FROM: R. BELLMAN "A BRIEF INTRODUCTION TO THETA FUNCTIONS"

30. Polya's Derivation

Let us now present a derivation of the fundamental transformation formula of the theta function due to Polya which utilizes nothing more than the binomial expansion and Stirling's asymptotic expansion for the factorial.

We start with the identity

$$(z^{1/2} + z^{-1/2})^{2m} = \sum_{v=-m}^{m} \binom{2m}{m+v} z^{v}. \tag{30.1}$$

Let $\omega = e^{2\pi i/l}$ be an lth root of unity. Then from 30.1 we derive the result

$$\sum_{-l/2 \le v \le l/2} [(\omega^v z)^{1/2} + (\omega^v z)^{-1/2}]^{2m} = l \sum_{v=-[m/l]}^{[m/l]} \binom{2m}{m+lv} z^{lv}. \tag{30.2}$$

(Here and below $[y]$ denotes the greatest integer less than or equal to y).

Let s and t be fixed quantities, with s an arbitrary complex number and t a real and positive quantity. Set

$$l = [(mt)^{1/2}], \qquad z = e^{s/l}. \tag{30.3}$$

Then, after division by 2^{2m}, the equation in 30.2 yields

$$\sum_{-l/2 \le v \le l/2} \left\{\frac{1}{2} [e^{(s+2\pi i v)/l} + e^{-(s+2\pi i v)/l}]\right\}^{2m} \tag{30.4}$$

$$= \sum_{-l/2 \le v \le l/2} \left\{1 + \frac{s+2\pi i v}{8l^2} + \dots\right\}^{8l^2(m/4l^2)}$$

$$= \sum_{v=-[m/l]}^{[m/l]} \frac{[(tm)^{1/2}]}{2^{2m}} \binom{2m}{m+[(tm)^{1/2}]v} e^{sv}. \tag{30.4}$$

We now wish to let l approach infinity and use the following two limit theorems.

(a) $\lim_{n\to\infty} \left(1 + \frac{x_n}{n}\right)^n = e^x$ if $\lim_{n\to\infty} x_n = x$,

(b) $\lim_{n\to\infty} \frac{n^{1/2}}{2^{2n}} \binom{2n}{n+r} = \frac{e^{-x^2}}{\pi^{1/2}}$ if $\lim_{n\to\infty} \frac{r}{\sqrt{n}} = x$, (30.5)

where n and r are positive integers.

The equation in 30.4 yields in the limit

$$\sum_{v=-\infty}^{\infty} e^{(s+2\pi i v)^2/4t} = \left(\frac{t}{\pi}\right)^{1/2} \sum_{v=-\infty}^{\infty} e^{-tv^2+sv}, \tag{30.6}$$

the desired theta function transformation, given in Section 9.