PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman, Douglas B. West**
with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Leonard Smiley, Kenneth Stolarsky, Richard Stong, Walter Stromquist, Daniel Velleman, Elizabeth Wilmer, Paul Zeitz, and Fuzhen Zhang.

Proposed problems should be submitted online at
http://www.americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by December 31, 2017 via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

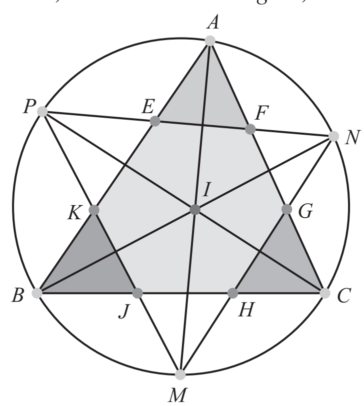
11992. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Prove that, for every positive integer n, there is a positive integer m such that $3^m + 5^m - 1$ is divisible by 7^n.

11993. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Prove

$$\int_0^1 \frac{\log(1-x)(\log(1+x))^2}{x}\,dx = -\frac{\pi^4}{240}.$$

11994. *Proposed by Miguel Ochoa Sanchez, Lima, Peru, and Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let ABC be a triangle with incenter I and circumcircle ω. Let M, N, and P be the second points of intersection of ω with lines AI, BI, and CI, respectively. Let E and F be the points of intersection of NP with AB and AC, respectively. Similarly, let G and H be the points of intersection of MN with AC and BC, respectively, and let J and K be the points of intersection of MP with BC and AB, respectively. Prove

$$EF + GH + JK \le KE + FG + HJ.$$

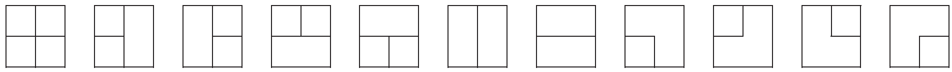


11995. *Proposed by Dan Ştefan Marinescu, National College "Iancu de Hunedoara," Hunedoara, Romania, and Mihai Monea, National College "Decebal," Deva, Romania.* Suppose $0 < x_0 < \pi$, and for $n \ge 1$ define $x_n = \frac{1}{n}\sum_{k=0}^{n-1} \sin x_k$. Find $\lim_{n\to\infty} x_n\sqrt{\ln n}$.

11996. *Proposed by Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* Consider all the tilings of a 2-by-n rectangle comprised of tiles that are either a unit square,

http://dx.doi.org/10.4169/amer.math.monthly.124.7.659

a domino, or a right tromino. Let f_n be the fraction of tiles among all such tilings that are unit squares. For example, $f_2 = 4/7$, because 16 out of the 28 tiles in the 11 tilings of a 2-by-2 rectangle are squares. What is $\lim_{n\to\infty} f_n$?



11997. *Proposed by Michael Drmota, Technical University of Vienna, Vienna, Austria; Christian Krattenthaler, University of Vienna, Vienna, Austria; and Gleb Pogudin, Johannes Kepler University, Linz, Austria.* Assume $|p| < 1$ and $pz \neq 0$. With $f(z) = \left(e^{(p-1)z} - e^{-z}\right)/(pz)$, define $f^*(z) = \prod_{k=0}^{\infty} f(p^k z)$, and then define $F_n(p)$ so that $f^*(z) = \sum_{n=0}^{\infty} F_n(p) z^n$. Prove the identity

$$\sum_{n=0}^{\infty} F_n(p)\, p^{\binom{n}{2}} = 0.$$

11998. *Proposed by Roger Cuculière, Lycée Pasteur, Neuilly-sur-Seine, France.* Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy $f(z) \le 1$ for some nonzero real number z and

$$f(x)^2 + f(y)^2 + f(x+y)^2 - 2f(x)f(y)f(x+y) = 1$$

for all real numbers x and y.

SOLUTIONS

Log-squared of the Catalan Generating Function

11832 [2015, 390]. *Proposed by Donald Knuth, Stanford University, Stanford, CA.* Let $C(z) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{z^n}{n+1}$ (thus $C(z)$ is the generating function of the Catalan numbers). Prove that

$$(\log C(z))^2 = \sum_{n=1}^{\infty} \binom{2n}{n} (H_{2n-1} - H_n) \frac{z^n}{n}.$$

Here $H_k = \sum_{j=1}^{k} 1/j$; that is, H_k is the kth harmonic number.

Solution by James Christopher Smith, Knoxville, TN. From the well-known formula $C(z) = \frac{1-\sqrt{1-4z}}{2z}$, it follows that $\frac{C'(z)}{C(z)} = \frac{1}{2z}\left(\frac{1}{\sqrt{1-4z}} - 1\right)$. The generating function for the central binomial coefficients is also well known; it is $\sum_{k=0}^{\infty} \binom{2k}{k} z^k = \frac{1}{\sqrt{1-4z}}$. This yields

$$\frac{C'(z)}{C(z)} = \frac{1}{2} \sum_{k=1}^{\infty} \binom{2k}{k} z^{k-1} \quad \text{and} \quad \log(C(z)) = \frac{1}{2} \sum_{k=1}^{\infty} \binom{2k}{k} \frac{z^k}{k},$$

where we obtain the second formula by integration. After squaring and collecting terms of degree n, we obtain

$$(\log C(z))^2 = \sum_{n=1}^{\infty} \left(\frac{1}{4} \sum_{k=1}^{n-1} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{k(n-k)} \right) z^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{4} \sum_{k=1}^{n-1} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{n} \left(\frac{1}{k} + \frac{1}{n-k} \right) \right) z^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2} \sum_{k=1}^{n-1} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{k} \right) \frac{z^n}{n}.$$

Thus it remains to show

$$\frac{1}{2} \sum_{k=1}^{n-1} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{k} = \binom{2n}{n} (H_{2n-1} - H_n).$$

Letting $H_n(x) = \sum_{k=1}^{n} \frac{1}{x+k}$ and using $\binom{x}{k} = \left(\prod_{i=0}^{k-1} (x-i) \right) / k!$, for $m \le n$ we obtain

$$\frac{d}{dx} \binom{x+n}{m} = \binom{x+n}{m} (H_n(x) - H_{n-m}(x)).$$

In D. Merlini, R. Sprugnoli, M. C. Verri, Lagrange inversion: when and how, *Acta Applicandae Mathematica* **94** (2006) 233–249, Lagrange inversion is used to prove $\sum_{k=0}^{n} \frac{a}{a+qk} \binom{a+qk}{k} \binom{qn-qk}{n-k} = \binom{a+qn}{n}$. Setting $a = x$ and $q = 2$ yields

$$\sum_{k=0}^{n} \frac{x}{x+2k} \binom{x+2k}{k} \binom{2n-2k}{n-k} = \binom{x+2n}{n}.$$

Differentiation of both sides with respect to x yields

$$\sum_{k=1}^{n} \frac{d}{dx} \left(\frac{x}{x+2k} \binom{x+2k}{k} \right) \binom{2n-2k}{n-k} = \binom{x+2n}{n} (H_{2n}(x) - H_n(x)).$$

Expanding the differentiation of the product on the left side, evaluating at $x = 0$, and using $\frac{x}{x+2k} \frac{d}{dx} \binom{x+2k}{k} \Big|_{x=0} = 0$ and $\frac{d}{dx} \frac{x}{x+2k} \Big|_{x=0} = \frac{1}{2k}$, we obtain

$$\frac{1}{2} \sum_{k=1}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{k} = \binom{2n}{n} (H_{2n} - H_n).$$

Moving the term for $k = n$ to the other side yields the desired equation

$$\frac{1}{2} \sum_{k=1}^{n-1} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{k} = \binom{2n}{n} (H_{2n} - \frac{1}{2n} - H_n) = \binom{2n}{n} (H_{2n-1} - H_n).$$

Editorial comment. Solvers used a variety of techniques, including integrations and Zeilberger's algorithm with telescoping sums. The proposer commented more generally that if $f(z) = \sum_{n=0}^{\infty} \binom{tn}{n} \frac{z^n}{(t-1)n+1}$, then the coefficient of z^n in $(\ln f(z))^p$ is the coefficient of x^{p-1} in $\frac{p!}{x+tn} \binom{x+tn}{n}$. Several solvers noted that $(\log C(z))^2$ was ambiguously typeset as $\log(C(z))^2$ in the original publication of the problem.

Also solved by T. Amdeberhan & V. H. Moll, D. Beckwith, R. Chapman (U. K.), H. Chen, R. Dutta (India), M. L. Glasser, P. Lalonde (Canada), K. D. Lathrop, K.-W. Lau (China), L. Matejíčka (Slovakia), M. Omarjee (France), B. Salvy (France), A. Stenger, R. Stong, R. Tauraso (Italy), C. I. Vălean (Romania), M. Wildon, and the proposer.

Napoleon into the Fray

11860 [2015, 801]. *Proposed by Dimitris Vartziotis, NIKI MEPE Digital Engineerings, Katsikas Ioannina, Greece.* Let ABC be a triangle. Let D, E, and F be the feet of the altitudes from A, B, and C, respectively. Extend the ray DA beyond A to a point A', and similarly extend EB to B' and FC to C', in such a way that $\sqrt{3}|AA'| = |BC|$, $\sqrt{3}|BB'| = |CA|$, and $\sqrt{3}|CC'| = |AB|$. Prove that $A'B'C'$ is an equilateral triangle.

Solution I by Irl C. Bivens and L. R. King, Davidson College, Davidson, NC. Let A^* be the reflection of A through the midpoint of the opposite side BC, and let B^* and C^* be defined similarly. The triangles A^*CB, CB^*A, and BAC^* are all congruent to the original triangle ABC. Therefore A, B, and C are the midpoints of the sides of triangle $A^*B^*C^*$, and so ABC is the medial triangle of $A^*B^*C^*$. The six triangles such as $A'AB^*$ are all $1:\sqrt{3}:2$ right triangles, so the points A', B', and C' defined in the problem are the centers of the equilateral triangles constructed outward on the sides of triangle $A^*B^*C^*$. Therefore, the Napoleon theorem guarantees that $A'B'C'$ is equilateral.

Solution II by TCDmath Problem Group, Trinity College, Dublin, Ireland. Set the construction in the complex plane, and use a corresponding lower-case letter to denote the complex number associated with the point whose name is an upper-case letter. For definiteness, let ABC be oriented positively. Let $\omega = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$, so that $\omega^3 = (\omega^2)^3 = 1$, $(\omega^2)^2 = \omega$, and $\omega^2 = -1/2 - i\sqrt{3}/2$.

Because of the positive orientation of ABC, the angle from BC to AA' is $\pi/2$. Since $\sqrt{3}|AA'| = |BC|$, we have $\sqrt{3}(a' - a) = i(c - b)$. Thus $\sqrt{3}a' = \sqrt{3}a - bi + ci$. Similarly, $\sqrt{3}b' = \sqrt{3}b - ci + ai$ and $\sqrt{3}c' = \sqrt{3}c - ai + bi$, so $(\sqrt{3}i/2)(b' - a') = \omega^2 a + \omega b + c$. Similarly, $(\sqrt{3}i/2)(c' - b') = \omega^2 b + \omega c + a$ and $(\sqrt{3}i/2)(a' - c') = \omega^2 c + \omega a + b$. Hence $b' - a' = \omega^2(c' - b') = \omega(a' - c')$, so $|A'B'| = |B'C'| = |C'A'|$, and thus $A'B'C'$ is equilateral.

Editorial comment. The problem did not specify that ABC is acute. When it is obtuse, the orthocenter lies outside of it. Some solutions did not allow for this possibility.

Also solved by A. Ali (India), H. Bailey, M. Bataille (France), B. S. Burdick, E. Chadraa, M. V. Channakeshava (India), R. Chapman (U. K.), K. Charatsaris (Greece), C. Curtis, N. Curwen (U. K.), P. P. Dályay (Hungary), S. N. Dinh (Germany), C. Effenberger (Germany), A. Fanchini (Italy), B. Fritzching (Germany), O. Geupel (Germany), M. Goldenberg & M. Kaplan, A. Gretsistas (Greece), A. Häcker (Germany), M. Hanselmann (Germany), I. Held (Germany), J. G. Heuver (Canada), S. Hitotumatu (Japan), S. Huggenberger (Germany), Y. J. Ionin, H. Jung (Korea), B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), O. Kouba (Syria), M. Kronenwelt (Germany), M. D. Meyerson, V. Mikayelyan (Armenia), J. Minkus, D. J. Moore, K. Nikolaou (Greece), P. Nüesch (Switzerland), C. R. Pranesachar (India), M. Sawhney, J. Schlosberg, N. Stanciu & T. Zvonaru (Romania), R. Stong, T. Toyonari (Japan), T. Viteam (Japan), Z. Vörös (Hungary), M. Vowe (Switzerland), T. Wiandt, J. Zacharias, GCHQ Problem Solving Group (U. K.), and the proposer.

A Symmetric Bound

11867 [2015, 899]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* For real numbers a, b, c, let

$$f(a,b,c) = \left(\frac{a^2}{a^2 - ab + b^2}\right)^{1/4}.$$

Prove $f(a,b,c) + f(b,c,a) + f(c,a,b) \le 3$.

Solution by Vazgen Mikayelyan, Yerevan State University, Armenia. For real numbers x, y, and z, we have

$$f(x,y,z) = \left(\frac{x^2}{x^2 - xy + y^2}\right)^{1/4} \le \left(\frac{|x|^2}{|x|^2 - |x||y| + |y|^2}\right)^{1/4}.$$

Hence we may assume that a, b, c are all nonnegative.

If one of the numbers a, b or c is zero, for example $c = 0$, then

$$f(a, b, c) + f(b, c, a) + f(c, a, b) = \left(\frac{a^2}{a^2 - ab + b^2}\right)^{1/4} + 1.$$

Thus, in this case, we need to prove

$$\frac{a^2}{a^2 - ab + b^2} \leq 2^4.$$

This holds because

$$\frac{a^2}{a^2 - ab + b^2} = \frac{1}{(\frac{b}{a} - \frac{1}{2})^2 + \frac{3}{4}} \leq \frac{4}{3} < 2^4.$$

Therefore, we may assume that a, b, c are all positive numbers. Note that for positive numbers x, y, and z,

$$\begin{aligned} f(x, y, z) = \left(\frac{x^2}{x^2 - xy + y^2}\right)^{1/4} &= \left(\frac{4x^2}{(x + y)^2 + 3(x - y)^2}\right)^{1/4} \\ &\leq \left(\frac{4x^2}{(x + y)^2}\right)^{1/4} \leq \sqrt{\frac{2x}{x + y}}. \end{aligned}$$

It follows that

$$f(a, b, c) + f(b, c, a) + f(c, a, b) \leq \sqrt{\frac{2a}{a + b}} + \sqrt{\frac{2b}{b + c}} + \sqrt{\frac{2c}{a + c}}.$$

Using the Cauchy–Schwarz inequality and the inequality $x + y \geq 2\sqrt{xy}$, we get

$$\begin{aligned} &\sqrt{\frac{2a}{a + b}} + \sqrt{\frac{2b}{b + c}} + \sqrt{\frac{2c}{a + c}} \\ &= \sqrt{a + c}\sqrt{\frac{2a}{(a + b)(a + c)}} + \sqrt{a + b}\sqrt{\frac{2b}{(b + c)(a + b)}} + \sqrt{b + c}\sqrt{\frac{2c}{(a + c)(b + c)}} \\ &\leq \sqrt{2(a + b + c)\left(\frac{2a}{(a + b)(a + c)} + \frac{2b}{(b + c)(a + b)} + \frac{2c}{(a + c)(b + c)}\right)} \\ &= 2\sqrt{\frac{2(a + b + c)(ab + bc + ac)}{(a + b)(b + c)(a + c)}} = 2\sqrt{\frac{2(a + b)(b + c)(a + c) + 2abc}{(a + b)(b + c)(a + c)}} \\ &= 2\sqrt{2 + \frac{2abc}{(a + b)(b + c)(a + c)}} \leq 2\sqrt{2 + \frac{2abc}{2\sqrt{ab} \cdot 2\sqrt{bc} \cdot 2\sqrt{ac}}} = 3. \end{aligned}$$

Also solved by A. Ali (India), T. Amdeberhan, P. Bracken, B. Bradie, M. V. Channakeshava (India), R. Chapman (U. K.), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), P. W. Gwanyama, T. Horine, O. Kouba (Syria), J. H. Lindsey II, J. Loverde, P. W. Lindstrom, O. P. Lossers (Netherlands), L. Matejíčka (Slovakia), T. L. McCoy, M. Omarjee (France) & R. Tauraso (Italy), P. Perfetti (Italy), J. Schlosberg, R. Stong, T. Wiandt, M. R. Yegan (Iran), GCHQ Problem Solving Group (U. K.), and the proposer.

Center of Mass of Multiplicative Orbits in a Grid

11868 [2015, 899]. *Proposed by James Propp, University of Massachusetts, Lowell, MA.* For fixed positive integers a and b, let $m = ab - 1$ and let R be the set $\{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\}$, indexed as p_0 through p_m in lexicographic order, so that $p_0 = (1, 1)$, $p_1 = (1, 2)$, and $p_m = (a, b)$. Define T from R to R as the map that sends p_0 to p_0 and p_m to p_m, and for $1 \le i \le m - 1$ sends p_i to p_j where $j \equiv ai \pmod m$. As a bijection, T partitions R into orbits. Show that the center of mass of each orbit lies on the line joining p_0 and p_m.

Solution by Jamie Simpson, Murdoch University, Perth, Australia. If $p_n = (x, y)$, then $n = b(x - 1) + y - 1$. Choose an orbit S under T and let $\overline{n}$, $\overline{x}$, and $\overline{y}$ be the average values of n, x, and y in S. The center of mass of S is then $(\overline{x}, \overline{y})$. Note that

$$\overline{n} = b(\overline{x} - 1) + (\overline{y} - 1). \tag{1}$$

Letting n' be the successor of n in the orbit, we have

$$n' \equiv ab(x - 1) + a(y - 1) \equiv x - 1 + a(y - 1) \pmod m.$$

Since the right side lies in the interval $[1, m]$, we have $n' = x - 1 + a(y - 1)$. The average value of n' over S is $\overline{n}$, so

$$\overline{n} = \overline{x} - 1 + a(\overline{y} - 1). \tag{2}$$

Together, (1) and (2) yield

$$\overline{y} = \frac{b - 1}{a - 1}(\overline{x} - 1) + 1,$$

and this is the equation of the line through $(1, 1)$ and (a, b).

Editorial comment. The solution shows that the average value of $(a - 1)y - (b - 1)x$ on each orbit is $a - b$. The phenomenon that prevails when the time-average of some quantity is the same for all orbits of a dynamical system is called *homomesy* and occurs in a broad range of contexts. For a catalog of examples, see Propp and Roby, Homomesy in products of two chains, *Electronic J. Combinatorics* **22** (2015) 3.4. The particular map T in this problem can be interpreted as a "re-reading map": for all k it maps the kth element of $\{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\}$ in lexicographic order to the kth element of $\{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\}$ in colexicographic order.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), A. Hammett & K. A. Roper, Y. J. Ionin, P. Lalonde (Canada), O. P. Lossers (Netherlands), R. Stong, the GCHQ Problem Solving Group (U. K.), TCDmath Problem Group (Ireland), and the proposer.

An Optimal Hölder Exponent

11869 [2015, 899]. *Proposed by George Stoica, University of New Brunswick, Saint John, Canada.* Prove that $|y \log y - x \log x| \le |y - x|^{1-1/e}$ for $0 < x < y \le 1$.

Solution by Bruce S. Burdick, Roger Williams University, Bristol, RI. Using elementary calculus, we see that the function $f(t) = t^{1/t}$ for $t > 0$ has its unique maximum when $t = e$. Therefore, for $t > 0$, we have $t^{1/t} \le e^{1/e}$, so $t \le e^{t/e}$ with equality only when $t = e$. Putting $t = \log u$, we have $\log u \le u^{1/e}$ for $u > 1$, with equality only when $u = e^e$. With $u = 1/v$ we have $-\log v \le v^{-1/e}$ for $0 < v < 1$, with equality only when $v = e^{-e}$.

Let x and y be real numbers satisfying $0 < x < y \le 1$, and consider the difference quotient $S = (g(y) - g(x))/(y - x)$, where $g(t) = t \log t$. Because g is convex on $[0, 1]$,

the lower bound of S for fixed y occurs as $x \to 0$. Also $S \le 1$ by the mean value theorem. So we have $\log y = g(y)/y < S \le 1$, which gives us

$$|S| \le \max\{-\log y, 1\} \le \max\{y^{-1/e}, 1\} < |y-x|^{-1/e}.$$

This yields the desired inequality.

Editorial comment. Omran Kouba noted that $1 - 1/e$ is the best possible exponent in the inequality. This is seen by choosing $y = e^{-e}$ and letting x tend to 0.

Also solved by K. F. Andersen (Canada), P. P. Dályay (Hungary), D. Fleischman, O. Geupel (Germany), O. Kouba (Syria), J. H. Lindsey II, P. W. Lindstrom, O. P. Lossers (Netherlands), L. Matejíčka (Slovakia), T. L. McCoy, V. Mikayelyan (Armenia), M. Omarjee (France) & R. Tauraso (Italy), E. Schmeichel, A. Stenger, R. Stong, GCHQ Problem Solving Group (U. K.), and NSA Problems Group.

Bounds Related to Convex Univalent Functions

11870 [2015, 900]. *Proposed by Finbarr Holland, University College Cork, Cork, Ireland.* Suppose $0 \le x \le 1$ and $y = 1 - x$, and let a and b be unimodular complex numbers. Let $c_1 = 2(xa + yb)$ and $c_2 = 2(xa^2 + yb^2)$. Prove $\big||c_1^2 + c_2| - 3|c_1|\big| \le 3$, with equality if and only if $x = y = 1/2$ and $b\overline{a} = e^{2\pi i/3}$.

Solution by the GCHQ Problem Solving Group, Cheltenham, UK. Correction: The last equation should read $b\overline{a} = e^{\pm 2\pi i/3}$. We can see that the stated equation is incorrect, since the inequality is invariant under swapping (a, x) and (b, y).

Lemma. $|c_1^2 + c_2| \ge 2\big||c_1|^2 - 1\big|$, *with equality if and only if* $(x, y) \in \big\{(0,1), (1/2, 1/2), (1, 0)\big\}$ *or* $a = b$.

Proof. Use $|z|^2 = z\overline{z}$, $x + y = 1$, $\overline{a} = a^{-1}$, $\overline{b} = b^{-1}$ to obtain

$$\begin{aligned}
|c_1^2 + c_2|^2 - 4\big(|c_1|^2 - 1\big)^2 &= (c_1^2 + c_2)(\overline{c_1}^2 + \overline{c_2}) - 4\big(c_1\overline{c_1} - 1\big)^2 \\
&= \big(4(xa + yb)^2 + 2(xa^2 + yb^2)\big)\big(4(x\overline{a} + y\overline{b})^2 + 2(x\overline{a}^2 + y\overline{b}^2)\big) \\
&\quad - 4\big(4(xa + yb)(x\overline{a} + y\overline{b}) - 1\big)^2 \\
&= \big(4(xa + yb)^2 + 2(x+y)(xa^2 + yb^2)\big)\big(4(xa^{-1} + yb^{-1})^2 \\
&\quad + 2(x+y)(xa^{-2} + yb^{-2})\big) - 4\big(4(xa + yb)(xa^{-1} + yb^{-1}) - (x+y)^2\big)^2 \\
&= 12x^3ya^2b^{-2} - 24x^2y^2a^2b^{-2} + 12xy^3a^2b^{-2} - 48x^3yab^{-1} + 96x^2y^2ab^{-1} \\
&\quad - 48xy^3ab^{-1} + 72x^3y - 144x^2y^2 + 72xy^3 - 48x^3ya^{-1}b + 96x^2y^2a^{-1}b \\
&\quad - 48xy^3a^{-1}b + 12x^3ya^{-2}b^2 - 24x^2y^2a^{-2}b^2 + 12xy^3a^{-2}b^2 \\
&= 12xy(x-y)^2(a-b)^2(a^{-1} - b^{-1})^2 = 12xy(x-y)^2(a-b)^2(\overline{a} - \overline{b})^2 \\
&= 12xy(x-y)^2|a-b|^4 \ge 0,
\end{aligned}$$

with equality if and only if $x = 0$, $y = 0$, $x = y$, or $a = b$. ■

Using the triangle inequality, we have $|c_1| \le 2x|a| + 2y|b| = 2$, $|c_2| \le 2x|a^2| + 2y|b^2| = 2$, and

$$|c_1^2 + c_2| - 3|c_1| \le |c_1|^2 + |c_2| - 3|c_1| \le 2|c_1| + 2 - 3|c_1| = 2 - |c_1| < 3.$$

Thus the upper bound 3 is always strict. Now if $|c_1| < 1$, then

$$|c_1^2 + c_2| - 3|c_1| > 0 - 3 \cdot 1 = -3.$$

Thus the lower bound of -3 is strict in this case. For the case $|c_1| \geq 1$, we use the lemma to obtain

$$|c_1^2 + c_2| - 3|c_1| \geq 2\big||c_1|^2 - 1\big| - 3|c_1| = 2\big(|c_1|^2 - 1\big) - 3|c_1|$$
$$= \big(2|c_1| - 1\big)\big(|c_1| - 1\big) - 3 \geq -3.$$

According to the lemma, equality holds in the first inequality here if and only if $(x, y) \in \big\{(0, 1), (1/2, 1/2), (1, 0)\big\}$ or $a = b$. However, since $|c_1| \geq 1$, equality holds in the second inequality if and only if $|c_1| = 1$. This rules out $(x, y) = (0, 1)$, $(x, y) = (1, 0)$, and $a = b$, because all these imply $|c_1| = 2$. Hence in the $|c_1| \geq 1$ case, the lower bound of -3 is achieved if and only if $x = y = 1/2$ and $|c_1| = 1$. Putting these together, we obtain

$$\big||c_1^2 + c_2| - 3|c_1|\big| \leq 3,$$

with equality if and only if $x = y = 1/2$ and $|c_1| = 1$. This equality condition is equivalent to $x = y = 1/2$ and $|a + b| = 1$. Equation $|a + b| = 1$ is equivalent to $a\overline{a} + a\overline{b} + \overline{a}b + b\overline{b} = -1$, i.e., $2\mathrm{Re}(b\overline{a}) = a\overline{b} + \overline{a}b = -1$. Since $b\overline{a}$ is unimodular, this is equivalent to $b\overline{a} = e^{\pm 2\pi i/3}$.

Editorial comment. The proposer notes that this is a toy version of an open problem: namely, the problem of finding sharp bounds for the functionals $\big||a_{n+1}| - |a_n|\big|$, $n = 0, 1, 2, \ldots$, where $a_0 = 0$, $a_1 = 1$, and $\sum_{n=0}^{\infty} a_n z^n$ is convex univalent on the open disk $\{z : |z| < 1\}$. The corresponding problem for starlike functions was resolved by Yuk Leung in 1977.

Also solved by P. Bracken, P. P. Dályay (Hungary), O. P. Lossers (Netherlands), V. Mikayelyan (Armenia), R. Stong, and the proposer.

Not All Angles are Rational Multiples of Pi

11871 [2015, 900]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA, and Ştefan Spătaru, Harvard University, Boston, MA.* Let ABC be a triangle in the Cartesian plane with vertices in $\mathbb{Z}^2$ (*lattice vertices*). Show that, if P is an interior lattice point of ABC, then at least one of the angles PAB, PBC, and PCA has a radian measure that is not a rational multiple of π.

Solution by L. R. King, Davidson, NC. The only rational values of $\tan(k\pi/n)$ when k/n is rational are 0 and ± 1. (See J. S. Calcut, Gaussian integers and arctangent identities for π, this MONTHLY **116** (2009) 515–530.) Since every angle of a triangle embedded in the integer lattice $\mathbb{Z}^2$ is either a right angle or has rational tangent, the possible angle measure for such an angle is reduced to $\pi/4$, $\pi/2$, and $3\pi/4$. As the sum of the radian measures of the named angles must be less than π, each must be $\pi/4$.

Assume $\angle PAB = \angle PBC = \pi/4$. We show $\angle PCA < \pi/4$. If $\angle CAP \geq \pi/4$ or $\angle ABP \geq \pi/4$, then we immediately get $\angle PCA < \angle BCA \leq \pi/4$, since the sum of the radian measures of the angles in triangle ABC is π. Therefore, assume $\angle CAP < \pi/4$ and $\angle ABP < \pi/4$, and let I be the incenter of triangle ABC, where its angle bisectors meet. Since $\angle CAP < \pi/4 = \angle PAB$, ray AP is internal to angle CAI. Likewise, $\angle PBA < \pi/4 = \angle PBC$, so ray BP is internal to angle ABI. We conclude that line CP is internal to $\angle ACI$, and thus $\angle PCA < \angle ACI = \angle ACB/2 < (\pi/2)/2 = \pi/4$. Therefore, at least one of the angles PAB, PBC, and PCA has a radian measure that is not a rational multiple of π.

Also solved by R. Chapman (U. K.), P. P. Dályay (Hungary), M. Goldenberg & M. Kaplan, T. Horine, Y. J. Ionin, O. P. Lossers (Netherlands), V. Pambuccian, R. Stong, R. Tauraso (Italy), J. Zacharias & R. Dempsey, GCHQ Problem Solving Group (U. K.), and the proposers.