E 2711. *Proposed by Frank Uhlig, Aachen, Germany*

Let A and B be $m \times m$ matrices over a field. If the characteristic polynomial of A is irreducible, show that $\operatorname{rank}(AB - BA) \neq 1$.

E 2712. *Proposed by A. Wilansky, Lehigh University*

Let A be a linear map from real bounded sequences to the real numbers, such that for each sequence x some subsequence of x converges to $A(x)$. Must $A(xy) = A(x)A(y)$?

SOLUTIONS OF ELEMENTARY PROBLEMS

Average Distance between Two Points in a Box

E 2629 [1977, 57]. *Proposed by David P. Robbins, Phillips Exeter Academy, New Hampshire*

Two points are chosen at random (uniform distribution) in the box $|x| \leqq a$, $|y| \leqq b$, $|z| \leqq c$ of $\mathbf{R}^3$. What is the expected distance between them?

(The cases when $\mathbf{R}^3$ is replaced by $\mathbf{R}$ or $\mathbf{R}^2$ are well known.)

Solution by Theodore S. Bolis, State University College at Oneonta. Let X_1, X_2; Y_1, Y_2; Z_1, Z_2 be independent random variables uniformly distributed in $[-a, a]$; $[-b, b]$; $[-c, c]$, respectively. Then the random variables $U = |X_1 - X_2|$, $V = |Y_1 - Y_2|$, $W = |Z_1 - Z_2|$ are also independent. It is easy to show that the density function of U, say, is

$$f(u) = (2a - u)/2a^2, \qquad 0 \leqq u \leqq 2a.$$

The sought expectation is given by

$$E = \frac{1}{8a^2b^2c^2} \int_0^{2c} \int_0^{2b} \int_0^{2a} \sqrt{u^2 + v^2 + w^2}\,(2a - u)(2b - v)(2c - w)\,du\,dv\,dw.$$

Let $P(a, b, c)$ be the pyramid determined by the planes $u = 2a$, $v = 0$, $w = 0$, $av = bu$, and $aw = cu$. Set

$$F(a, b, c) = \int\int\int_{P(a,b,c)} \sqrt{u^2 + v^2 + w^2}\,(2a - u)(2b - v)(2c - w)\,du\,dv\,dw.$$

Then it is clear that $F(a, b, c) = F(a, c, b)$ and that

$$E = \frac{1}{8a^2b^2c^2}\,(F(a, b, c) + F(c, a, b) + F(b, c, a)). \tag{1}$$

Thus it suffices to compute $F(a, b, c)$. By using spherical coordinates we obtain

$$F(a, b, c) = \int_0^{\tan^{-1}(b/a)} \int_{\cot^{-1}((c/a)\cos\theta)}^{\pi/2} \int_0^{2a \csc\phi \sec\theta} \Phi\, d\rho\, d\phi\, d\theta,$$

where

$$\Phi = \rho^3(2a - \rho \sin\phi \cos\theta)(2b - \rho \sin\phi \sin\theta)(2c - \rho\cos\phi)\sin\phi.$$

By tedious but routine successive integration one finds that

$$F(a, b, c) = \frac{64}{315}a^7 - \frac{8}{315}a^2(8a^4 - 19a^2b^2 - 6b^4)r_3 - \frac{8}{315}a^2(8a^4 - 19a^2c^2 - 6c^4)r_2$$
$$+ \frac{8}{315}a^2(8a^4 - 6b^4 - 6c^4 - 19a^2b^2 - 19a^2c^2 + 30b^2c^2)r$$

$$+\frac{8}{15}a^6 b\sinh^{-1}\frac{b}{a}+\frac{8}{15}a^6 c\sinh^{-1}\frac{c}{a}$$

$$-\frac{8}{15}a^2 b(a^4-4a^2c^2-c^4)\sinh^{-1}\frac{b}{r_2}$$

$$-\frac{8}{15}a^2 c(a^4-4a^2b^2-b^4)\sinh^{-1}\frac{c}{r_3}-\frac{32}{15}a^5 bc\sin^{-1}\frac{bc}{r_2r_3},$$

where

$$r=\sqrt{a^2+b^2+c^2},\qquad r_1=\sqrt{b^2+c^2};$$

$$r_2=\sqrt{c^2+a^2},\qquad r_3=\sqrt{a^2+b^2}.$$

Substituting in (1) we obtain

$$E=\frac{2}{15}r-\frac{7}{45}\left[(r-r_1)\left(\frac{r_1}{a}\right)^2+(r-r_2)\left(\frac{r_2}{b}\right)^2+(r-r_3)\left(\frac{r_3}{c}\right)^2\right]$$

$$+\frac{8}{315a^2b^2c^2}(a^7+b^7+c^7-r_1^7-r_2^7-r_3^7+r^7)$$

$$+\frac{1}{15ab^2c^2}\left(b^6\sinh^{-1}\frac{a}{b}+c^6\sinh^{-1}\frac{a}{c}-r_1^2(r_1^4-8b^2c^2)\sinh^{-1}\frac{a}{r_1}\right)$$

$$+\frac{1}{15a^2bc^2}\left(c^6\sinh^{-1}\frac{b}{c}+a^6\sinh^{-1}\frac{b}{a}-r_2^2(r_2^4-8c^2a^2)\sinh^{-1}\frac{b}{r_2}\right)$$

$$+\frac{1}{15a^2b^2c}\left(a^6\sinh^{-1}\frac{c}{a}+b^6\sinh^{-1}\frac{c}{b}-r_3^2(r_3^4-8a^2b^2)\sinh^{-1}\frac{c}{r_3}\right)$$

$$-\frac{4}{15abc}\left(a^4\sin^{-1}\frac{bc}{r_2r_3}+b^4\sin^{-1}\frac{ca}{r_3r_1}+c^4\sin^{-1}\frac{ab}{r_1r_2}\right).$$

In the case of the unit cube, $a=b=c=\frac{1}{2}$, we obtain

$$E=\frac{1}{105}[4+17\sqrt{2}-6\sqrt{3}+21\log(1+\sqrt{2})+42\log(2+\sqrt{3})-7\pi], \tag{2}$$

i.e., $E\doteq 0.661707$.

Formula (2) was found also by Günter Bach & Frank Piefke (West Germany), and by the proposer.

Comments. Bach and Piefke use in their solution a formula due to J. F. C. Kingman (Journal of Applied Probability, 6 (1969), p. 668) for the expectation of the kth power of the distance between two random points of a convex body in $\mathbf{R}^n$, and a formula due to R. Coleman (ibid., p. 439) for the density function of secants of fixed length of the unit cube.

The solution of the analogous problem for an n-dimensional ball is known, see L. A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley, 1976, p. 212.

Polyhedral Models

E 2630 [1976, 57]. *Proposed by Edward T. Ordman, University of Kentucky, Lexington*

Suppose that a polyhedral model (made, say, of cardboard) is slit along certain edges and unfolded to lie flat in the plane. The cuts may not be made so as to disconnect the figure. Now suppose that the resulting plane figure is again folded up to make a polyhedron (folding is allowed only on the original