For $m \neq n$ (the off-diagonal terms) we get by partial integration

$$\int f(t)\left(\frac{m}{n}\right)^{it} \mathrm{d}t = \left(\log\frac{m}{n}\right)^{-2}\int_{-X}^{T+X} f'(t)\,\mathrm{d}\left(\frac{m}{n}\right)^{it} \ll X^{-1}\left(\log\frac{m}{n}\right)^{-2}.$$

Hence

$$\sum_{m\neq n} a_m\overline{a_n}\int f(t)\left(\frac{m}{n}\right)^{it}\mathrm{d}t \ll X^{-1}\sum_{m\neq n}|a_m a_n|\left(\log\frac{m}{n}\right)^{-2} \leqslant X^{-1}\sum_{m\neq n}|a_n|^2\frac{n}{m}\left(\log\frac{m}{n}\right)^{-2} \ll X^{-1}H,$$

because

$$\sum_{m\neq n} m^{-1}\left(\log\frac{m}{n}\right)^{-2} \ll n.$$

Adding the diagonal contribution we see that the integral (13.12) is estimated from above by

$$TG + O(XG + X^{-1}H). \tag{13.14}$$

Similarly we get (13.14) as a lower bound for the integral (13.12). Hence, choosing $X = (H/G)^{1/2}$ we get the formula (13.12). □

Estimating H by GN^2 we get

COROLLARY 13.2. *For $T > 0$ we have*

$$\int_0^T |A(it)|^2\,\mathrm{d}t = \big(T + O(N)\big)G \tag{13.15}$$

where the implied constant is absolute.

This result shows that G is the mean value of $|A(it)|^2$ on the segment $0 < t < T$, provided T is somewhat larger than the length of the polynomial.

Sometimes we need estimates for sums $A(s)$ whose coefficients a_n depend on s in a steady fashion. By (13.15) we derive

COROLLARY 13.3. *Suppose for $T \leqslant t \leqslant 2T$ and $n \leqslant N$ that we have*

$$|a_n(t)| \leqslant a_n, \qquad t|a_n'(t)| \leqslant a_n. \tag{13.16}$$

Then

$$\int_T^{2T}\left|\sum_{n\leqslant N} a_n(t)n^{it}\right|^2 \mathrm{d}t \leqslant \big(2T + O(N)\big)\sum_{n\leqslant N}|a_n|^2. \tag{13.17}$$

PROOF. Apply the Cauchy-Schwarz inequality to

$$\left|\sum_n a_n(t)n^{it}\right| \leqslant \left|\sum_n a_n(T)n^{it}\right| + \int_T^{2T}\left|\sum_n a_n'(\tau)n^{it}\right|\mathrm{d}\tau. \qquad \square$$

COROLLARY 13.4. *For $T \geqslant 2$ we have*

$$\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \mathrm{d}t = T\log T + O\big(T(\log T)^{\frac{1}{2}}\big). \tag{13.18}$$

PROOF. By the approximation (6.5) for $s = \frac{1}{2} + it$ with $\frac{1}{2}T < t < T$ we have

$$\zeta(s) = \sum_{n\leqslant T} n^{-s} + O(T^{-\frac{1}{2}}).$$

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