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Problems and Solutions

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PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Paul Zeitz, and Fuzhen Zhang.

Proposed problems should be submitted online at
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Proposed solutions to the problems below should be submitted by May 31, 2018 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12013. *Proposed by David Stoner, student, Harvard University, Cambridge, MA.* Suppose that a, b, c, d, e, and f are nonnegative real numbers that satisfy $a+b+c=d+e+f$. Let t be a real number greater than 1. Prove that at least one of the inequalities

$$\begin{aligned} a^t+b^t+c^t &> d^t+e^t+f^t, \\ (ab)^t+(bc)^t+(ca)^t &> (de)^t+(ef)^t+(fd)^t, \text{ and} \\ (abc)^t &> (def)^t \end{aligned}$$

is false.

12014. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let a, b, c, and d be real numbers with $bc>0$. Calculate

$$\lim_{n\to\infty}\begin{bmatrix}\cos(a/n) & \sin(b/n)\\ \sin(c/n) & \cos(d/n)\end{bmatrix}^n.$$

12015. *Proposed by Dao Thanh Oai, Kien Xuong, Vietnam.* Let ABC be a triangle, let G be its centroid, and let D, E, and F be the midpoints of BC, CA, and AB, respectively. For any point P in the plane of ABC, prove

$$PA+PB+PC \le 2(PD+PE+PF)+3PG,$$

and determine when equality holds.

12016. *Proposed by Hideyuki Ohtsuka, Saitama, Japan, and Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* For nonnegative integers m, n, r, and s, prove

$$\sum_{k=0}^{s}\binom{m+r}{n-k}\binom{r+k}{k}\binom{s}{k}=\sum_{k=0}^{r}\binom{m+s}{n-k}\binom{s+k}{k}\binom{r}{k}.$$

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12017. *Proposed by Mowaffaq Hajja, Philadelphia University, Amman, Jordan.* For $n \geq 2$, let R be the ring $F[t_1, \ldots, t_n]$ of polynomials in n variables over a field F. For j with $1 \leq j \leq n$, let $s_j = \sum \prod_{i=1}^{j} t_{m_i}$, where the sum is taken over all j-element subsets $\{m_1, \ldots, m_j\}$ of $\{1, \ldots, n\}$. This is the *elementary symmetric polynomial* of degree j in the variables $t_1, \ldots, t_n$. Let $f = \sum_{i=0}^{n} c_i s_i$ for some $c_0, \ldots, c_n$ in F with $c_1, \ldots, c_n$ not all 0. Show that f is reducible in R if and only if either $c_0 = \cdots = c_{n-1} = 0$ or $(c_0, \ldots, c_n)$ is a geometric progression, meaning that there is $r \in F$ such that $c_i = rc_{i-1}$ for all i with $1 \leq i \leq n$.

12018. *Proposed by Zachary Franco, Houston, TX.* For $n > 1$, let $k(n)$ be the largest integer k for which there exists a triangle with sides of length n^k, $(n+4)^k$, and $(n+5)^k$. Find $\lim_{n\to\infty} k(n)/n$.

12019. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Find all positive integers n such that $(2^n - 1)(5^n - 1)$ is a perfect square.

SOLUTIONS

Almost-Binary Expansions

11883 [2016, 97]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For $|q| > 1$, prove that

$$\sum_{k=0}^{\infty} \frac{1}{(q^{2^0}+q)(q^{2^1}+q)\cdots(q^{2^k}+q)} = \frac{1}{q-1}\prod_{i=0}^{\infty}\frac{1}{q^{1-2^i}+1}.$$

Solution I by Adnan Ali (student), Atomic Energy Central School–4, Mumbai, India. Setting $q = 1/x$ converts the assertion to

$$\sum_{k=0}^{\infty}\prod_{j=0}^{k}\frac{x^{2^j}}{(1+x^{2^j-1})} = \frac{x}{1-x}\prod_{i=0}^{\infty}\frac{1}{1+x^{2^i-1}}$$

for $|x| < 1$. After rearranging and summing the exponents in the numerator, we seek

$$\frac{1}{x}\left(\prod_{i=0}^{\infty}(1+x^{2^i-1})\right)\sum_{k=0}^{\infty}\frac{x^{2^{k+1}-1}}{\prod_{j=0}^{k}(1+x^{2^j-1})} = \frac{1}{1-x}.$$

The left side of this equation simplifies to $\sum_{k=0}^{\infty} x^{2^{k+1}-2}\prod_{j=k+1}^{\infty}(1+x^{2^j-1})$.

Letting $F_n(x) = \sum_{k=0}^{n} x^{2^{k+1}-2}\prod_{j=k+1}^{n}(1+x^{2^j-1})$, where an empty product is 1, observe that $F_0(x) = 1$ and that $F_n(x) = (1+x^{2^n-1})F_{n-1}(x) + x^{2^{n+1}-2}$ for $n \geq 1$. Hence it follows by induction on n that $F_n(x) = \sum_{k=0}^{2^{n+1}-2} x^k$. Letting $n \to \infty$ yields both sides of the desired identity.

Solution II by GCHQ Problem Solving Group, Cheltenham, U. K. In Solution I, the identity is reduced to

$$\sum_{k=0}^{\infty} x^{2^{k+1}-1}\prod_{j=k+1}^{\infty}(1+x^{2^j-1}) = \frac{x}{1-x}.$$

As a formal power series, this is the statement that every positive integer has a unique expression as a sum of distinct numbers of the form $2^j - 1$ for $j \geq 1$, except that the smallest number used (expressed as $2^{k+1} - 1$) can appear once or twice. We establish this by partitioning the positive integers into blocks of the form $[2^k - 1, 2^{k+1} - 2]$ for $k \geq 1$ and using induction on k.

For $k = 1$, the block is $[1, 2]$, and the partitions are 1 and $1 + 1$. For the block $[2^k - 1, 2^{k+1} - 2]$, one obtains such partitions by using $2^k - 1$ and $(2^{k+1} - 1) + (2^{k+1} - 1)$ for the two extreme elements and adding the part $2^k - 1$ to the partitions already found for 1 through $2^k - 2$. For uniqueness, note that the largest number that can be so partitioned without using a number at least $2^k - 1$ is in fact $2^k - 2$. Since $2^{k+1} - 1$ is too big for numbers in the block $[2^k - 1, 2^{k+1} - 2]$, partitions of numbers in this block must use one copy of $2^k - 1$, and then uniqueness follows inductively.

Given the identity as a formal power series, it then suffices to observe that the Taylor series for $x/(1 - x)$ converges when $|x| < 1$.

Also solved by T. Amdeberhan, P. Bracken, B. Bradie, R. Chapman (U. K.), P. P. Dályay (Hungary), R. S. Dubey, R. Dutta (India), O. Geupel (Germany), W. P. Johnson, O. Kouba (Syria), H. Kwong, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France), M. Sawhney, J. C. Smith, R. Stong, R. Tauraso (Italy), L. Zhou, and the proposer.

It's a Quartic Equation

11890 [2016, 197]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Find all x in $(1, \infty)$ such that

$$\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{1}{x^{2k-1}} + \left(\frac{x-1}{x+1}\right)^{2k-1}\right) = \frac{1}{2}\int_0^x \frac{dt}{\sqrt{1+t^2}}.$$

Solution by Thomas Horine, Indiana University Southeast, New Albany, IN. The right side equals $\frac{1}{2}\sinh^{-1} x$, which equals $\frac{1}{2}\ln(x + \sqrt{x^2+1}\,)$. For $|x| < 1$, let $f(x) = \sum_{k=1}^{\infty} x^{2k-1}/(2k-1)$. This series converges absolutely, and $f'(x) = \sum_{k=0}^{\infty} x^{2k} = 1/(1 - x^2)$, so

$$f(x) = \int_0^x \frac{dt}{1-t^2} = \tanh^{-1} x = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|.$$

When $x > 1$, both $\frac{1}{x}$ and $\frac{x-1}{x+1}$ are in $(0, 1)$, so $f(\frac{1}{x}) = \frac{1}{2}\ln|\frac{x+1}{x-1}|$ and $f(\frac{x-1}{x+1}) = \frac{1}{2}\ln|x|$. Since the left side of the original equation is $f(\frac{1}{x}) + f(\frac{x-1}{x+1})$, we need to solve

$$\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + \frac{1}{2}\ln|x| = \frac{1}{2}\ln(x + \sqrt{x^2+1}\,),$$

which reduces to $x^4 - 2x^3 - 2x^2 - 2x + 1 = 0$. Dividing this equation by x^2 yields $x^2 + \frac{1}{x^2} - 2(x + \frac{1}{x}) - 2 = 0$. With $u = x + \frac{1}{x}$, we have $u^2 - 2u - 4 = 0$. Since $u > 1$, this implies $u = 1 + \sqrt{5}$. Finally, the only solution of $1 + \sqrt{5} = x + \frac{1}{x}$ with $x > 1$ is $x = \frac{1+\sqrt{5}}{2} + \sqrt{\frac{1+\sqrt{5}}{2}}$.

Also solved by A. Ali (India), R. Amdeberhan, K. F. Andersen (Canada), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), B. S. Burdick, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), B. E. Davis, P. De (India), O. Geupel (Germany), M. L. Glasser, M. Goldenberg & M. Kaplan, N. Grivaux (France), A. Hannan (India), E. A. Herman, R. Howard, B. Karaivanov (U. S. A.) & T. Vassilev (Canada), O. Kouba (Syria), D. López-Aguayo (Mexico), O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), M. Omarjee (France), M. Panchatcharam (Ireland), R. Pratt, M. Sawhney, A. Stenger, R. Stong, R. Tauraso (Italy), D. B. Tyler, M. Vowe (Switzerland), T. Wiandt, J. Zacharias, L. Zhou, Armstrong Problem Solvers, GCHQ Problem Solving Group (U. K.), GWstat Problem Solving Group, NSA Problems Group, San Francisco University High School Problem Solving Group, and the proposer.

A Mean-Value Point

11892 [2016, 198]. *Proposed by Francisco Perdomo and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.* Let f be a real-valued continuously differentiable function on $[a, b]$ with positive derivative on (a, b). Prove that for all pairs (x_1, x_2) with $a \le x_1 < x_2 \le b$ and $f(x_1)f(x_2) > 0$, there exists $\xi \in (x_1, x_2)$ such that

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \xi - \frac{f(\xi)}{f'(\xi)}.$$

Solution by Henry Ricardo, Tappan, NY. Suppose that (x_1, x_2) is a pair satisfying the given conditions. We note that $f'(x) > 0$ on (a, b) and $f(x_1)f(x_2) > 0$ imply $f(x) \neq 0$ for $x \in [x_1, x_2]$. Thus, if we define $F(x) = -x/f(x)$ and $G(x) = -1/f(x)$, then F and G are continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), with $G'(x) \neq 0$ for $x \in (x_1, x_2)$. Therefore, we may apply Cauchy's extended mean value theorem to conclude that there exists $\xi \in (x_1, x_2)$, such that

$$\frac{F(x_2) - F(x_1)}{G(x_2) - G(x_1)} = \frac{F'(\xi)}{G'(\xi)}.$$

This yields

$$\frac{-\dfrac{x_2}{f(x_2)} + \dfrac{x_1}{f(x_1)}}{-\dfrac{1}{f(x_2)} + \dfrac{1}{f(x_1)}} = \frac{\dfrac{-f(\xi) + \xi f'(\xi)}{(f(\xi))^2}}{\dfrac{f'(\xi)}{(f(\xi))^2}},$$

which simplifies to the desired equation.

Also solved by A. Ali (India), T. Amdeberhan, K. F. Andersen (Canada), G. Apostolopoulos (Greece), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Doncal (Spain), R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), R. Dutta (India), J. Grivaux (France), J. W. Hagood, L. Han, O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Markoe, V. Mikayelyan (Armenia), M. Omarjee (France), C. G. Petalas (Greece), J. C. Smith, R. Stong, R. Tauraso (Italy), E. I. Verriest, Z. Vőrős (Hungary), T. Wiandt, M. R. Yegan (Iran), J. Zacharias, L. Zhou, GCHQ Problem Solving Group (U. K.), Northwestern University Math Problem Solving Group, NSA Problems Group, and the proposers.

Constructing an Inscribed Quadrilateral

11893 [2016, 198]. *Proposed by Florin S. Pârvănescu, Slatina, Romania.* Let O be the center of a circle, let AB and CD be the perpendicular chords of this circle that do not contain O, let M be the intersection of these chords, and suppose that MA is longer than MB and MC is longer than MD. Give a compass and straightedge construction of a quadrilateral inscribed in the circle with sides of lengths $|MA| + |MB|$, $|MC| + |MD|$, $|MA| - |MB|$, and $|MC| - |MD|$.

Solution by James Christopher Smith, Knoxville, TN. Using a straightedge, draw ray AO and let E be its other intersection with the circle. Use a compass spiked at M to produce point F on AM such that $|MF| = |MB|$. Then use a compass spiked at A to produce point G on the circle such that $|AG| = |AF|$. Now $ABEG$ is a quadrilateral with the desired side lengths. (There are two choices for point G; one choice leads to a self-intersecting polygon, whereas the other does not. Either will meet the required conditions.)

Note that AB has length $|MA| + |MB|$ and that AG has the same length as AF, which is $|MA| - |MB|$. We show next that $|BE| = |MC| - |MD|$ and $|EG| = |MC| + |MD|$.

Let ray BO intersect the circle again at P, and let PE intersect CD at N. By symmetry, $|NC| = |MD|$, so $|MN| = |MC| - |MD|$. Since ABE is a right angle, BE and $DMNC$ are

parallel (both being perpendicular to AMB). Also PNE is parallel to AMB; thus, $MNEB$ is a rectangle and $|BE| = |MN| = |MC| - |MD|$.

Since AOE is a diameter, both AGE and ABE are right triangles. Let the radius of the circle be r. Applying the Pythagorean theorem to AGE yields

$$4r^2 = |EG|^2 + |AG|^2 = |EG|^2 + (|MA| - |MB|)^2.$$

Applying the Pythagorean theorem to ABE yields

$$4r^2 = |AB|^2 + |BE|^2 = (|MA| + |MB|)^2 + (|MC| - |MD|)^2.$$

Thus,

$$\begin{aligned} |EG|^2 &= (|MC| - |MD|)^2 + (|MA| + |MB|)^2 - (|MA| - |MB|)^2 \\ &= |MC|^2 + |MD|^2 - 2|MC|\,|MD| + 4|MA|\,|MB| \\ &= |MC|^2 + |MD|^2 + 2|MC|\,|MD| \\ &= (|MC| + |MD|)^2, \end{aligned}$$

using for the penultimate equality the power-of-the-point theorem, which asserts that $|MA|\,|MB|$ is equal to $|MC|\,|MD|$. Thus, $|EG| = |MC| + |MD|$.

Editorial comment. O. P. Lossers pointed out that if chords AB and CD do not meet inside the circle, then they can be extended to cross at point M outside the circle, and the construction required is still possible. Oliver Geupel proved the following converse: For every inscribed convex quadrilateral $PQRS$ such that PR is a diameter, there are two perpendicular chords AB and CD (not containing O) with intersection M such that $|MA| - |MB| > 0$, $|MC| - |MD| > 0$, and the sides of $PQRS$ have lengths $|MA| + |MB|$, $|MC| + |MD|$, $|MA| - |MB|$, and $|MC| - |MD|$.

There was a misprint in the published statement of the problem. The third side length $|MA| - |MB|$ was inadvertently misprinted as $|MA| - |MD|$.

Also solved by R. Chapman (U. K.), O. Geupel (Germany), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), M. D. Meyerson, R. Stong, L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

Orthogonal Projection of Ellipsoids

11896 [2016, 296]. *Proposed by Ron Evans, University of California, San Diego, CA.* Let $n \geq 2$, and let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional ellipsoid, by which we mean that E has n orthogonal semi-axis vectors. (For instance, E is an ellipse in $\mathbb{R}^3$ when $n = 2$.) Show that the projection of E onto an n-dimensional subspace of $\mathbb{R}^{n+1}$ is either an n-dimensional ellipsoid or a solid $(n-1)$-dimensional set bounded by an $(n-1)$-dimensional ellipsoid (when $n = 2$, the solid is a line segment.)

Solution by Richard Stong, Center for Communications Research, San Diego, CA. More generally, we show that any image of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ under any affine map is either an ellipsoid of dimension $n-1$ (if the map is nonsingular) or the convex hull of an ellipsoid of dimension $n-k-1$ (if the map is singular with a k-dimensional kernel). Since the given ellipsoid E is an affine image of S^{n-1} and the projection is affine with at most a 1-dimensional kernel, this implies the requested result.

Note that by composing with a translation, we may assume that the affine map is simply a linear map $L : \mathbb{R}^n \to \mathbb{R}^N$ defined by a matrix A. Also, note that by restricting to the image $L(\mathbb{R}^n)$, we may assume the linear map is onto.

First, suppose $n = N$, and hence L is nonsingular. In that case, $E = L(S^{n-1}) = \{Y : Y^T(A^{-1})^T A^{-1} Y = 1\}$. Since $\Sigma = (A^{-1})^T A^{-1}$ is a symmetric positive-definite matrix, it has an orthonormal basis of eigenvectors. It follows that E is an ellipsoid.

Next, suppose that $L: \mathbb{R}^n \to \mathbb{R}^{n-k}$ has a k-dimensional kernel for some $k \geq 1$. Let K be the kernel of L, and let $\pi: \mathbb{R}^n \to K^\perp$ be the orthogonal projection of $\mathbb{R}^n$ onto the orthogonal complement $K^\perp$ of K. We can write $L = L' \circ \pi$ for some nonsingular linear map $L': K^\perp \to \mathbb{R}^{n-k}$. The image of S^{n-1} under π is the closed unit ball $B^{n-k} \subset K^\perp$. By the previous case, L' sends the boundary S_{n-k-1} of this ball to an ellipsoid of dimension $n-k-1$, and hence it sends the convex hull B^{n-k} to the convex hull of the ellipsoid.

Also solved by J.-P. Grivaux (France), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Sawhney, J. C. Smith, GCHQ Problem Solving Group (U. K.), and the proposer.

A Product of Catalan Numbers

11897 [2016, 296]. *Proposed by Pál Péter Dályay, Szeged, Hungary.* Prove for $n \geq 0$ that

$$\sum_{k+l=n,\, k\geq 0,\, l\geq 0} \frac{\binom{2k}{k}\binom{2l+2}{l+1}}{k+1} = 2\binom{2n+2}{n}. \qquad (*)$$

Solution I by Yury J. Ionin, Central Michigan University, Mount Pleasant, MI. Let $C_k = \frac{1}{k+1}\binom{2k}{k}$. It is well known that $\sum_{k+l=n} C_k C_l = C_{n+1}$ for $n \geq 0$. Note that

$$\sum_{k+l=n} l C_k C_l = \frac{1}{2} \sum_{k+l=n} (k+l) C_k C_l = \frac{n}{2} \sum_{k+l=n} C_k C_l = \frac{n}{2} C_{n+1}.$$

Now set $j = l+1$ (canceling the term $j = 0$ from the summation) to conclude

$$\sum_{k+l=n} \frac{\binom{2k}{k}\binom{2l+2}{l+1}}{k+1} = \sum_{k+l=n} (l+2) C_k C_{l+1} = \sum_{k+j=n+1} (j+1) C_k C_j - C_{n+1}$$
$$= \frac{n+1}{2} C_{n+2} + C_{n+2} - C_{n+1} = \frac{(n+3)(2n+4)!}{2(n+2)!(n+3)!} - \frac{(2n+2)!}{(n+1)!(n+2)!} = 2\binom{2n+2}{n}.$$

Solution II by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. Let

$$C(x) = \sum_{k=0}^{\infty} C_k x^{k+1} = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^{k+1}}{(k+1)}.$$

The recurrence $\sum_{k+l=n} C_k C_l = C_{n+1}$ for $n \geq 0$ with $C_0 = 1$ yields $C(x) - x = C(x)^2$. Hence, $C(x)C'(x) = \frac{1}{2}(C'(x) - 1)$. The summand on the left side of $(*)$ is the product of the coefficients of x^{k+1} in $C(x)$ and x^{l+1} in $C'(x)$, summed over $(k+1)+(l+1) = n+2$, but lacking the term for $l = -1$. Since the constant term in $C'(x)$ is 1, the sum is the coefficient of x^{n+2} in $C(x)(C'(x) - 1)$. We compute

$$C(x)(C'(x)-1) = \frac{1}{2}(C'(x)-1) - C(x) = \frac{1}{2}\sum_{k=1}^{\infty}\binom{2k}{k}x^k - \sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^{k+1}}{k+1}.$$

The coefficient of x^{n+2} is $\frac{1}{2}\binom{2n+4}{n+2} - \frac{1}{n+2}\binom{2n+2}{n+1}$, which equals $2\binom{2n+2}{n}$.

Solution III by John H. Smith, Needham, Massachusetts. Consider lattice paths in the xy-plane, consisting of unit steps in the positive x- or y-direction. It is well known that the number of such paths from $(0, 0)$ to (n, n) not rising above the line $y = x$ is the Catalan number C_n, equal to $\frac{1}{n+1}\binom{2n}{n}$.

The total number of lattice paths from $(0, -1)$ to $(n+1, n+1)$ is $\binom{2n+3}{n+1}$. Among these, the number that first meet the line $y = x$ at the point (k, k) is $\frac{1}{k+1}\binom{2k}{k}\binom{2(n+1-k)}{n+1-k}$, since the initial portion does not rise above the line $y = x - 1$. Therefore, the left side of (*) counts all paths from $(0, -1)$ to $(n+1, n+1)$ except those that first meet the line $y = x$ at $(n+1, n+1)$, since the term for $k = n+1$ is missing. Note also that $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1} = \binom{2n+2}{n+1} - \binom{2n+2}{n}$. Thus, to evaluate the sum we compute

$$\begin{aligned}\binom{2n+3}{n+1} - C_{n+1} &= \left(\binom{2n+2}{n+1} + \binom{2n+2}{n}\right) - \left(\binom{2n+2}{n+1} - \binom{2n+2}{n}\right) \\ &= 2\binom{2n+2}{n}.\end{aligned}$$

Also solved by U. Abel (Germany), A. Ali (India), T. Amdeberhan, M. Apagodu, M. Arakelian (Armenia), N. Balachandran & P. De (India), D. Beckwith, M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal, B. Bradie, R. Chapman (U. K.), H. Chen, J. Cigler (Austria), C. Georghiou (Greece), O. Geupel (Germany), M. Goldenberg & M. Kaplan, J.-P. Grivaux (France), A. Hannan (India), M. Hoffman, O. Kouba (Syria), H. Kwong, P. Lalonde (Canada), L. Mannion, R. Nandan, M. Omarjee (France), S. Pathak (Canada), Á. Plaza & S. Falcón (Spain), J. Schlosberg, E. Schmeichel, J. C. Smith, A. Stenger, M. Štofka (Slovakia), D. Stoner, R. Stong, M. Tang, R. Tauraso (Italy), Z. Vőrős (Hungary), M. Vowe (Switzerland), M. Wildon (U. K.), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

No Matter How You Slice It

11898 [2016, 297]. *Proposed by Richard Stanley, University of Miami, Coral Gables, FL.* Let n and k be integers, with $n \geq k \geq 2$. Let G be a graph with n vertices whose components are cycles of length greater than k. Let $f_k(G)$ be the number of k-element independent sets of vertices of G. Show that $f_k(G)$ depends only on k and n. (A set of vertices is independent if no two of them are adjacent.)

Solution by Edward Schmeichel, San Jose State University, San Jose, California. It suffices to show that the number of independent k-sets is the same when G consists of two cycles as when G is just one cycle. When G has more components, one can then repeatedly merge two cycles to reach a single cycle without changing the number of independent k-sets.

Let $V(C_s) = \{v_1, \ldots, v_s\}$ and $V(C_t) = \{w_1, \ldots, w_t\}$. Form C_{s+t} by replacing $v_s v_1$ and $w_t w_1$ with $v_s w_1$ and $w_t v_1$. The independent sets in C_{s+t} that do not contain $\{v_1, v_s\}$ or $\{w_1, w_t\}$ are the same as the independent sets in $C_s + C_t$ that do not contain $\{v_s, w_1\}$ or $\{w_t, v_1\}$. It suffices to pair the remaining independent k-sets in C_{s+t} and $C_s + C_t$.

Let $S = \{v_1, v_s, w_1, w_t\}$. Let I be an independent k-set in C_{s+t}. In the remaining case, $I \cap S$ is $\{v_1, v_s\}$ or $\{w_1, w_t\}$. Let m be the least index such that $v_m, w_m \notin I$; note that m exists and is less than $\min\{s, t\}$, since otherwise $|I| \geq \min\{s, t\} > k$. Define I' by exchanging the incidence vector of I over $(v_1, \ldots, v_m)$ with its incidence vector over $(w_1, \ldots, w_m)$.

The result is an independent k-set in $C_s + C_t$, since $v_m, w_m \notin I'$ and $I' \cap S$ is $\{v_s, w_1\}$ or $\{w_t, v_1\}$. The map is also an involution. Hence, it produces a one-to-one correspondence between the two desired families of independent k-sets.

Editorial comment. The proposer and most solvers used generating functions. A substantial generalization has been proved inductively by Hannah Spinoza and Douglas West. They

showed that the conclusion about independent k-sets also holds for every k-vertex subgraph, over all n-vertex graphs whose components are cycles with more than k vertices or paths with at least $k-1$ vertices, as long as the number of components that are paths is the same. They used this in determining when a graph with maximum degree 2 can be reconstructed from its multiset of k-vertex induced subgraphs.

Also solved by M. Arakelian (Armenia), D. Beckwith, R. Chapman (U. K.), Y. J. Ionin, P. Lalonde (Canada), J. H. Lindsey II, J. C. Smith, J. H. Smith, R. Stong, R. Tauraso (Italy), L. Zhou, GCHQ Problem Solving Group (U. K.), and the proposer.

A Couple of Convolutions

11899 [2016, 297]. *Proposed by Julien Sorel, PNI, Piatra Neamt, Romania.* Show that for every positive integer n,

$$\sum_{k=0}^{n}\binom{2n}{k}\binom{2n+1}{k}+\sum_{k=n+1}^{2n+1}\binom{2n+1}{k}\binom{2n}{k-1}=\binom{4n+1}{2n}+\binom{2n}{n}^2.$$

Solution I by Li Zhou, Polk State College, Winter Haven, FL. We begin with the well-known Vandermonde convolution, comparing coefficients of x^{2n} in the expansions of $(1+x)^{4n+1}$ and $(1+x)^{2n}(1+x)^{2n+1}$:

$$\binom{4n+1}{2n}=\sum_{k=0}^{2n}\binom{2n}{2n-k}\binom{2n+1}{k}=\sum_{k=0}^{n}\binom{2n}{k}\binom{2n+1}{k}+\sum_{k=n+1}^{2n}\binom{2n}{k}\binom{2n+1}{k}.$$

Manipulating the second term, we obtain

$$\begin{aligned}\sum_{k=n+1}^{2n}\binom{2n}{k}\binom{2n+1}{k}&=\sum_{k=n+1}^{2n}\binom{2n}{k}\left(\binom{2n}{k-1}+\binom{2n}{k}\right)\\&=\sum_{k=n+1}^{2n}\binom{2n}{k}\binom{2n}{k-1}+\sum_{j=n+2}^{2n+1}\binom{2n}{j-1}\binom{2n}{j-1}\\&=\sum_{k=n+1}^{2n}\binom{2n}{k}\binom{2n}{k-1}+\sum_{k=n+1}^{2n+1}\binom{2n}{k-1}\binom{2n}{k-1}-\binom{2n}{n}^2\\&=\sum_{k=n+1}^{2n+1}\binom{2n+1}{k}\binom{2n}{k-1}-\binom{2n}{n}^2,\end{aligned}$$

completing the proof.

Solution II by John H. Smith, Needham, MA. It suffices to show

$$\sum_{k=0}^{n}\binom{2n}{k}\binom{2n+1}{2n+1-k}+\sum_{k=n+1}^{2n+1}\binom{2n+1}{k}\binom{2n}{2n-k+1}-\binom{2n}{n}^2=\binom{4n+1}{2n+1}.$$

We count the ways of choosing $2n+1$ objects from $\{1,\dots,4n+1\}$. The first sum counts the choices with at most n objects from the first $2n$. The second counts those having at least $n+1$ objects from the first $2n+1$. Each choice is counted in at least one of these sums. Those counted twice are the choices having exactly n from the first $2n$, plus n from the last $2n$, plus the element $2n+1$. There are thus $\binom{2n}{n}^2$ choices counted twice.

Also solved by A. Ali (India), T. Amdeberhan, B. Bradie, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), P. De (India), N. Fontes-Merz, N. Grivaux (France), T. Guan (China), M. Hoffman, Y. J. Ionin, B. Karivanov (U. S. A) & T. S. Vassilev (Canada), O. Kouba (Syria), P. Lalonde (Canada), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Madhyastha (India), R. Nandan, M. Nathanson, S. Pathak (Canada), Á. Plaza

(Spain), M. Sawhney, E. Schmeichel, J. C. Smith, A. Stenger, R. Stong, R. Tauraso (Italy), Z. Vőrős (Hungary), M. Vowe (Switzerland), S. Y. Wang (Korea), G. Whieldon, M. Wildon (U. K.), Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), and the proposer.

Circles Next after Incircles

11900 [2016, 297]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let ABC be a triangle, and let I and r be the center and radius of its incircle. The circle with center and radius (I_A, r_A) is externally tangent to the incircle and internally tangent to sides AB and AC of ABC. Define (I_B, r_B) and (I_C, r_C) similarly. Prove for $n \geq 1$ that

$$\left(\frac{r+r_A}{r-r_A}\right)^n + \left(\frac{r+r_B}{r-r_B}\right)^n + \left(\frac{r+r_C}{r-r_C}\right)^n \geq 3 \cdot 2^n.$$

Solution by Oleh Faynshteyn, Leipzig, Germany. Let D and E be the feet of the perpendiculars dropped onto side AB from I and I_B, respectively. Let F be the foot of the perpendicular dropped onto segment ID from I_B. We have $IF = r - r_B$ and $II_B = r + r_B$. From the right triangle ΔIFI_B, we have

$$\frac{II_B}{IF} = \frac{r+r_B}{r-r_B} = \csc(B/2).$$

Similarly,

$$\frac{r+r_A}{r-r_A} = \csc(A/2) \qquad \text{and} \qquad \frac{r+r_C}{r-r_C} = \csc(C/2).$$

Since $x^n + y^n + z^n \geq \frac{1}{3^{n-1}}(x+y+z)^n$,

$$\begin{aligned}\left(\frac{r+r_A}{r-r_A}\right)^n + \left(\frac{r+r_B}{r-r_B}\right)^n + \left(\frac{r+r_C}{r-r_C}\right)^n &= \csc^n(A/2) + \csc^n(B/2) + \csc^n(C/2) \\ &\geq \frac{1}{3^{n-1}}\big(\csc(A/2) + \csc(B/2) + \csc(C/2)\big)^n. \qquad (1)\end{aligned}$$

From the harmonic-geometric mean inequality, the identity

$$\sin(A/2)\sin(B/2)\sin(C/2) = \frac{r}{4R},$$

and Euler's inequality $R \geq 2r$, we obtain

$$\frac{3}{\csc(A/2)+\csc(B/2)+\csc(C/2)} \leq \sqrt[3]{\sin(A/2)\sin(B/2)\sin(C/2)} \leq \sqrt[3]{\frac{1}{8}} = \frac{1}{2}.$$

This implies

$$\csc(A/2) + \csc(B/2) + \csc(C/2) \geq 6. \qquad (2)$$

Substituting (2) into (1), we obtain

$$\csc^n(A/2) + \csc^n(B/2) + \csc^n(C/2) \geq \frac{1}{3^{n-1}} \cdot 6^n = 3 \cdot 2^n,$$

as required.

Also solved by J. J. Ahn (Korea), A. Ali (India), R. Bagby, R. Boukharfane (France), M. V. Channakeshava (India), R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), M. Goldenberg & M. Kaplan, N. & J.-P. Grivaux (France), T. Guan (China), B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), W.-K. Lai & J. Risher, K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), R. Nandan, D. Pispinis (Saudi Arabia), M. Sawhney, V. Schindler (Germany), E. Schmeichel, J. C. Smith, N. Stanciu & T. Zvonaru (Romania), D. Stoner, R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), T. Wiandt, M. R. Yegan (Iran), J. Zacharias, L. Zhou, Armstrong Problem Solvers, Con Amore Problem Group (Denmark), GCHQ Problem Solving Group (U. K.), and the proposer.