m\log n\left|\log\frac{m}{n}\right|} = O\left(\frac{T}{\sqrt{\log T}}\right), \text{[1])}$$

and

$$\sum_{m \geq \xi}\frac{c_m}{m\log^2 m} = \sum_{k=0}^{\infty}\sum_{\xi^k}^{\xi^{k+1}}\frac{c_m}{m\log^2 m} < \frac{1}{\log^2 \xi}\sum_{k=0}^{\infty}4^{-k}\sum_1^{\xi^{k+1}}\frac{c_m}{m}.$$

Now for $x \geq \xi$ we have

$$\sum_1^x \frac{c_m}{m} < e\sum_1^{\infty}\frac{c_m}{m^{1+\frac{1}{\log x}}} = e\zeta\left(1+\frac{1}{\log x}\right)\prod_{p \leq \xi}\left(1-\frac{1}{p^{1+\frac{1}{\log x}}}\right) =$$

$$= O\left(\log x \cdot e^{-\sum_{p \leq \xi}p^{-1-\frac{1}{\log x}}}\right) = O\left(\log x \cdot e^{-\sum_{p \leq \xi}\frac{1}{p} + O\left(\frac{1}{\log x}\sum_{p \leq \xi}\frac{\log p}{p}\right)}\right)$$

$$= O\left(\log x \cdot e^{-\log\log \xi + O(1) + O\left(\frac{\log \xi}{\log x}\right)}\right) = O\left(\frac{\log x}{\log \xi}\right),$$

hence

$$(26) \qquad \sum_{m > \xi}\frac{c_m}{m\log^2 m} = O\left(\frac{1}{\log^2 \xi}\sum_{k=0}^{\infty}2^{1-k}\right) = O\left(\frac{1}{\log^2 \xi}\right).$$

Inserting (25) and (26) in (24), we get

$$\int_T^{2T}|g(t)|^2\,dt = O\left(\frac{T}{\log^2 \xi}\right) + O\left(\frac{T}{\sqrt{\log T}}\right) = O\left(\frac{T}{\log^2 \xi}\right),$$

obviously also

$$\int_T^{2T}|g(t+H)|^2\,dt = O\left(\frac{T}{\log^2 \xi}\right).$$

(20) now follows immediately, since

$$\psi = |g(t)| + |g(t+H)| + O(T^{-\frac{1}{4}}).$$

Now let S be the sub-set of the interval $(T, 2T)$ where $|I| = J$. Then if $m = m(S)$ is the measure of S

[1]) Hardy and Littlewood. Loc. cit. Lemma 6.

$$\int_S^{} |I|\, dt \leqq m^{\frac{1}{2}} \left\{ \int_T^{2T} I^2 dt \right\}^{\frac{1}{2}} < A_7 \frac{H^{\frac{1}{2}} T^{\frac{1}{4}} m^{\frac{1}{2}}}{\sqrt{\log \xi}},$$

by (8); on the other hand, by (19) and (20),

$$\int_S J\, dt > A_3 T^{\frac{1}{4}} \int_S (H - \psi)\, dt > A_3 H T^{-\frac{1}{4}} m - A_3 T^{-\frac{1}{4}} \int_S |\psi|\, dt \geqq$$

$$\geqq A_3 H T^{-\frac{1}{4}} m - A_3 T^{-\frac{1}{4}} m^{\frac{1}{2}} \left\{ \int_T^{2T} |\psi|^2 dt \right\}^{\frac{1}{2}} > A_3 H T^{-\frac{1}{4}} m - A_8 \frac{T^{\frac{1}{4}} m^{\frac{1}{2}}}{\log \xi}.$$

Hence

$$m^{\frac{1}{2}} < A_9 \frac{T^{\frac{1}{4}}}{\sqrt{H \log \xi}} + A_{10} \frac{T^{\frac{1}{4}}}{H \log \xi},$$

or

$$m(S) < A_{11} \left(\frac{1}{H \log \xi} + \frac{1}{H^2 \log^2 \xi} \right) T < \frac{A_{12}}{H \log \xi} T,$$

since $H > \dfrac{1}{\log \xi}$. Now divide the interval $(T, 2T)$ into $\left[\dfrac{T}{2H} \right]$ pairs of abutting intervals j_1, j_2, each, except the last j_2 of length H, and each j_2 lying immediately to the right of the corresponding j_1. Then either j_1 or j_2 contains a zero of $Z_1(t)$, unless j_1 consists entirely of points of S. Suppose the latter occurs for νj_1's. Then

$$\nu H \leqq m(S) < \frac{A_{12}}{H \log \xi} T.$$

Hence there are, in $(T, 2T)$, at least

$$\left[\frac{T}{2H} \right] - \nu > \frac{T}{H} \left(\frac{1}{3} - \frac{A_{12}}{H \log \xi} \right)$$

zeros, now choose H so great that

$$\frac{A_{12}}{H \log \xi} = \frac{1}{6}, \quad H = \frac{A_{13}}{\log \xi} = \frac{A_{14}}{\log \log \log T}.$$

Thus we see that the number of zeros are, at least

$$\frac{T}{6H} > A_{15} T \log \log \log T,$$

since the real zeros of $Z_1(t)$ obviously also are the zeros of $\zeta(\frac{1}{2} + it)$, this proves the theorem stated in the introduction.

§ 2.
Remarks on the Proof.

The main idea of the proof given above, is the introduction of the factor $|\eta(\frac{1}{2} + it)|^2$, which to a certain extent neutralizes the variation of $|\zeta(\frac{1}{2} + it)|$. It is clear that we should expect a better result, if we could choose ξ greater in comparison with T. Now we see easily, that the proof of (8) will not be affected, if we put $\xi = \sqrt{\log T}$; the proof of (20) however, would not hold in this case. This is caused by the approximate formula we have used for $\zeta(s)$ on $\sigma = \frac{1}{2}$. If we instead of this use the «approximate functional equation» of Hardy and Littlewood it is possible to arrange the proof of (20) so that it still holds for $\xi = \sqrt{\log T}$, the proof being of course much more complicated. So we could replace the factor $\log \log \log T$ in our theorem by $\log \log T$; indeed we may, by using the approximate functional equation, go a good deal further and prove the following theorem:[1] Let $U = T^a$, where $a > \frac{1}{2}$. Then there is a $K = K(a) > 0$ and a $T_0 = T_0(a)$ such that

$$(27) \qquad N_0(T + U) - N_0(T) > K U \log \log T \qquad (T > T_0).$$

It seems not improbable, that still further progresses can be made, if one uses another function instead of $\eta(s)$.[2] I'll return to these questions in a later paper.

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[1] This corresponds to the Theorem B of Hardy and Littlewood. Loc. cit.

[2] In course of the proof-correction I have succeeded in proving that the factor $\log \log T$ in (27) may be replaced by $\log T$.

11.

On the zeros of the zeta-function of Riemann

(Innlevert til Generalsekretæren 23de mai 1942 av herr Brun)

Let $N_0(T)$ denote the number of zeros of $\zeta(s) = \zeta(\sigma + it)$, for which $\sigma = \frac{1}{2}$, $0 < t < T$. Hardy and Littlewood [1] have then proved that there exist positive constants A and T_0, so that

(1) $$N_0(T) > AT \quad (T > T_0).$$

This result may be improved to the
Theorem 1:
There is an $A > 0$ and a T_0, so that

(2) $$N_0(T) > AT \log T \quad (T > T_0).$$

In the following we shall sketch the main ideas of the proof.
We write when $s = \frac{1}{2} + it$,

$$X(t) = \frac{1}{2} t^{\frac{1}{4}} e^{\frac{1}{4}\pi t} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

so that $X(t)$ is real for real t. Also put when T is positive

$$\eta(t) = \sum_{\substack{v < T^{\frac{1}{100}}}} \frac{\alpha_v}{v^{\frac{1}{2}+it}} \left(1 - \frac{100 \log v}{\log T}\right),$$

where the α_v are the coefficients in the expansion

$$\frac{1}{\sqrt{\zeta(s)}} = \sum_1^\infty \frac{\alpha_v}{v^s}, \quad \alpha_1 = 1,$$

for $\sigma > 1$. Further let

$$\frac{1}{\log T} < H < \frac{1}{\sqrt{\log T}}.$$

Then we put

$$I = I(t, H) = \int_t^{t+H} X(t) \, |\eta(t)|^2 \, dt,$$

and

$$M = M(t, H) = \int_t^{t+H} \zeta\left(\frac{1}{2} + it\right) \eta^2(t) \, dt - H,$$

It is then possible to show that [2]

(3) $$\int_T^{2T} I^2 \, dt = O\left(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

and similarly

(4) $$\int_T^{2T} |M|^2 \, dt = O\left(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

Now put

$$J = J(t, H) = \int_t^{t+H} |X(t)| \cdot |\eta(t)|^2 \, dt,$$

then we can show that for $T > T_0$

(5) $$J > H - |M|.$$

Now let S denote the sub-set of $(T, 2T)$, where $|I| = J$. Then

(6) $$\int_S |I| \, dt = \int_S J \, dt.$$

By (3)

$$\int_S |I| \, dt < m^{\frac{1}{2}} \left\{\int_T^{2T} |I|^2 \, dt\right\}^{\frac{1}{2}} = O\left(m^{\frac{1}{2}} T^{\frac{1}{2}} \frac{H^{\frac{3}{4}}}{(\log T)^{\frac{1}{4}}}\right),$$

where $m = m(S)$ is the measure of S. On the other hand, from (5) and (4)

$$\int_S J \, dt > H \, m - \int_S |M| \, dt > H \, m - m^{\frac{1}{2}} \left\{ \int_S^{2T} |M|^2 \, dt \right\}^{\frac{1}{2}} .$$

$$= H \, m - O \left(m^{\frac{1}{2}} T^{\frac{1}{2}} \frac{H^{\frac{3}{4}}}{(\log T)^{\frac{1}{4}}} \right) .$$

Comparing these inequalities with (6), we find

$$m = O \left(\frac{T}{\sqrt{H \log T}} \right) ,$$

or replacing the O-relation by an inequality, we get

$$m < A_1 \frac{T}{\sqrt{H \log T}} \quad (T > T_0),$$

where A_1 is a positive constant. Now choosing

$$H = \frac{16 A_1^2}{\log T} ,$$

we get

(7)
$$m < \frac{T}{4} .$$

Now [3] divide the interval $(T, 2T)$ into $\left[\frac{T}{2H} \right]$ pairs of abutting intervals j_1, j_2, each, except the last j_2 of length H, and each j_2 lying immediately to the right of the corresponding j_1. Then since $X(t) |\eta(t)|^2$ can only change the sign if t is passing through a zero of $X(t)$, that is of $\zeta\left(\frac{1}{2} + it \right)$, either j_1 or j_2 must contain a zero of $\zeta\left(\frac{1}{2} + it \right)$, unless j_1 consists entirely of points of S. Suppose that that is the case for $\nu \, j_1$'s, then from (7)

$$\nu H \le m < \frac{T}{4}$$

or

$$\nu < \frac{T}{4H} .$$

Hence there are in $(T, 2T)$ at least

$$\left[\frac{T}{2H} \right] - \nu > \frac{T}{3H} - \frac{T}{4H} = \frac{T}{12H} = \frac{T \log T}{12 \cdot 16 A_1^2} ,$$

zeros of $\zeta\left(\frac{1}{2} + it \right)$, which obviously proves theorem 1.

Another new theorem on the zeros of $\zeta(s)$ is
Theorem 2.

If $\Phi(t)$ is positive and increases to infinity with t, then all but an infinitesimal proportion of the zeros of $\zeta(s)$ in the upper half-plane lie in the region

(8)
$$\left| \sigma - \frac{1}{2} \right| < \Phi(t) \frac{1}{\log t} \quad (t > 3) .$$

This is an improvement of a theorem of LITTLEWOOD, which instead of (8) has the weaker

$$\left| \sigma - \frac{1}{2} \right| < \Phi(t) \frac{\log \log t}{\log t} \quad (t > 3) .$$

Full proofs of these and more general results will appear in a later paper.

[1] G. H. HARDY and z. E. LITTLEWOOD, The zeros of Riemann's Zeta-function on the critical line, Math. Zeitschr., vol. 10 (1922), pp. 281-317.
[2] The constants implied in the O's are here and in the following absolute constants.
[3] As usual [x] means the greatest integer $\le x$.