Landsberg, M., "Zur Theorie der Gauss'schen Summen und der linearen Transformationen der Thetafunctionen." *Crelle's J. Mathematik,* 111 (1893).

For further history and a different method of proof, see

Lindelof, E., *Le Calcul des Residus.* . . . New York: Chelsea Publ. Co., 1947.

The simplest proof using contour integration is given by Mordell in his paper in *Acta math.*, vol. 61. The precise reference is given at the end of Section 43.

See also

Lerch, M., "Zur Theorie der Gausschen Summen." *Math. Ann.* 57 (1903), 554.
Mordell, L. J., "On the reciprocity formula for the Gauss's sums in the quadratic field." *Proc. Lond. math. Soc.*, 20 (1921-2), 289.

The Gauss sum is a particular trigonometric sum arising in the study of cyclotomic sums—sums of roots of unity occurring naturally in the problem of constructing a regular polygon of n sides by means of ruler and compass. It is interesting to see that this problem, extended to the lemniscate, led Gauss to his independent discovery of elliptic functions.

30. Polya's Derivation

Let us now present a derivation of the fundamental transformation formula of the theta function due to Polya which utilizes nothing more than the binomial expansion and Stirling's asymptotic expansion for the factorial.

We start with the identity

$$(z^{1/2}+z^{-1/2})^{2m} = \sum_{v=-m}^{m} \binom{2m}{m+v} z^{v}. \tag{30.1}$$

Let $\omega = e^{2\pi i/l}$ be an lth root of unity. Then from 30.1 we derive the result

$$\sum_{-l/2\leq v\leq l/2} [(\omega^v z)^{1/2} + (\omega^v z)^{-1/2}]^{2m} = l \sum_{v=-[m/l]}^{[m/l]} \binom{2m}{m+lv} z^{lv}. \tag{30.2}$$

(Here and below $[y]$ denotes the greatest integer less than or equal to y).

Let s and t be fixed quantities, with s an arbitrary complex number and t a real and positive quantity. Set

$$l = [(mt)^{1/2}], \qquad z = e^{s/l}. \tag{30.3}$$

Then, after division by 2^{2m}, the equation in 30.2 yields

$$\sum_{-l/2\leq v\leq l/2} \left\{\frac{1}{2}\,[e^{(s+2\pi iv)/l} + e^{-(s+2\pi iv)/l}]\right\}^{2m} \tag{30.4}$$

$$= \sum_{-l/2\leq v\leq l/2} \left\{1 + \frac{s+2\pi iv}{8l^2} + \dots\right\}^{8l^2(m/4l^2)}$$

$$= \sum_{v=-[m/l]}^{[m/l]} \frac{[(tm)^{1/2}]}{2^{2m}} \binom{2m}{m+[(tm)^{1/2}]v} e^{sv}. \tag{30.4}$$

We now wish to let l approach infinity and use the following two limit theorems.

$$\text{(a)}\quad \lim_{n\to\infty}\left(1+\frac{x_n}{n}\right)^n = e^x \qquad \text{if} \qquad \lim_{n\to\infty} x_n = x,$$

$$\text{(b)}\quad \lim_{n\to\infty}\frac{n^{1/2}}{2^{2n}}\binom{2n}{n+r} = \frac{e^{-x^2}}{\pi^{1/2}} \qquad \text{if} \qquad \lim_{n\to\infty}\frac{r}{\sqrt{n}} = x, \tag{30.5}$$

where n and r are positive integers.

The equation in 30.4 yields in the limit

$$\sum_{v=-\infty}^{\infty} e^{(s+2\pi iv)^2/4t} = \left(\frac{t}{\pi}\right)^{1/2} \sum_{v=-\infty}^{\infty} e^{-tv^2+sv}, \tag{30.6}$$

the desired theta function transformation, given in Section 9.

Comments and References

For additional details required to justify the limiting processes we refer the reader to Polya's paper,

Polya, G., "Elementarer Beweis einer Thetaformel." *Sitz. der Phys.-Math. Klasse.* Berlin (1927) 158-161.

Although the foregoing result at first may seem like a tour de force, in actuality it is closely connected with the fact that the continuous diffusion process may be considered to be a limit of a discrete random walk process. Since the random walk is ruled by the binomial expansion, and the diffusion process by the heat equation which gives rise to the Gaussian distribution, we see that it is not at all surprising that a modification of binomial expansions should yield the theta function formula.

31. Discussion

This brings us to the end of the first part of the monograph, devoted to the proof of the transformation formula for the theta function and related topics. The second part will be devoted to results of quite different nature, established by the use of a variety of methods.

32. A Fundamental Infinite Product

Let us begin our foray into a different area with the consideration of the expression of $\theta_4(z)$ as an infinite product. From this result some interesting infinite series can be obtained.