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Problems and Solutions

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PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Paul Zeitz, and Fuzhen Zhang.

Proposed problems should be submitted online at
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Proposed solutions to the problems below should be submitted by June 30, 2018 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12020. *Proposed by Erhard Braune, Linz, Austria*. Let α, β, and γ be the radian measures of the three angles of a triangle, and let ω be the radian measure of its Brocard angle. (The *Brocard angle* of triangle ABC is the angle TAB, where T is the unique point such that $\angle TAB$, $\angle TBC$, and $\angle TCA$ are congruent.) Yff's inequality asserts that $8\omega^3$ is a lower bound for $\alpha\beta\gamma$. Show that $\omega\pi^3/4$ is an upper bound for the same product.

12021. *Proposed by Omar Sonebi, Lycée Technique, Settat, Morroco*. Let ϕ be the Euler totient function. Given $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}^+$, show that there exists $n \in \mathbb{Z}^+$ such that $an + b$ is not in the range of ϕ.

12022. *Proposed by Mircea Merca, University of Craiova, Craiova, Romania*. Let n be a positive integer, and let x be a real number not equal to -1 or 1. Prove

$$\sum_{k=0}^{n-1} \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k})}{1-x^{k+1}} = n$$

and

$$\sum_{k=0}^{n-1}(-1)^k \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k})}{1-x^{k+1}} x^{\binom{n-1-k}{2}} = nx^{\binom{n}{2}}.$$

12023. *Proposed by Vazgen Mikayelyan, Yerevan State University, Yerevan, Armenia*. Let α be a positive real number. Prove

$$\int_0^{\pi} x^{\alpha-2}\,\sin x\,dx \ge \pi^{\alpha}\frac{\alpha+6}{\alpha(\alpha+2)(\alpha+3)}.$$

12024. *Proposed by Marian Cucoaneş, Mărăşeşti, Romania, Marius Drăgan, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania*. Let x, y, and z be positive real numbers satisfying $xyz = 1$. Prove

$$(x^{10}+y^{10}+z^{10})^2 \ge 3(x^{13}+y^{13}+z^{13}).$$

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12025. *Proposed by Askar Dzhumadil'daev, S. Demirel University, Almaty, Kazakhstan.* The Chebyshev polynomials of the second kind are defined by the recurrence relation $U_0(x) = 1$, $U_1(x) = 2x$, and $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$ for $n \geq 2$. For an integer n with $n \geq 2$, prove

$$\det \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ x & 0 & 1 & \cdots & 1 & 1 \\ x^2 & x & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{n-2} & x^{n-3} & x^{n-4} & \cdots & 0 & 1 \\ x^{n-1} & x^{n-2} & x^{n-3} & \cdots & x & 0 \end{bmatrix} = (-1)^{n-1}x^{n/2}U_{n-2}(\sqrt{x}).$$

12026. *Proposed by Michel Bataille, Rouen, France.* For $n \in \mathbb{N}$, let $H_n = \sum_{k=1}^{n} 1/k$ and $S_n = \sum_{k=1}^{n}(-1)^{n-k}(H_1 + \cdots + H_k)/k$. Find $\lim_{n\to\infty} S_n/\ln n$ and $\lim_{n\to\infty}(S_{2n} - S_{2n-1})$.

SOLUTIONS

Expressing the Sum of Three Squares as the Sum of Two

11894 [2016, 296]. *Proposed by Eugen J. Ionascu, Columbus State University, Columbus, GA.* Let a, b, c, and d be integers such that $a^2 + b^2 + c^2 = d^2$ and $d \neq 0$. Let x, y, and z be three integers such that $ax + by + cz = 0$.

(**a**) Prove that $x^2 + y^2 + z^2$ can be written as the sum of two squares.

(**b**) Let $ABCD$ be a square in $\mathbb{R}^3$ with integer vertices A, B, C, and D. Show that the side lengths of $ABCD$ have the form $\sqrt{l}$, where l is the sum of two squares.

Solution by James Christopher Smith, Knoxville, TN.

(**a**) Since $d \neq 0$, at least one of a, b, and c is nonzero, so we may assume $c \neq 0$. Using $cz = -(ax + by)$ and algebraic manipulation, we obtain

$$\begin{aligned} (a^2 + c^2)c^2(x^2 + y^2 + z^2) &= (a^2 + c^2)\left(c^2x^2 + c^2y^2 + (ax + by)^2\right) \\ &= ((a^2 + c^2)x + aby)^2 + (a^2 + b^2 + c^2)c^2y^2 \\ &= ((a^2 + c^2)x + aby)^2 + (dcy)^2. \end{aligned}$$

It is well known that an integer can be written as the sum of two squares if and only if every prime congruent to 3 modulo 4 occurs in its prime factorization with even exponent. Also, every prime occurs with even exponent in the factorization of a square. Thus, each of the quantities c^2, $a^2 + c^2$, and $\left((a^2 + c^2)x + aby\right)^2 + (dcy)^2$ has the property that every prime congruent to 3 modulo 4 occurs with even exponent in its prime factorization. Also, none of these quantities equals 0. Hence after cancelation the same property holds for $x^2 + y^2 + z^2$, which proves that it can be written as the sum of two squares.

(**b**) Translate the square so that one of its vertices is at the origin; still all vertices are at integer points. Let (α, β, γ) and (x, y, z) be the coordinates of the two vertices of the square adjacent to the vertex at the origin, so $\alpha x + \beta y + \gamma z = 0$ and $l = \alpha^2 + \beta^2 + \gamma^2 = x^2 + y^2 + z^2$. Let (a, b, c) be the cross-product of (α, β, γ) and (x, y, z) (so $a = \beta z - \gamma y$, $b = \gamma x - \alpha z$, $c = \alpha y - \beta x$). We observe that (a, b, c) is orthogonal to (x, y, z) and has length l, so the claim follows from part (**a**).

Also solved by B. S. Burdick, R. Chapman (U. K.), R. Dempsey, Y. J. Ionin, O. P. Lossers (Netherlands), M. Omarjee (France), J. P. Robertson, M. Sawhney, A. Stenger, R. Stong, R. Tauraso (Italy), T. Viteam (Denmark), M. Wildon (U. K.), GCHQ Problem Solving Group (UK), and the proposer.

On the Definition of "Regularly Varying Function"

11895 [2016, 296]. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada*. Let f be a *regularly varying* function from $(0, \infty)$ into $(0, \infty)$, with index $\rho > 0$, and let g be a function from $(0, \infty)$ into $(0, \infty)$ such that $\lim_{x\to\infty} g(x) = \infty$. (A function L on $\mathbb{R}^+$ is *regularly varying with index* ρ if $\lim_{x\to\infty} L(ax)/L(x) = a^\rho$.) Prove

$$\lim_{x\to\infty} \frac{f(x)}{f(g(x))} = L \text{ if and only if } \lim_{x\to\infty} \frac{x}{g(x)} = L^{1/\rho}.$$

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The problem is not correct without some additional hypothesis that $\lim_{x\to\infty} f(ax)/f(x) = a^\rho$ occurs "uniformly in a." We provide a counterexample.

A function $L : \mathbb{R} \to \mathbb{R}$ is $\mathbb{Q}$-*linear* if $L(ax + by) = aL(x) + bL(y)$ for all $x, y \in \mathbb{R}$ and $a, b \in \mathbb{Q}$. Let $L : \mathbb{R} \to \mathbb{R}$ be a discontinuous $\mathbb{Q}$-linear function. The graph of any such function is dense in the plane. Define $h : \mathbb{R} \to (-1, 1)$ by $h(x) = \tanh(L(x)/x)$ for $x \neq 0$, and define $h(0)$ arbitrarily. The graph of h is dense in $\mathbb{R} \times [-1, 1]$. Fix a real α, and let $\beta = L(\alpha)$. We have

$$\begin{aligned} h(x+\alpha) - h(x) &= \tanh \frac{L(x+\alpha)}{x+\alpha} - \tanh \frac{L(x)}{x} \\ &= \left(\tanh \frac{L(x+\alpha)}{x+\alpha} - \tanh \frac{L(x)}{x+\alpha}\right) - \left(\tanh \frac{L(x)}{x} - \tanh \frac{L(x)}{x+\alpha}\right) \\ &= \frac{\beta}{x+\alpha} \operatorname{sech}^2 \xi_1 - \frac{\alpha L(x)}{x(x+\alpha)} \operatorname{sech}^2 \xi_2 \end{aligned}$$

for some ξ_1 between $L(x)/(x+\alpha)$ and $(L(x)+\beta)/(x+\alpha)$ and some ξ_2 between $L(x)/(x+\alpha)$ and $L(x)/x$. The first term goes to 0 as $x \to \infty$ since the hyperbolic secant is bounded by 1. The second term also goes to 0 since $\xi_2 \operatorname{sech}^2 \xi_2$ is bounded and

$$\left|\frac{\alpha L(x)}{x(x+\alpha)\xi_2}\right| \le \frac{|\alpha|}{\min\big(|x|, |x+\alpha|\big)} \to 0.$$

We conclude that $\lim_{x\to\infty} \big(h(x+\alpha) - h(x)\big) = 0$ for any α.

Now let

$$f(x) = x^\rho e^{h(\log x)}.$$

For any $a > 0$, we have

$$\frac{f(ax)}{f(x)} = a^\rho e^{h(\log x + \log a) - h(\log x)} \to a^\rho$$

as $x \to \infty$. Thus, f is "regularly varying" according to the definition given in the problem statement. Note that since β is unbounded, the rate of this convergence depends on a in a complicated way. Now from the density of the graph of h, there is a sequence $(x_n)_{n\ge1}$ of positive reals that increases monotonically to ∞ such that $(-1)^n h(\log x_n) \to 1$. Again using the density, let y_n be chosen so that y_n is within 1 of x_n and $(-1)^n h(\log y_n) \to -1$. Define g by making its graph linearly interpolate the points $(0, 0)$, (x_1, y_1), (x_2, y_2), (x_3, y_3), ... We have $\lim_{x\to\infty} x/g(x) = 1$, but

$$\frac{f(x_n)}{f(g(x_n))} = \frac{f(x_n)}{f(y_n)} = \left(\frac{x_n}{y_n}\right)^\rho e^{h(\log x_n) - h(\log y_n)} \approx e^{2(-1)^n},$$

so $\lim_{x\to\infty} f(x)/f(g(x))$ fails to exist.

Editorial comment. Most solvers noted that the textbook definition of "regularly varying" usually includes an additional condition on f such as measurability. With that additional condition, the required assertion follows from a representation theorem of Karamata found in all textbooks on the subject. The GCHQ Problem Solving Group showed that the claimed conclusion holds for all functions g going to ∞ and all $L \in [0, \infty]$ if and only if, for all compact subsets $K \subseteq (0, \infty)$, $\lim_{x\to\infty} f(ax)/f(x)$ converges uniformly to a^ρ for $a \in K$.

Also solved by P. Bracken, P. J. Fitzsimmons, O. P. Lossers (Netherlands), M. Omarjee (France), E. Omey (Belgium), J. M. Sanders, J. C. Smith, GCHQ Problem Solving Group (U. K.), and the proposer.

A Finite Semigroup of Endofunctions

11901 [2016, 399]. *Proposed by Donald Knuth, Stanford, CA.* For $n \in \mathbb{Z}^+$, let $[n] = \{1, 2, \ldots, n\}$. Define the functions $\uparrow$ and $\downarrow$ on $[n]$ by $\uparrow x = \min\{x+1, n\}$ and $\downarrow x = \max\{x-1, 1\}$. How many distinct mappings from $[n]$ to $[n]$ occur as compositions of $\uparrow$ and $\downarrow$?

Solution by Rob Pratt, Washington, DC. We show that every such mapping f has the form

$$(f(1), \ldots, f(n)) = (\underbrace{i, i, \ldots, i}_{j}, \underbrace{i+1, i+2, \ldots i+d}_{d}, \underbrace{i+d, \ldots, i+d}_{n-d-j})$$

with $d \in \{0, 1, \ldots, n-1\}$, $i \in [n-d]$, and $j \in [n-d]$. Clearly composition with both $\uparrow$ and $\downarrow$ preserves this form. From the empty composition (itself the case $i = j = 1$ and $d = n-1$), application of $\uparrow^{i-1} \circ \downarrow^{n-d-1} \circ \uparrow^{n-d-j}$ achieves the specified function, given d, i, and j.

The allowable mappings are easily counted: for $d = 0$ there are n, and for $d \in [n-1]$ both i and j may take any value in $[n-d]$. Hence, the computation is

$$n + \sum_{d=1}^{n-1} (n-d)^2 = n + \sum_{m=1}^{n-1} m^2 = n + \frac{(n-1)n(2n-1)}{6} = \frac{n(2n^2-3n+7)}{6}.$$

Editorial comment. Many solvers excluded the empty composition from their count (for $n \neq 1$). The proposer's solution included it.

Also solved by J. Bartz, C. Blatter (Switzerland), N. Caro (Spain), P. P. Dályay (Hungary), K. David, F. Eckstrom (Sweden), S. Gagola, O. Geupel (Germany), J. Grossman & S. Kruk, Y. Ionin, O. Kouba (Syria), M. Kuczma (Poland), M. Lafond (France), J. H. Lindsey II, O. P. Lossers (Netherlands), M. Meyerson, J. Olson, M. Patnott, J. Schlosberg, J. C. Smith, J. H. Smith, R. Stong, R. Tauraso (Italy), M. Wildon (U. K.), Armstrong Problem Solvers Group, Con Amore Problem Group, FAU Problem Solving Group, GCHQ Problem Solving Group, and the proposer.

A Row of Zetas

11902 [2016, 399]. *Proposed by Cornel Ioan Vălean, Teremia Mare, Timiş, Romania.* Let $\{x\}$ denote $x - \lfloor x \rfloor$, the fractional part of x. Prove

$$\int_0^1 \int_0^1 \int_0^1 \left(\left\{ \frac{x}{y} \right\} \left\{ \frac{y}{z} \right\} \left\{ \frac{z}{x} \right\} \right)^2 dx\, dy\, dz$$
$$= 1 - \frac{\zeta(2)}{2} - \frac{\zeta(3)}{2} + \frac{7\zeta(6)}{48} + \frac{\zeta(2)\zeta(3)}{18} + \frac{\zeta(3)^2}{18} + \frac{\zeta(3)\zeta(4)}{12}.$$

Solution by Rituraj Nandan, SunEdison, Maryland Heights, MO. For $(x, y, z) \in [0, 1]^3$, there are six possibilities for the ordering of x, y, and z. We write the requested integral as the sum of the integrals corresponding to these six orderings.

For $z \le y \le x$, we have $\{x/y\} = x/y - n$ whenever $x/(n+1) < y \le x/n$ and n is a positive integer, $\{y/z\} = y/z - m$ whenever $y/(m+1) < z \le y/m$ and m is a positive integer, and $\{z/x\} = z/x$. Therefore

$$\begin{aligned}
&\iiint_{0\le z\le y\le x\le 1} \left(\left\{\frac{x}{y}\right\} \left\{\frac{y}{z}\right\} \left\{\frac{z}{x}\right\} \right)^2 dx\,dy\,dz \\
&\quad = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \int_0^1 \int_{x/(n+1)}^{x/n} \int_{y/(m+1)}^{y/m} \left(\left(\frac{x}{y} - n\right) \cdot \left(\frac{y}{z} - m\right) \cdot \frac{z}{x} \right)^2 dz\,dy\,dx \\
&\quad = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{4n+1}{108m(m+1)^3 n^2 (n+1)^4} \\
&\quad = \frac{1}{108} \sum_{n=1}^{\infty} \frac{4n+1}{n^2(n+1)^4} \sum_{m=1}^{\infty} \frac{1}{m(m+1)^3}. \qquad (1)
\end{aligned}$$

We evaluate the summations by partial fractions, obtaining

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{4n+1}{n^2(n+1)^4} &= \sum_{n=1}^{\infty} \left(\left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) - \frac{2}{(n+1)^3} - \frac{3}{(n+1)^4} \right) \\
&= 1 - 2(\zeta(3) - 1) - 3(\zeta(4) - 1) = 6 - 2\zeta(3) - 3\zeta(4)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{1}{m(m+1)^3} &= \sum_{m=1}^{\infty} \left(\left(\frac{1}{m} - \frac{1}{m+1} \right) - \frac{1}{(m+1)^2} - \frac{1}{(m+1)^3} \right) \\
&= 1 - (\zeta(2) - 1) - (\zeta(3) - 1) = 3 - \zeta(2) - \zeta(3).
\end{aligned}$$

Substituting into (1), we obtain

$$\begin{aligned}
&\iiint_{0\le z\le y\le x\le 1} \left(\left\{\frac{x}{y}\right\} \left\{\frac{y}{z}\right\} \left\{\frac{z}{x}\right\} \right)^2 dx\,dy\,dz \\
&\quad = \frac{1}{108} \big(6 - 2\zeta(3) - 3\zeta(4)\big)\big(3 - \zeta(2) - \zeta(3)\big). \qquad (2)
\end{aligned}$$

For $x \le z \le y$ and $y \le x \le z$, the cyclic permutations of $z \le y \le x$, the integrals have the same form and therefore the same value.

For $y \le z \le x$, we have $\{x/y\} = x/y - n$ whenever $x/(n+1) < y \le x/n$ and n is a positive integer, $\{y/z\} = y/z$, and $\{z/x\} = z/x$. Therefore

$$\begin{aligned}
&\iiint_{0\le y\le z\le x\le 1} \left(\left\{\frac{x}{y}\right\} \left\{\frac{y}{z}\right\} \left\{\frac{z}{x}\right\} \right)^2 dx\,dy\,dz \\
&\quad = \sum_{n=1}^{\infty} \int_0^1 \int_{x/(n+1)}^{x/n} \int_y^x \left(\left(\frac{x}{y} - n\right) \cdot \frac{y}{z} \cdot \frac{z}{x} \right)^2 dz\,dy\,dx \\
&\quad = \sum_{n=1}^{\infty} \frac{4n^2 - 1}{36n^2(n+1)^4}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{36}\sum_{n=1}^{\infty}\left(4\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{n^2}-\frac{3}{(n+1)^2}-\frac{2}{(n+1)^3}+\frac{3}{(n+1)^4}\right)\\
&= \frac{1}{36}\big(4-\zeta(2)-3(\zeta(2)-1)-2(\zeta(3)-1)+3(\zeta(4)-1)\big)\\
&= \frac{1}{36}\big(6-4\zeta(2)-2\zeta(3)+3\zeta(4)\big). \qquad (3)
\end{aligned}
$$

As before, for $x \le y \le z$ and $z \le x \le y$, the cyclic permutations of $y \le z \le x$, the integrals have the same value.

Using (2) and (3), we obtain

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^2 dx\,dy\,dz\\
&= 3\cdot\frac{1}{108}\big(6-2\zeta(3)-3\zeta(4)\big)\big(3-\zeta(2)-\zeta(3)\big)+3\cdot\frac{1}{36}\big(6-4\zeta(2)-2\zeta(3)+3\zeta(4)\big)\\
&= 1-\frac{\zeta(2)}{2}-\frac{\zeta(3)}{2}+\frac{\zeta(2)\zeta(4)}{12}+\frac{\zeta(2)\zeta(3)}{18}+\frac{\zeta(3)^2}{18}+\frac{\zeta(3)\zeta(4)}{12}\\
&= 1-\frac{\zeta(2)}{2}-\frac{\zeta(3)}{2}+\frac{7\zeta(6)}{48}+\frac{\zeta(2)\zeta(3)}{18}+\frac{\zeta(3)^2}{18}+\frac{\zeta(3)\zeta(4)}{12}.
\end{aligned}
$$

Editorial comment. Several solvers derived the more general formula

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^n dx\,dy\,dz\\
&\quad = 1-\frac{3\sum_{j=1}^n \zeta(j+1)}{2(n+1)}+\frac{\sum_{j=1}^n \zeta(j+1)}{(n+1)^2(n+2)}\left(\sum_{j=1}^{n}(j+1)\zeta(j+2)\right).
\end{aligned}
$$

Also solved by T. Amdeberhan & V. H. Moll, K. F. Andersen (Canada), R. Boukharfane (France), R. Dempsey, R. Dutta (India), D. Fritze (Germany), M. L. Glasser, M. Hoffman, O. Kouba (Syria), O. P. Lossers (Netherlands), M. Omarjee (France), J. C. Smith, R. Stong, R. Tauraso (Italy), GCHQ Problem Solving Group (UK), and the proposer.

An Integral Identity

11903 [2016, 399]. *Proposed by Paolo Perfetti, Universitá Degli Studi di Roma "Tor Vergata," Rome, Italy*. Find a homogeneous polynomial p of degree 2 in a, b, c, and d such that for $0 < -d < a < b < c$,

$$
\int_0^a \frac{\sqrt{x(a-x)(x-b)(x-c)}}{x+d}\,dx = \int_b^c \frac{\sqrt{x(a-x)(x-b)(x-c)}}{x+d}\,dx
$$

if and only if $\sqrt{-d(a+d)(b+d)(c+d)} = p(a,b,c,d)$.

Solution by GCHQ Problem Solving Group, Cheltenham, U. K. The integral written on the left does not exist due to its singularity at $x = -d$. We assume that instead of $0 < -d < a < b < c$, the condition on a, b, c, d is $0 < a < b < c$ and $d > 0$, and the condition we are aiming for is $\sqrt{d(a+d)(b+d)(c+d)} = p(a,b,c,d)$.

Let

$$
p(a,b,c,d) = \frac{2ab+2bc+2ca+4ad+4bd+4cd-a^2-b^2-c^2+8d^2}{8}.
$$

We show that, for $0 < a < b < c$ and $d > 0$, we have

$$\int_0^a \frac{\sqrt{x(a-x)(x-b)(x-c)}}{x+d}dx - \int_b^c \frac{\sqrt{x(a-x)(x-b)(x-c)}}{x+d}dx$$
$$= \pi \cdot p(a,b,c,d) - \pi\sqrt{d(a+d)(b+c)(c+d)},$$

which gives the result.

Let D be the complex plane, removing vertical half-lines starting at $0, a, b, c \in \mathbb{R}$ and going through the lower half-plane. As D is a simply-connected, open subset of the plane and $z \mapsto z(z-a)(z-b)(z-c)$ is a nonzero holomorphic function on D, it has a holomorphic square root, so there is a holomorphic s on D with $s(z)^2 = z(z-a)(z-b)(z-c)$ for $z \in D$. Restricting to the real axis, the argument of s decreases by $\pi/2 \pmod{2\pi}$ as s goes from 0^- to 0^+, from a^- to a^+, from b^- to b^+, and from c^- to c^+, with the argument locally constant at other points. Hence, if we take s to be positive real on $(-\infty, 0)$, then s is negative imaginary on $(0, a)$, negative real on (a, b), positive imaginary on (b, c), and positive real again on $(c, +\infty)$.

Now consider $f(z) = s(z)/(z+d)$. Note that f is holomorphic on $D \setminus \{-d\}$ with a simple pole at $-d$. Choose $R > \max\{c, d\}$ and $\varepsilon > 0$ small. Let C be the closed contour defined as follows. The contour goes from $-R$ to R on the real axis, taking semicircular detours $C_0, C_a, C_b, C_c, C_{-d}$ of radius ε around $0, a, b, c, -d$ into the upper half-plane, with semicircle C_R in the upper half-plane from R to $-R$. In calculating the counterclockwise contour integral of f around C, the contributions from C_0, C_a, C_b, C_c are $O(\varepsilon)$ and the contribution from C_{-d} is

$$-\pi i\sqrt{d(a+d)(b+d)(c+d)} + O(\varepsilon)$$

as $\varepsilon \to 0^+$. Hence, the integral along the part of the contour from $-R$ to R along the real axis with detours $C_{-d}, C_0, C_a, C_b, C_c$ is

$$\left(\int_{-R}^{-d-\varepsilon} + \int_{-d+\varepsilon}^{-\varepsilon} - i\int_{\varepsilon}^{a-\varepsilon} - \int_{a+\varepsilon}^{b-\varepsilon} + i\int_{b+\varepsilon}^{c-\varepsilon} + \int_{c+\varepsilon}^{R}\right) \frac{\sqrt{|x|\;|x-a|\;|x-b|\;|x-c|}}{x+d}dx$$
$$-\pi i\sqrt{d(a+d)(b+d)(c+d)} + O(\varepsilon).$$

Next, we determine the behavior of $f(z)$ as $z \to \infty$ in the upper half-plane. Since

$$s(z)^2 = z^4 + (-a-b-c)z^3 + (ab+bc+ca)z^2 - abcz,$$

we have

$$s(z) = z^2 + \frac{-a-b-c}{2}z + \frac{2ab+2bc+2ca-a^2-b^2-c^2}{8} + O(z^{-1}).$$

Since

$$\frac{1}{z+d} = z^{-1} - dz^{-2} + d^2z^{-3} + O(z^{-4}),$$

we obtain

$$f(z) = z + \frac{-a-b-c-2d}{2}$$
$$+ \frac{2ab+2bc+2ca+4ad+4bd+4cd-a^2-b^2-c^2+8d^2}{8}z^{-1} + O(z^{-2})$$
$$= z + \frac{-a-b-c-2d}{2} + p(a,b,c,d)z^{-1} + O(z^{-2}).$$

The integral of f along C_R is then $(a+b+c+2d)R + p(a,b,c,d)\pi i + O(R^{-1})$. By Cauchy's theorem, the contour integral of f around C is 0. Taking the imaginary part with $\varepsilon \to 0^+$ and $R \to \infty$, we get the claimed identity and hence the result.

Also solved by R. Stong, R. Tauraso (Italy), and the proposer.

The Triangle Inequality from the Parallelogram Law

11904 [2016, 399]. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Let f be a function from $\mathbb{R}$ into $[0,\infty)$ such that $f^2(x+y)+f^2(x-y) = 2f^2(x) + 2f^2(y)$ for all x and y. Prove $f(x+y) \le f(x) + f(y)$ for all x and y.

Solution by Henry Ricardo, Westchester Area Math Circle, Purchase, NY. Setting $x = y = 0$ in the given functional equation yields $2f(0)^2 = 4f(0)^2$, implying $f(0) = 0$. Letting $x = 0$, we conclude that $f^2(y) + f^2(-y) = 2f^2(y)$, or (since f is nonnegative) $f(y) = f(-y)$ for all $y \in \mathbb{R}$.

If we define $g(x,y) = \left(f^2(x+y) - f^2(x-y)\right)/4$, then we see from the original equation that $g(x,x) = f^2(2x)/4 = f^2(x)$. Furthermore, the result of the previous paragraph gives $g(x,y) = g(y,x)$. Subtracting the functional equation for $(y-z,x)$ from the functional equation for $(x+y,z)$, then adding the functional equation for (y,z) and subtracting it for (x,y), we obtain

$$\begin{aligned} f^2(x+y+z) - f^2(x-y+z) &= 2f^2(x+y) - 2f^2(y-z) + 2f^2(z) - 2f^2(x) \\ &= f^2(x+y) - f^2(x-y) + f^2(y+z) - f^2(y-z). \end{aligned}$$

Thus, we conclude $g(x+z,y) = g(x,y) + g(z,y)$. Since g is symmetric, we conclude that it is also additive in its second argument. The additivity of g implies $g(nx,y) = ng(x,y)$ for all $n \in \mathbb{Z}$. It follows that $g(rx,y) = rg(x,y)$ for all $r \in \mathbb{Q}$. Thus for rational r we have

$$0 \le f^2(rx+y) = g(rx+y, rx+y) = r^2 g(x,x) + 2rg(x,y) + g(y,y).$$

Since the right-hand side is a polynomial in r, it follows that the inequality holds for all real r. Hence, the discriminant of this quadratic must be nonpositive, that is, $g^2(x,y) \le g(x,x)g(y,y)$ or equivalently $|g(x,y)| \le f(x)f(y)$. Hence

$$f^2(x+y) = g(x,x) + 2g(x,y) + g(y,y) \le (f(x)+f(y))^2,$$

which implies $f(x+y) \le f(x) + f(y)$.

Editorial comment. Allen Stenger noted that the problem appeared (in a slightly more general form) in this MONTHLY, Problem 5264 [1965, 193; 1966, 211], proposed by D. E. Knuth, with solutions by E. O. Buchman and W. G. Dotson, Jr.

Also solved by D. Bailey & E. Campbell & C. Diminnie, P. P. Dályay (Hungary), R. Ger (Poland), M. Goldenberg & M. Kaplan, J. W. Hagood, O. Kouba (Syria), O. P. Lossers (Netherlands), J. C. Smith, A. Stenger, R. Stong, GCHQ Problem Solving Group (U. K.), and the proposer.

Strengthening the Mordell–Oppenheim Inequality

11905 [2016, 400]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA.* From a point P inside a triangle ABC, the perpendiculars PP_A, PP_B, and PP_C are drawn to its sides. Let R be the circumradius and r the inradius of the triangle. Prove

$$\frac{R}{2r} \le \frac{|PA|\,|PB|\,|PC|}{(|PP_B|+|PP_C|)(|PP_A|+|PP_C|)(|PP_A|+|PP_B|)}.$$

Solution by Mohammad Reza Yegan, Central Tehran Branch, Islamic Azad University, Tehran, Iran. We assume that P_A lies on segment BC, P_B lies on segment CA, and P_C lies on segment AB. Let PA divide angle A into A_1 and A_2 so that $|PP_B| = |PA|\sin A_1$ and $|PP_C| = |PA|\sin A_2$. It follows that $|PP_B| + |PP_C| = |PA|(\sin A_1 + \sin A_2)$. Similarly, $|PP_C| + |PP_A| = |PB|(\sin B_1 + \sin B_2)$ and $|PP_A| + |PP_B| = |PC|(\sin C_1 + \sin C_2)$. Hence

$$\begin{aligned}(|PP_A| &+ |PP_B|)(|PP_B| + |PP_C|)(|PP_C| + |PP_A|)\\ &= |PA|\ |PB|\ |PC|(\sin A_1 + \sin A_2)(\sin B_1 + \sin B_2)(\sin C_1 + \sin C_2),\end{aligned}$$

or equivalently

$$\begin{aligned}(\sin A_1 &+ \sin A_2)(\sin B_1 + \sin B_2)(\sin C_1 + \sin C_2)\\ &= \frac{(|PP_A| + |PP_B|)(|PP_B| + |PP_C|)(|PP_C| + |PP_A|)}{|PA|\ |PB|\ |PC|}.\end{aligned}$$

Since

$$\sin A_1 + \sin A_2 = 2(\sin(A_1 + A_2)/2)\,(\cos(A_1 - A_2)/2) \le 2\sin A/2,$$

this implies

$$\frac{(|PP_A| + |PP_B|)(|PP_B| + |PP_C|)(|PP_C| + |PP_A|)}{|PA|\ |PB|\ |PC|} \le 8\sin A/2 \sin B/2 \sin C/2.$$

Taking reciprocals and using $\sin A/2 \sin B/2 \sin C/2 = r/4R$, we obtain the desired result.

Editorial comment. This inequality is (12.28) on p. 111 in *Geometric Inequalities* by O. Bottema et al. (1969). They reference L. J. Mordell, "On geometric problems of Erdös and Oppenheim," *Math. Gazette* **46** (1962) 213–215.

Using Euler's inequality $R \ge 2r$, we may conclude

$$|PA|\ |PB|\ |PC| \ge (|PP_A| + |PP_B|)(|PP_B| + |PP_C|)(|PP_C| + |PP_A|),$$

which is the Mordell–Oppenheim inequality.

The problem statement is correct even if "side" is interpreted as "extended side," the angle at A, say, is obtuse, and the projection P_B of P onto line AC falls outside the segment AC. Define angles A_1 and A_2 as above. Note that $\angle PAP_B$ is equal not to $\angle PAC$ but rather to its supplement. However, the sine of an angle and the sine of its supplement are equal, so it remains true that $|PP_C| + |PP_B| = |PA|(\sin A_1 + \sin A_2)$, and therefore Yegan's proof continues to be valid.

Also solved by A. Ali (India), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), R. Bosch, M. V. Channakeshava (India), P. P. Dályay (Hungary), P. De (India), M. Dincă (Romania), D. Fleischman, S. Gayen (India), O. Geupel (Germany), B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), W. Liu, O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), R. Nandan, P. Nüesch (Switzerland), M. Sawhney, J. C. Smith, R. Stong, R. Tauraso (Italy), Z. Vőrős (Hungary), M. Vowe (Switzerland), T. Wiandt, L. Wimmer (Germany), J. Zacharias, GCHQ Problem Solving Group (UK), and the proposer.