Taylor & Francis
Taylor & Francis Group

The American Mathematical Monthly

ISSN: 0002-9890 (Print) 1930-0972 (Online) Journal homepage: http://www.tandfonline.com/loi/uamm20

Problems and Solutions

Gerald A. Edgar, Daniel H. Ullman & Douglas B. West

To cite this article: Gerald A. Edgar, Daniel H. Ullman & Douglas B. West (2018) Problems and Solutions, The American Mathematical Monthly, 125:3, 276-283, DOI: 10.1080/00029890.2018.1424478

To link to this article: https://doi.org/10.1080/00029890.2018.1424478

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Paul Zeitz, and Fuzhen Zhang.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit
Proposed solutions to the problems below should be submitted by July 31, 2018 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12027. *Proposed by Abdul Hannan, Chennai Mathematical Institute, Chennai, India.* Let ABC be a triangle with circumradius R and inradius r. Let D, E, and F be the points where the incircle of ABC touches BC, CA, and AB, respectively, and let X, Y, and Z be the second points of intersection between the incircle of ABC and AD, BE, and CF, respectively. Prove

$$\frac{|AX|}{|XD|} + \frac{|BY|}{|YE|} + \frac{|CZ|}{|ZF|} = \frac{R}{r} - \frac{1}{2}.$$

12028. *Proposed by Michael Elgersma, Minneapolis, MN, Ramin Naimi, Occidental College, Los Angeles, CA, and Stan Wagon, Macalester College, St. Paul, MN.* We have n coins, where $n = d + p + q$ for positive integers d, p, and q. Suppose that whenever any d of the coins are removed, the rest can be split into sets of size p and q that balance when placed on a balance with arm lengths q and p, respectively. That is, q times the weight of the p coins equals p times the weight of the q coins. Must all n coins have the same weight?

12029. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For $a > 0$, evaluate

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(a + \frac{k}{n}\right).$$

12030. *Proposed by Jonathan Sondow, New York, NY.* Let S be the set of positive integers d such that, for some multiple m of d,

$$\binom{m+d}{d} \equiv 1 \pmod{m}.$$

(a) Does S contain a prime number?
(b) Does S contain a number with distinct prime factors?
(c)* Does S contain a nontrivial prime power?

doi.org/10.1080/00029890.2018.1424478

12031. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*
(a) Prove

$$\int_0^1 \int_0^1 \left\{ \frac{x}{1-xy} \right\} dx\, dy = 1 - \gamma,$$

where $\{a\}$ denotes the fractional part of a, and γ is Euler's constant.
(b) Let k be a nonnegative integer. Prove

$$\int_0^1 \int_0^1 \left\{ \frac{x}{1-xy} \right\}^k dx\, dy = \int_0^1 \left\{ \frac{1}{x} \right\}^k dx.$$

12032. *Proposed by David Galante (student) and Ángel Plaza, University of Las Palmas de Gran Canaria, Las Palmas, Spain.* For a positive integer n, compute

$$\sum_{p=0}^{n} \sum_{k=p}^{n} (-1)^{k-p} \binom{k}{2p} \binom{n}{k} 2^{n-k}.$$

12033. *Proposed by Dao Thanh Oai, Thai Binh, Vietnam, and Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let $ABCD$ be a convex quadrilateral with area S. Prove

$$AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2 \geq 8S + AB \cdot CD + BC \cdot AD - AC \cdot BD.$$

SOLUTIONS

A Radical Bound

11906 [2016, 400]. *Proposed by Robert Bosch, Archimedean Academy, FL.* Let x, y, and z be positive numbers such that $xyz = 1$. Prove

$$\sqrt{\frac{x+1}{x^2-x+1}} + \sqrt{\frac{y+1}{y^2-y+1}} + \sqrt{\frac{z+1}{z^2-z+1}} \leq 3\sqrt{2}.$$

Solution by Ramya Dutta, Chennai Mathematical Institute, Chennai, India. Since $xyz = 1$, we can choose $a, b, c \in \mathbb{R}^+$ such that $x = a/b$, $y = b/c$, and $z = c/a$. The inequality then becomes

$$\sum_{\text{cyc}} \sqrt{\frac{b(a+b)}{a^2-ab+b^2}} \leq 3\sqrt{2},$$

where $\sum_{\text{cyc}} \tau(a,b,c)$ denotes the cyclic sum $\tau(a,b,c) + \tau(b,c,a) + \tau(c,a,b)$. Since

$$\sum_{\text{cyc}} \sqrt{\frac{b(a+b)}{a^2-ab+b^2}} = \sum_{\text{cyc}} \sqrt{\frac{b(a+b)}{\frac{1}{4}(a+b)^2 + \frac{3}{4}(a-b)^2}} \leq \sum_{\text{cyc}} 2\sqrt{\frac{b}{a+b}}$$

and since

$$\sum_{\text{cyc}} \sqrt{\frac{b}{a+b}} = \sum_{\text{cyc}} \sqrt{b+c} \cdot \sqrt{\frac{b}{(a+b)(b+c)}}$$

$$\leq \left(\sum_{\text{cyc}}(b+c)\right)^{1/2} \cdot \left(\sum_{\text{cyc}} \frac{b}{(a+b)(b+c)}\right)^{1/2} = \sqrt{2} \cdot \left[\sum_{\text{cyc}} \frac{b(a+b+c)}{(a+b)(b+c)}\right]^{1/2},$$

by the Cauchy–Schwarz inequality, it suffices to prove

$$\sum_{\text{cyc}} \frac{b(a+b+c)}{(a+b)(b+c)} \leq \frac{9}{4}.$$

This is equivalent to

$$\sum_{\text{cyc}} \left(1 - \frac{b(a+b+c)}{(a+b)(b+c)}\right) \geq \frac{3}{4}, \qquad \text{or} \qquad \sum_{\text{cyc}} \frac{ac}{(a+b)(b+c)} \geq \frac{3}{4},$$

which is in turn equivalent to

$$\sum_{\text{cyc}} ac(a+c) \geq \frac{3}{4}(a+b)(b+c)(a+c) = \frac{3}{4}\left(2abc + \sum_{\text{cyc}} ac(a+c)\right).$$

Solving this for the cyclic sum, we obtain

$$\sum_{\text{cyc}} ac(a+c) \geq 6abc.$$

This follows from the AM–GM inequality applied to the six terms a^2b, ab^2, a^2c, ac^2, b^2c, and bc^2. Therefore,

$$\sum_{\text{cyc}} \sqrt{\frac{b(a+b)}{a^2-ab+b^2}} \leq 2\sum_{\text{cyc}} \sqrt{\frac{b}{a+b}} \leq 2\sqrt{2} \cdot \sqrt{\frac{9}{4}} = 3\sqrt{2}.$$

Also solved by R. A. Agnew, T. Amdeberhan & V. H. Moll, R. Boukharfane (France), P. Bracken, M. V. Channakeshava (India), H. Chen, P. P. Dályay (Hungary), M. Dincă (Romania), H. Evans, D. Fleischman, S. Gayen (India), J.-P. Grivaux (France), N. Grivaux (France), O. Kouba (Syria), M. E. Kuczma (Poland), K.-W. Lau (China), J. H. Lindsey II, S. Malikić (Canada), V. Mikayelyan (Armenia), M. Omarjee (France), Á. Plaza (Spain), J. C. Smith, A. Stenger, R. Stong, R. Tauraso (Italy), T. Wiandt, M. R. Yegan (Iran), Con Amore Problem Group (Denmark), FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), GWstat Problem Solving Group, NSA Problems Group, and the proposer.

An Inequality Applied To Eigenvalues

11907 [2016, 400]. *Proposed by Xiang-Qian Chang, Massachusetts College of Pharmacy and Health Sciences, Boston, MA.* Let A be an $n \times n$ positive-definite Hermitian matrix, with minimum and maximum eigvenvalues λ and ω, respectively. Prove

$$\left(\frac{1}{\omega}\frac{\mathrm{Tr}(A)}{n} + \frac{\omega n}{\mathrm{Tr}(A)}\right)^n \leq \det\left(\frac{1}{\omega}A + \omega A^{-1}\right),$$

$$\left(\frac{1}{\lambda}\frac{n}{\mathrm{Tr}(A^{-1})} + \lambda\frac{\mathrm{Tr}(A^{-1})}{n}\right)^n \leq \det\left(\frac{1}{\lambda}A + \lambda A^{-1}\right).$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The function $f(t) = \log(t + 1/t) = \log(t^2+1) - \log t$ has

$$f''(t) = \frac{1}{t^2} + \frac{2(1-t^2)}{(1+t^2)^2} \geq 0$$

for $0 < t \leq 1$. Hence by Jensen's inequality, for any $x_1, \ldots, x_n \in (0, 1]$, we have

$$\left(\frac{x_1 + \cdots + x_n}{n} + \frac{n}{x_1 + \cdots + x_n}\right)^n \leq \prod_{k=1}^{n}(x_k + x_k^{-1}).$$

Let the eigenvalues of A (with multiplicities) be $\lambda_1, \ldots, \lambda_n > 0$. Applying this inequality to $x_k = \lambda_k/\omega$, we obtain the first inequality requested. Applying it to $x_k = \lambda/\lambda_k \leq 1$, we obtain the second inequality.

Editorial comment. The problem statement contained a typographical error: The exponent n was missing from the second inequality.

Also solved by H. Chen, P. P. Dályay (Hungary), R. Dutta (India), D. Fleischman, N. Grivaux (France), E. A. Herman, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France) & R. Tauraso (Italy), J. C. Smith, FAU Problem Solving Group, GCHQ Problem Solving Group (U. K.), and the proposer.

A Generalized Bijection for Partitions

11908 [2016, 504]. *Proposed by George E. Andrews, The Pennsylvania State University, University Park, PA, and Emeric Deutsch, Polytechnic Institute of New York University, Brooklyn, NY.* Let n and k be nonnegative integers. Show that the number of partitions of n having k even parts is the same as the number of partitions of n in which the largest repeated part is k (defined to be 0 if the parts are all distinct). For example, 7 has three partitions with two even parts $(4 + 2 + 1 = 3 + 2 + 2 = 2 + 2 + 1 + 1 + 1)$ and also three partitions in which the largest repeated part is 2 $(3 + 2 + 2 = 2 + 2 + 2 + 1 = 2 + 2 + 1 + 1 + 1)$.

Solution I by Meghana Madhyastha, International Institute of Information Technology, Bangalore, India. Fixing k, we find the generating functions of the two quantities, indexed by n. In a partition where k is the largest repeated part, each part smaller than k can appear any number of times, k appears at least twice, and parts larger than k appear at most once. Hence, the generating function is

$$\left(\prod_{i=1}^{k-1} \frac{1}{1 - x^i}\right) \frac{x^{2k}}{1 - x^k} \prod_{i=k+1}^{\infty} (1 + x^i).$$

For a partition with exactly k even parts, consider the even and odd parts separately. In the conjugate of the partition using the even parts, each part occurs an even number of times, and the largest part is k (occurring at least twice). There is no restriction on the use of odd parts. Hence, the generating function is

$$\left(\prod_{i=1}^{k-1} \frac{1}{1 - x^{2i}}\right) \frac{x^{2k}}{1 - x^{2k}} \prod_{i=1}^{\infty} \frac{1}{1 - x^{2i-1}}.$$

Straightforward manipulation shows that both generating functions equal

$$x^{2k} \prod_{i=1}^{\infty} \frac{1 - x^{2(k+i)}}{1 - x^i}.$$

Solution II by Nicolas Allen Smoot, Georgia Southern University, Statesboro, GA. We prove the following generalization: Given nonnegative integers n and k and a positive integer d, the number of partitions of n having exactly k parts divisible by d is the same as the number of partitions of n in which k is the largest part that occurs at least d times.

When $n = 0$, the claim is trivial, so assume $n > 0$. We construct a bijection. Let λ be a partition of n having exactly k parts divisible by d. Let A consist of all the parts in λ that

are divisible by d, and let B consist of the other parts (A is empty when $k = 0$). Map A to its conjugate partition A^*, in which the largest part is k and every part occurs a multiple of d times.

We map B, which has no part divisible by d, to a partition B', in which no part occurs at least d times, bijectively. For this we use Glaisher's bijection (J. W. L. Glaisher, A theorem in partitions, *Messenger of Math.* **12** (1883) 158–170). This turns B into B' by iteratively combining d equal parts into one part until no instance of d identical parts remains. The proof that this is a bijection relies on the fact that every positive integer is expressible as a power of d times a number not divisible by d in a unique way.

Note that in the union of A^* and B', the largest part occurring at least d times is k. To invert the map, we separate a partition μ in which k is the largest part occurring at least d times into the contributions A^* and B', where A^* will have each part occurring a multiple of d times (k being the largest part) and B' will have no part occurring at least d times.

For each part i, occurring m_i times in μ, put $d\lfloor m_i/d\rfloor$ of the copies of i into A^*. Put the remaining copies into B'; no part occurs at least d times among these. This is the only way that μ can be separated into two partitions in the specified families. We can now invert the two maps separately and recombine the outcomes to retrieve the only partition λ that maps to μ under the given function.

Editorial comment. Mingjia Yang also proved the generalization in Solution II.

Also solved by D. Beckwith, K. David, Y. J. Ionin, P. Lalonde (Canada), P. W. Lindstrom, G. Lord, O. P. Lossers (Netherlands), R. Nandan, M. Sawhney, J. H. Smith, R. Stong, R. Tauraso (Italy), V. Walavalkar (India), E. T. White, M. Wildon (U. K.), M. Yang, GCHQ Problem Solving Group (U. K.), and the proposers.

Reciprocal Fibonacci Arctangents

11910 [2016, 504]. *Proposed by Cornel Ioan Vălean, Teremia Mare, Romania.* Let G_k be the reciprocal of the kth Fibonacci number, for example, $G_4 = 1/3$ and $G_5 = 1/5$. Find

$$\sum_{n=1}^{\infty}\left(\arctan G_{4n-3} + \arctan G_{4n-2} + \arctan G_{4n-1} - \arctan G_{4n}\right).$$

Solution by Ramya Dutta, Chennai Mathematical Institute, Chennai, India. The sum is $\pi/2 + \arctan((\sqrt{5}-1)/2)$. To see this, we write F_n for the nth Fibonacci number, and we make use of Catalan's identity $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$ and d'Ocagne's identity $F_nF_{n+1} + F_nF_{n-1} = F_{2n}$. Since

$$\arctan\frac{1}{F_{2n}} - \arctan\frac{1}{F_{2n+2}} = \arctan\frac{F_{2n+2}-F_{2n}}{F_{2n}F_{2n+2}+1} = \arctan\frac{F_{2n+1}}{F_{2n+1}^2} = \arctan\frac{1}{F_{2n+1}},$$

we have

$$\begin{aligned}\arctan\frac{1}{F_{4n-3}} + \arctan\frac{1}{F_{4n-1}} &= \arctan\frac{1}{F_{4n-4}} - \arctan\frac{1}{F_{4n-2}}\\ + \arctan\frac{1}{F_{4n-2}} - \arctan\frac{1}{F_{4n}} &= \arctan\frac{1}{F_{4n-4}} - \arctan\frac{1}{F_{4n}}.\end{aligned} \tag{1}$$

Equation (1) holds for all positive integers n, including $n = 1$, provided that we interpret $\arctan(1/0)$ to be $\pi/2$. We also have

$$\begin{aligned}\arctan\frac{F_{n-1}}{F_n} - \arctan\frac{F_n}{F_{n+1}} &= \arctan\frac{F_{n+1}F_{n-1} - F_n^2}{F_nF_{n-1} + F_nF_{n+1}}\\ &= \arctan\frac{(-1)^n}{F_{2n}} = (-1)^n\arctan\frac{1}{F_{2n}}.\end{aligned} \tag{2}$$

Thus,

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\arctan\frac{1}{F_{4n-3}}+\arctan\frac{1}{F_{4n-2}}+\arctan\frac{1}{F_{4n-1}}-\arctan\frac{1}{F_{4n}}\right)\\
&\quad=\sum_{n=1}^{\infty}\left(\arctan\frac{1}{F_{4n-4}}-\arctan\frac{1}{F_{4n}}\right)+\sum_{n=1}^{\infty}\left(\arctan\frac{1}{F_{4n-2}}-\arctan\frac{1}{F_{4n}}\right)\quad\text{by (1)}\\
&\quad=\frac{\pi}{2}+\sum_{n=1}^{\infty}(-1)^{n-1}\arctan\frac{1}{F_{2n}}=\frac{\pi}{2}-\sum_{n=1}^{\infty}\left(\arctan\frac{F_{n-1}}{F_n}-\arctan\frac{F_n}{F_{n+1}}\right)\quad\text{by (2)}\\
&\quad=\frac{\pi}{2}+\lim_{n\to\infty}\arctan\frac{F_n}{F_{n+1}}=\frac{\pi}{2}+\arctan\frac{1}{\varphi},
\end{aligned}$$

where $\varphi=(1+\sqrt{5})/2$. This gives the claimed result.

Also solved by K. Adegoke (Nigeria) & Á. Plaza (Spain), B. Bradie, M. V. Channakeshava (India), P. P. Dályay (Hungary), D. Fleischman, D. Fritze (Germany), M. Goldenberg & M. Kaplan, S. Hitotumatu (Japan), O. Kouba (Syria), M. E. Kuczma (Poland), P. Lalonde (Canada), O. P. Lossers (Netherlands), R. Nandan, M. Omarjee (France), A. Rajkumar & F. Mawyer, M. Sawhney, A. Stenger, R. Stong, R. Tauraso (Italy), D. Terr, D. B. Tyler, M. Wildon (U. K.), J. Zacharias, L. Zhou, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

A Symmetric Inequality for Real Triples

11911 [2016, 504]. *Proposed by Marian Cucoanes, Focşani, Romania, and Leonard Giugiuc, Drobeta-Turnu Severin, Romania.* Let a, b, and c be positive real numbers such that $1+ab+bc+ca=a+b+c+2abc$. Prove $a^3+b^3+c^3+5abc\geq 1$, and determine when equality holds.

Solution by Marcin E. Kuczma, University of Warsaw, Poland. The claim holds trivially when $\max\{a,b,c\}\geq 1$. Hence we assume $a,b,c\in(0,1)$, which yields $abc<1$. In terms of the elementary symmetric polynomials $A=a+b+c$, $B=ab+bc+ca$, and $C=abc$, the constraint says $1+B=A+2C$. Let

$$X=a^3+b^3+c^3+5abc-1=A^3-3AB+8C-1=A^3+(4-3A)B+(3-4A).$$

We must show that X is nonnegative.

The AM–GM inequality implies $B/3\geq C^{2/3}$. Combining this with $C<1$ yields $B\geq 3C^{2/3}>3C>2C$, which implies $A=1+B-2C>1$. If $4-3A>0$, then $B=A+2C-1>A-1$ yields

$$X>A^3+(4-3A)(A-1)+(3-4A)=(A-1)^3>0.$$

If $4-3A\leq 0$, then the Cauchy–Schwarz inequality yields $A^2\leq 3(a^2+b^2+c^2)=3(A^2-2B)$, and thus $B\leq A^2/3$. Therefore,

$$X\geq A^3+(4-3A)(A^2/3)+(3-4A)=(2A-3)^2/3\geq 0.$$

Equality requires $2A-3=0$ as well as equality in the Cauchy–Schwarz application; the latter occurs when $a=b=c$. Thus equality holds if and only if $a=b=c=1/2$.

Also solved by A. Alt, P. P. Dályay (Hungary), M. Dincă (Romania), D. Fleischman, N. Grivaux (France), Y. Ionin, K.-W. Lau (China), J. H. Lindsey II, T. L. McCoy, R. Stong, T. Wiandt, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.

Bounds for the Sum of the Mixtilinear Radii

11912 [2016, 505]. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Let ω be the circumscribed circle of triangle ABC, and let R and r be the radii of its circumcircle and incircle,

respectively. Let r_A, r_B, and r_C be the radii of the A-, B-, and C-mixtilinear incircles of ABC and ω, respectively. Prove $4r \le r_A + r_B + r_C \le (5R + 6r)/4$.

Solution by Tamas Wiandt, Rochester Institute of Technology, Rochester, NY.

Lower bound: Let the angles at A, B, and C be α, β, and γ, respectively. We have $r_A = r\sec^2(\alpha/2)$, and similarly for r_B and r_C. (See, for example, L. Bankoff, A mixtilinear adventure, *Crux Mathematicorum* **9** (1983) 2–7.) Now we have

$$\sec^2\left(\frac{\alpha/2+\beta/2+\gamma/2}{3}\right) = \sec^2\frac{\pi}{6} = \frac{4}{3}.$$

For $x \in (0, \pi)$, let $f(x) = \sec^2(x/2)$. Since $f''(x) \ge 0$, we may apply Jensen's inequality to f and obtain

$$\begin{aligned} 4r &= 3r\sec^2\left(\frac{\alpha/2+\beta/2+\gamma/2}{3}\right) \\ &\le 3r\frac{\sec^2(\alpha/2)+\sec^2(\beta/2)+\sec^2(\gamma/2)}{3} = r_A + r_B + r_C. \end{aligned}$$

Upper bound: Let a, b, c denote the side lengths opposite angles A, B, C, respectively. Let x, y, z be the distances from A, B, C, respectively, to the points of tangency of the incircle. Since $a = y + z$, $b = z + x$, and $c = z + y$, the semiperimeter s of the triangle is $x + y + z$. Now $\sec^2(\alpha/2) = \tan^2(\alpha/2) + 1 = r^2/x^2 + 1$, and similarly for β and γ. So $r_A = r(r^2/x^2 + 1)$, $r_B = r(r^2/y^2 + 1)$, and $r_C = r(r^2/z^2 + 1)$, and the desired inequality becomes

$$r^2\left(\frac{r^2}{x^2}+\frac{r^2}{y^2}+\frac{r^2}{z^2}\right) + 3r^2 \le \frac{3r^2}{2} + \frac{5Rr}{4}.$$

The area T of the triangle is given by any of the three formulae $T = rs$, $T = abc/4R$, or $T^2 = s(s-a)(s-b)(s-c) = sxyz$ (Heron's formula). From the first and the third, we obtain $r^2 = xyz/s$. From the first and second, we obtain $Rr = abc/4s$. Substituting these expressions into the desired inequality yields

$$\frac{xyz}{s}\left(\frac{xyz}{sx^2}+\frac{xyz}{sy^2}+\frac{xyz}{sz^2}\right) + \frac{3xyz}{2s} \le \frac{5abc}{16s}.$$

Expressing a, b, c in terms of x, y, z and rearranging, we produce the equivalent inequality

$$\begin{aligned} &3(2x^2y^2 + 2y^2z^2 + 2z^2x^2) + 2(2x^2yz + 2y^2zx + 2z^2xy) \\ &\quad \le 5(x^3y + y^3x + y^3z + z^3y + z^3x + x^3z). \end{aligned} \tag{$*$}$$

We now recall Muirhead's inequality, which asserts the following (in the case of three variables). Let (a, b, c) and (p, q, r) be two nonnegative triples satisfying the conditions $a \ge b \ge c$, $p \ge q \ge r$, $a \ge p$, $a + b \ge p + q$, and $a + b + c = p + q + r$. For all nonnegative real numbers x, y, and z, we have $\sum x^a y^b z^c \ge \sum x^p y^q z^r$, where the sums are taken over all $3! = 6$ permutations of the three variables x, y, z.

Applying Muirhead's inequality with $(a, b, c) = (3, 1, 0)$ and $(p, q, r) = (2, 2, 0)$, we get

$$x^3y + y^3x + y^3z + z^3y + z^3x + x^3z \ge 2x^2y^2 + 2y^2z^2 + 2z^2x^2.$$

Applying Muirhead's inequality with $(a, b, c) = (3, 1, 0)$ and $(p, q, r) = (2, 1, 1)$, we get

$$x^3y + y^3x + y^3z + z^3y + z^3x + x^3z \ge 2x^2yz + 2y^2zx + 2z^2xy.$$

Together these give the required inequality ($*$).

Also solved by O. Geupel (Germany), O. Kouba (Syria), M. E. Kuczma (Poland), R. Stong, J. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposer.

An Integral Inequality

11918 [2016, 613]. *Proposed by Phu Cuong Le Van, College of Education, Hue University, Hue City, Vietnam.* Let f be n times continuously differentiable on [0, 1], with $f(1/2) = 0$ and $f^{(i)}(1/2) = 0$ when i is even and at most n. Prove

$$\left(\int_0^1 f(x)\,dx\right)^2 \le \frac{1}{(2n+1)2^{2n}(n!)^2}\int_0^1 (f^{(n)}(x))^2\,dx.$$

Solution by Patrick J. Fitzsimmons, University of California, San Diego, La Jolla, CA. Let F be an antiderivative of f. Using Taylor's theorem with remainder in integral form, we expand F in powers of $t - 1/2$ to obtain

$$F(t) = F(1/2) + \sum_{k=0}^{n-1}\frac{f^{(k)}(1/2)}{(k+1)!}\left(t-\frac{1}{2}\right)^{k+1} + \int_{1/2}^{t}\frac{f^{(n)}(x)}{n!}(t-x)^n dx$$

for any t in [0, 1]. In particular, with $t = 1$,

$$\int_{1/2}^1 f(x)\,dx = \sum_{k=0}^{n-1}\frac{f^{(k)}(1/2)}{(k+1)!}\left(\frac{1}{2}\right)^{k+1} + \int_{1/2}^{1}\frac{f^{(n)}(x)}{n!}(1-x)^n dx,$$

and with $t = 0$,

$$\int_0^{1/2} f(x)\,dx = -\sum_{k=1}^{n-1}\frac{f^{(k)}(1/2)}{(k+1)!}\left(-\frac{1}{2}\right)^{k+1} + \int_0^{1/2}\frac{f^{(n)}(x)}{n!}(-x)^n dx.$$

When we add these, the terms for odd k cancel, while the terms for even k vanish by hypothesis. It follows that

$$\int_0^1 f(x)\,dx = \int_0^1 g(x)f^{(n)}(x)\,dx,$$

where

$$g(x) = \begin{cases}(-x)^n/n! & \text{when } 0 \le x \le 1/2;\\ (1-x)^n/n! & \text{when } 1/2 \le x \le 1.\end{cases}$$

Now the desired inequality follows from the Cauchy–Schwarz inequality, because

$$\int_0^1 g(x)^2\,dx = \int_0^{1/2}\frac{x^{2n}}{(n!)^2}\,dx + \int_{1/2}^1\frac{(1-x)^{2n}}{(n!)^2}\,dx = \frac{1}{(2n+1)2^{2n}(n!)^2}.$$

Also solved by U. Abel (Germany), K. F. Andersen (Germany), P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), R. Dutta (India), N. Grivaux (France), A. Harnist (France), E. A. Herman, K. Koo (China), O. Kouba (Syria), M. E. Kuczma (Poland), J. H. Lindsey II, O. P. Lossers (Netherlands), F. Marino (Italy), V. Mikayelyan (Armenia), R. Nandan, M. Omarjee (France), Á. Plaza & F. Perdomo (Spain), M. A. Prasad (India), M. Sawhney, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, T. Wiandt, L. Zhou, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposer.