VON NEUMANN MINIMAX THEOREM

Theorem: Let $\mathbf{A}$ be a $m \times n$ matrix representing the payoff matrix for a two-person, zero-sum game. Then the game has a value and there exists a pair of mixed strategies which are optimal for the two players.

Let us recall the following definition where, for a mixed strategy pair (x, y), we define $V(x,y) := \sum_{i=1}^{m} \sum_{j=1}^{n} x_i \, a_{i,j} \, y_j$.

Definition: A pair of mixed strategies $(x^\star, y^\star)$ is said to be an equilibrium point for a two-person, zero-sum game provided

$$V(x, y^\star) \le V(x^\star, y^\star) \text{ for all } x \in X_m, \text{ and } V(x^\star, y^\star) \le V(x^\star, y) \text{ for all } y \in Y_n\,.$$

Note that this is equivalent to the assertion that

$$\max_{x \in X_m} V(x, y^\star) = V(x^\star, y^\star) = \min_{y \in Y_n} V(x^\star, y)\,.$$

Using the notation that we introduced in class, we begin with the following theorem

Theorem: Each of the following three conditions are equivalent:

(a) An equilibrium pair exists.

(b)
$$v_A := \max_{x \in X_m} \min_{y \in Y_n} V(x,y) = \min_{y \in Y_n} \max_{x \in X_m} V(x,y) := v_B\,.$$

(c) There exists a $\mathbf{v} \in \mathbb{R}$ and $x^{(o)} \in X_m$, $y^{(o)} \in Y_n$ such that

(i) $\sum_{i=1}^{m} a_{i,j} \, x_i^{(o)} \ge \mathbf{v}, \quad j = 1, 2, \ldots, n\,,$

(ii) $\sum_{j=1}^{n} a_{i,j} \, y_j^{(o)} \le \mathbf{v}, \quad i = 1, 2, \ldots, m\,.$

Proof: To see that (a) $\Rightarrow$ (b) consider the equilibrium pair $(x^\star, y^\star)$. Then

$$\begin{aligned} v_B &:= \min_{y \in Y_n} \max_{x \in X_m} V(x,y) \le \max_{x \in X_m} V(x, y^\star) = V(x^\star, y^\star) \\ &= \min_{y \in Y_n} V(x^\star, y) \le \max_{x \in X_m} \min_{y \in Y_n} V(x,y) =: v_A\,. \end{aligned}$$

But, since we always have $v_A \le v_B$ we must have equality throughout.

To see that (b) $\Rightarrow$ (c) suppose that $\mathbf{v} = v_A = v_B$. Let $x^{(o)}$ be the maximin and $y^{(o)}$ be the minimax. Then for all $j = 1, 2, \ldots, n$ and for all $i = 1, 2, \ldots, m$ we have

$$
\begin{aligned}
\sum_{i=1}^{m} a_{i,j}\, x_i^{(o)} &= V(x^{(o)},\beta_j) \,\geq\, \min_{y\in Y_n} V(x^{(o)},y) \,=\, \max_{x\in X_m}\,\min_{y\in Y_n} V(x,y) \\
&= \mathbf{v} \,=\, \min_{y\in Y_n}\,\max_{x\in X_m} V(x,y) \,=\, \max_{x\in X_m} V(x,y^{(o)}) \\
&\geq V(\alpha_i,y^{(o)}) \,=\, \sum_{j=1}^{n} a_{i,j}\, y_j^{(o)}\,.
\end{aligned}
$$

Finally, to see that (c) $\Rightarrow$ (a) we see from (i) and (ii) that

$$
V(x^{(o)},y) \,\geq\, \mathbf{v} \,\geq\, V(x,y^{(o)}) \quad \text{for all} \quad x\in X_m \quad \text{and} \quad y\in Y_n\,.
$$

Putting $x = x^{(o)}$ and $y = y^{(o)}$ in this inequality we see that $\mathbf{v} = V(x^{(o)},y^{(o)})$ and hence that $(x^{(o)},y^{(o)})$ is an equilibrium pair. ■

Now we ask the question whether, given a game described by a payoff matrix $\mathbf{A}$, any one (and hence all) of these conditions hold. That one does is the force of von Neumann's theorem. We present Nash's proof of the theorem which uses the Brouwer fixed point theorem. Recall that the Brouwer theorem says the following.

Theorem: Let $K \subset \mathbb{R}^p$ be a closed, bounded, and convex set. Then, if $f : K \longrightarrow K$ is continuous, there is an $\hat{x}\in K$ such that $f(\hat{x}) = \hat{x}$.

Here is Nash's proof of von Neumann's theorem.

Proof: We know that the set $X_m \times Y_n$ of pairs of mixed strategies in a closed, bounded, and convex subset of $\mathbb{R}^{m+n}$. We first define a transformation $T : X_m \times Y_n \longrightarrow X_m \times Y_n$. Let

$$
\begin{aligned}
c_i(x,y) &:= \begin{cases} V(\alpha_i,y) - V(x,y) & \text{if this quantity is positive} \\ 0 & \text{otherwise} \end{cases}\,, \\
d_j(x,y) &:= \begin{cases} V(x,y) - V(x,\beta_j) & \text{if this quantity is positive} \\ 0 & \text{otherwise} \end{cases}\,.
\end{aligned}
$$

For each $(x,y)\in X_m \times Y_n$ we define $T(x,y) = (x',y')$ by

$$
x_i' \,:=\, \frac{x_i + c_i(x,y)}{1 + \sum\limits_{k=1}^{m} c_k(x,y)} \quad \text{and} \quad y_j' \,:=\, \frac{y_j + d_j(x,y)}{1 + \sum\limits_{k=1}^{n} d_k(x,y)}\,.
$$

Note that $x_i' \geq 0$ since $x_i \geq 0$, $c_i \geq 0$, and $1 + \sum_k c_k \geq 0$. Moreover we have

$$\sum_{i=1}^{m} x_i' \; = \; \left(\frac{1}{1 + \sum\limits_{k=1}^{m} c_k(x,y)} \right) \sum_{i=1}^{m} (x_i + c_i(x,y)) \; = \; 1\,.$$

Likewise $y_j' \geq 0$ and $\sum_j y_j' = 1$. Hence T maps $X_m \times Y_n$ into itself.

We first show that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is an equilibrium pair if and only if it is a fixed point for the mapping T. To do this, note first that $c_i(x,y)$ measures the amount that α_i is better than x, if at all, as a response against y, and that $d_j(x,y)$ measures the amount that β_j is better than y as a response against x. Now suppose that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are an equilibrium pair. Since $\hat{\mathbf{x}}$ is good against $\hat{\mathbf{y}}$, it follows that $c_i(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = 0$ for all i, so $\hat{\mathbf{x}}_i' = \hat{\mathbf{x}}_i$ for all i. Similarly, $\hat{\mathbf{y}}_j' = \hat{\mathbf{y}}_j$. Thus, if $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is an equilibrium pair, $T(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = (\hat{\mathbf{x}}, \hat{\mathbf{y}})$.

To show the converse, suppose that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a fixed point of T. We first show that there must be at least one index i_o for which both $\hat{\mathbf{x}}_{i_o} > 0$ and $c_i(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = 0$.

To see this, recall the definition of $V(x,y)$ and note that

$$V(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \; = \; \sum_{i=1}^{m} \hat{\mathbf{x}}_i \, V(\alpha_i, \hat{\mathbf{y}})\,,$$

and we may conclude that $V(\hat{\mathbf{x}}, \hat{\mathbf{y}}) < V(\alpha_i, \hat{\mathbf{y}})$ *cannot* hold for all i such that $\hat{\mathbf{x}}_i > 0$. Thus, for at least one i, say i_o we have $c_{i_o}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = V(\alpha_{i_o}, \hat{\mathbf{y}}) - V(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = 0$. But then, the fact that $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is a fixed point for T implies that

$$x_{i_o} \; = \; \frac{\hat{\mathbf{x}}_{i_o}}{1 + \sum\limits_{k=1}^{m} c_k(\hat{\mathbf{x}}, \hat{\mathbf{y}})}\,,$$

so that $\sum_k c_k(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = 0$. But the terms $c_k(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \geq 0$ so they must all be zero.

From this we can conclude that $\hat{\mathbf{x}}$ is at least as good a response against $\hat{\mathbf{y}}$ as any α_k so $\hat{\mathbf{x}}$ is good against $\hat{\mathbf{y}}$. Similarly, $\hat{\mathbf{y}}$ can be shown to be good against $\hat{\mathbf{x}}$. So $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is an equilibrium pair.

Now that we know that the equilibrium pairs are fixed points of T we need only check that T does in fact have fixed points. This follows from the Brouwer theorem since $T : X_m \times Y_n \longrightarrow X_m \times Y_n$ and, as is easily checked from the definition, T is continuous. ■