FROM: "TEXTBOOK OF ALGEBRA"

Now (8′) gives us $q_n = 1/(1 - p_n/q_n)$. Hence the following remarkable transformation theorem :—

COR. *If $b_2, \ldots, b_n$ be any quantities whatsoever, then*

$$1 + b_2 + b_2 b_3 + \ldots + b_2 b_3 \ldots b_n$$
$$= \frac{1}{1-}\,\frac{b_2}{b_2+1-}\,\frac{b_3}{b_3+1-}\cdots\frac{b_n}{b_n+1} \quad (9),$$

from which, putting $u_1 = b_2$, $u_2 = b_2 b_3$, $\ldots$, $u_n = b_2 b_3 \ldots b_{n+1}$, we readily derive

$$1 + u_1 + u_2 + \ldots + u_n$$
$$= \frac{1}{1-}\,\frac{u_1}{1+u_1-}\,\frac{u_2}{u_1+u_2-}\,\frac{u_1 u_3}{u_2+u_3-}\,\frac{u_2 u_4}{u_3+u_4-}\cdots$$
$$\frac{u_{n-3}u_{n-1}}{u_{n-2}+u_{n-1}-}\,\frac{u_{n-2}u_n}{u_{n-1}+u_n} \quad (10),$$

an important theorem of Euler's to which we shall return presently.

INCOMMENSURABILITY OF CERTAIN CONTINUED FRACTIONS.

§ 17.] *If $a_2, a_3, \ldots, a_n$, $b_2, b_3, \ldots, b_n$ be all positive integers, then*

I. *The infinite continued fraction*

$$\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\cdots\frac{b_n}{a_n+}\cdots \quad (1)$$

converges to an incommensurable limit provided that after some finite value of n the condition $a_n \geqq b_n$ be always satisfied.

II. *The infinite continued fraction*

$$\frac{b_2}{a_2-}\,\frac{b_3}{a_3-}\cdots\frac{b_n}{a_n-}\cdots \quad (2)$$

converges to an incommensurable limit provided that after some finite value of n the condition $a_n \geqq b_n + 1$ be always satisfied, where the sign $>$ need not always occur but must occur infinitely often.*

To prove II., let us first suppose that the condition $a_n \geqq b_n + 1$ holds from the first. Then (2) converges, by § 16,

* These theorems are due to Legendre, *Éléments de Géométrie*, note IV.

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to a positive value < 1. Let us assume that it converges to a commensurable limit, say λ_2/λ_1, where λ_1, λ_2 are positive integers, and $\lambda_1 > \lambda_2$.

Let now

$$\rho_3 = \frac{b_3}{a_3-}\,\frac{b_4}{a_4-}\cdots.$$

Since the sign $>$ must occur among the conditions $a_3 \geqq b_3 + 1$, $a_4 \geqq b_4 + 1$, $\ldots$, ρ_3 must be a positive quantity < 1. Now, by our hypothesis,

$$\lambda_2/\lambda_1 = b_2/(a_2 - \rho_3),$$

therefore

$$\rho_3 = (a_2\lambda_2 - b_2\lambda_1)/\lambda_2,$$
$$= \lambda_3/\lambda_2, \text{ say,}$$

where $\lambda_3 = a_2\lambda_2 - b_2\lambda_1$ is an integer, which must be positive and $< \lambda_2$, since ρ_3 is positive and < 1.

Next, put

$$\rho_4 = \frac{b_4}{a_4-}\,\frac{b_5}{a_5-}\cdots.$$

Then, exactly as before, we can show that $\rho_4 = \lambda_4/\lambda_3$, where λ_4 is a positive integer $< \lambda_3$.

Since the sign $>$ occurs infinitely often among the conditions $a_n \geqq b_n + 1$, this process can be repeated as often as we please. The hypothesis that the fraction (2) is commensurable therefore requires the existence of an infinite number of positive integers $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \ldots$ such that $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \ldots$; but this is impossible, since λ_1 is finite. Hence (2) is incommensurable.

Next suppose the condition $a_n \geqq b_n + 1$ to hold after $n = m$. Then, by what has been shown,

$$y = \frac{b_{m+1}}{a_{m+1}-}\,\frac{b_{m+2}}{a_{m+2}-}\cdots$$

is incommensurable.

Now we have

$$F = \frac{b_2}{a_2-}\,\frac{b_3}{a_3-}\cdots\frac{b_m}{a_m - y},$$

consequently

$$F = \frac{(a_m - y)\,p_{m-1} - b_m p_{m-2}}{(a_m - y)\,q_{m-1} - b_m q_{m-2}},$$
$$= \frac{p_m - y p_{m-1}}{q_m - y q_{m-1}} \quad (3),$$

where p_m/q_m, p_{m-1}/q_{m-1} are the ultimate and penultimate convergents of

$$\frac{b_2}{a_2-}\frac{b_3}{a_3-}\cdots\frac{b_m}{a_m}.$$

It results from (3) that

$$y(Fq_{m-1}-p_{m-1})=Fq_m-p_m \qquad (4).$$

Now $Fq_{m-1}-p_{m-1}$ and Fq_m-p_m cannot both be zero, for that would involve the equality $p_m/q_m=p_{m-1}/q_{m-1}$, which is inconsistent with the equation (2) of § 3. Hence, if F were commensurable, (4) would give a commensurable value for the incommensurable y. F must therefore be incommensurable.

The proof of I. is exactly similar, for the condition $a_n \geqq b_n$ secures that each of the residual fractions of (1) shall be positive and less than unity.

These two theorems do not by any means include all cases of incommensurability in convergent infinite continued fractions.

Brouncker's fraction, for example, $1+\frac{1^2}{2+}\frac{3^2}{2+}\frac{5^2}{2+}\cdots$, converges to the incommensurable value $4/\pi$, and yet violates the condition of Proposition I.

CONVERSION OF SERIES AND CONTINUED PRODUCTS INTO CONTINUED FRACTIONS.

§ 18.] *To convert the series*

$$u_1+u_2+\ldots+u_n+\ldots$$

into an "equivalent" continued fraction of the form

$$\frac{b_1}{a_1-}\frac{b_2}{a_2-}\cdots\frac{b_n}{a_n-}\cdots \qquad (1).$$

A continued fraction is said to be "equivalent" to a series when the nth convergent of the former is equal to the sum of n terms of the latter for all values of n.

Since the convergents merely are given, we may leave the denominators $q_1, q_2, \ldots, q_n$ arbitrary (we take $q_0=1$, as usual).

For the fraction (1) we have

$$p_n/q_n-p_{n-1}/q_{n-1}=b_1b_2\ldots b_n/q_{n-1}q_n \qquad (2);$$

$$q_1=a_1,\quad q_2=a_2q_1-b_2,\quad \ldots,\quad q_n=a_nq_{n-1}-b_nq_{n-2} \qquad (3);$$

$$p_1/q_1=b_1/q_1 \qquad (4).$$

Since

$$p_n/q_n=u_1+u_2+\ldots+u_n \qquad (5),$$

we get from (2) and (5)

$$\left.\begin{aligned} u_n&=b_1b_2\ldots b_n/q_{n-1}q_n,\\ u_{n-1}&=b_1b_2\ldots b_{n-1}/q_{n-2}q_{n-1},\\ &\cdots\cdots\\ u_2&=b_1b_2/q_1q_2,\\ u_1&=b_1/q_1.\end{aligned}\right\} \qquad (6).$$

From (6), by using successive pairs of the equations, we get

$$b_1=q_1u_1,\quad b_2=q_2u_2/u_1,\quad b_3=q_3u_3/q_1u_2,\quad \ldots,\quad b_n=q_nu_n/q_{n-2}u_{n-1} \qquad (7).$$

Combining (3) with (7), we also find

$$a_1=q_1,\quad a_2=q_2(u_1+u_2)/q_1u_1,\quad a_3=q_3(u_2+u_3)/q_2u_2,\quad \ldots,$$
$$a_n=q_n(u_{n-1}+u_n)/q_{n-1}u_{n-1} \qquad (8).$$

Hence

$$S_n=u_1+u_2+\ldots+u_n,$$
$$=\frac{q_1u_1}{q_1-}\frac{q_2u_2/u_1}{q_2(u_1+u_2)/q_1u_1-}\frac{q_3u_3/q_1u_2}{q_3(u_2+u_3)/q_2u_2-}\cdots$$
$$\frac{q_nu_n/q_{n-2}u_{n-1}}{q_n(u_{n-1}+u_n)/q_{n-1}u_{n-1}} \qquad (9).$$

It will be observed that the q's may be cleared out of the fraction. Thus, for example, we get rid of q_1 by multiplying the first and second numerators and the first denominator by $1/q_1$, and the second and third numerators and the second denominator by q_1; and so on. We thus get for S_n the equivalent fraction

$$S_n=\frac{u_1}{1-}\frac{u_2/u_1}{(u_1+u_2)/u_1-}\frac{u_3/u_2}{(u_2+u_3)/u_2-}\cdots\frac{u_n/u_{n-1}}{(u_{n-1}+u_n)/u_{n-1}} \qquad (10),$$

which may be thrown into the form

$$S_n=\frac{u_1}{1-}\frac{u_2}{u_1+u_2-}\frac{u_1u_3}{u_2+u_3-}\cdots\frac{u_{n-2}u_n}{u_{n-1}+u_n} \qquad (11).$$

If only n be taken large enough, the fraction inside the brackets satisfies the condition of § 16 throughout: its value is therefore <1, however great m may be; and it follows from (12) that $Lq_m/q_{m-1}=1$ when $m=\infty$.

Since $LG(n+m, -y)=1$ when $m=\infty$, it follows that all the requisite conditions are fulfilled in the present case also.

We have thus shown that

$$\frac{F(1,x)}{F(0,x)}=\frac{1}{1+}\,\frac{x/\gamma(\gamma+1)}{1+}\,\frac{x/(\gamma+1)(\gamma+2)}{1+}\cdots\frac{x/(\gamma+n-1)(\gamma+n)}{1+}\cdots \text{ ad } \infty \quad (13),$$

whence, by an obvious reduction,

$$\frac{F(1,x)}{F(0,x)}=\frac{\gamma}{\gamma+}\,\frac{x}{\gamma+1+}\,\frac{x}{\gamma+2+}\cdots\frac{x}{\gamma+n+}\cdots \quad (14),$$

a result which holds for all finite real values of x, except such as render $F(0, x)$ zero*, and for all values of γ, except zero and negative integers.

If we put $\pm x^2/4$ in place of x in the functions $F(0, x)$ and $F(1, x)$, and at the same time put $\gamma=\frac{1}{2}$, we get

$$F(0, -x^2/4)=\cos x, \quad F(1, -x^2/4)=\sin x/x;$$
$$F(0, x^2/4)=\cosh x, \quad F(1, x^2/4)=\sinh x/x.$$

Cor. 1. *Hence, from* (14), *we get at once*

$$\tan x=\frac{x}{1-}\,\frac{x^2}{3-}\,\frac{x^2}{5-}\cdots\frac{x^2}{2n+1-}\cdots \quad (15);$$

$$\tanh x=\frac{x}{1+}\,\frac{x^2}{3+}\,\frac{x^2}{5+}\cdots\frac{x^2}{2n+1+}\cdots \quad (16).$$

Cor. 2. *The numerical constants π and π^2 are incommensurable.*

For, if π were commensurable, $\pi/4$ would be commensurable, say $=\lambda/\mu$. Hence we should have, by (15),

* In a sense it will hold even then, for the fraction

$$\frac{1}{\gamma}\left\{\gamma+\frac{x}{\gamma+1+}\,\frac{x}{\gamma+2+}\cdots\right\}$$

which represents $F(0,x)/F(1,x)$ will converge to 0. Of course, two consecutive functions $F(n, x)$, $F(n+1, x)$ cannot vanish for the same value of x; otherwise we should have $F(\infty, x)=0$, which is impossible, since $F(\infty, x)=1$.

$$1=\frac{\lambda/\mu}{1-}\,\frac{\lambda^2/\mu^2}{3-}\,\frac{\lambda^2/\mu^2}{5-}\cdots\frac{\lambda^2/\mu^2}{2n+1-}\cdots,$$
$$=\frac{\lambda}{\mu-}\,\frac{\lambda^2}{3\mu-}\,\frac{\lambda^2}{5\mu-}\cdots\frac{\lambda^2}{(2n+1)\mu-}\cdots \quad (17).$$

Now, since λ and μ are fixed finite integers, if we take n large enough we shall have $(2n+1)\mu>\lambda^2+1$. Hence, by § 17, the fraction in (17) converges to an incommensurable limit, which is impossible since 1 is commensurable.

That π^2 is also incommensurable follows in like manner very readily from (15).

By using (16) in a similar way we can easily show that

Cor. 3. *Any commensurable power of e is incommensurable**.

§ 22.] The development of last paragraph is in reality a particular case of the following general theorem regarding the hypergeometric series, given by Gauss in his classical memoir on that subject (1812)† :—

If

$$F(\alpha, \beta, \gamma, x)=1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\ldots,$$

and

$$G(\alpha, \beta, \gamma, x)=F(\alpha, \beta+1, \gamma+1, x)/F(\alpha, \beta, \gamma, x),$$

then

$$G(\alpha, \beta, \gamma, x)=\frac{1}{1-}\,\frac{\beta_1 x}{1-}\,\frac{\beta_2 x}{1-}\cdots\frac{\beta_{2n}x}{1/G(\alpha+n, \beta+n, \gamma+2n)} \quad (18),$$

where

$$\beta_1=\frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}, \qquad \beta_2=\frac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)},$$
$$\beta_3=\frac{(\alpha+1)(\gamma+1-\beta)}{(\gamma+2)(\gamma+3)}, \qquad \beta_4=\frac{(\beta+2)(\gamma+2-\alpha)}{(\gamma+3)(\gamma+4)},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\beta_{2n-1}=\frac{(\alpha+n-1)(\gamma+n-1-\beta)}{(\gamma+2n-2)(\gamma+2n-1)}, \quad \beta_{2n}=\frac{(\beta+n)(\gamma+n-\alpha)}{(\gamma+2n-1)(\gamma+2n)}.$$

* The results of this paragraph were first given by Lambert in a memoir which is very important in the history of continued fractions (*Hist. d. l'Ac. Roy. d. Berlin*, 1761). The arrangement of the analysis is taken from Legendre (*l.c.*), the general idea of the discussion of the convergence of the fraction from Schlömilch.

† *Werke*, Bd. III., p. 134.