7 Bigger models

Summary

Having applied our basic Black–Scholes model to the pricing of some exotic options, we now turn to more general market models.

In §7.1 we replace the (constant) parameters that characterised our basic Black–Scholes model by previsible processes. Under appropriate boundedness assumptions, we then repeat our analysis of Chapter 5 to obtain the fair price of an option as the discounted expected value of the claim under a martingale measure. In general this expectation must be evaluated numerically. We also make the connection with a generalised Black–Scholes equation via the Feynman–Kac Stochastic Representation Theorem.

Our models so far have assumed that the market consists of a single stock and a riskless cash bond. More complex equity products can depend on the behaviour of several separate securities and, in general, the prices of these securities will not evolve independently. In §7.2 we extend some of the fundamental results of Chapter 4 to allow us to manipulate systems of stochastic differential equations driven by *correlated* Brownian motions. For markets consisting of many assets we have much more freedom in our choice of 'reference asset' or numeraire and so we revisit this issue before illustrating the application of the 'multifactor' theory by pricing a 'quanto' product.

We still have no satisfactory justification for the geometric Brownian motion model. Indeed, there is considerable evidence that it does not capture all features of stock price evolution. A first objection is that stock prices occasionally 'jump' at unpredictable times. In §7.3 we introduce a Poisson process of jumps into our Black–Scholes model and investigate the implications for option pricing. This approach is popular in the analysis of credit risk. In §1.5 we saw that, if a model is to be free from arbitrage and complete, there must be a balance between the number of sources of randomness and the number of independent stocks. We reiterate this here. We see more evidence that the Black–Scholes model does not reflect the true behaviour of the market in §7.4. It seems a little late in the day to condemn the model that has been the subject of all our efforts so far and so we ask how much it matters

A word of warning is in order. In order to 'calibrate' such a model to the market we must choose the parameters $\{r_t\}_{t\geq 0}$, $\{\mu\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$ from an infinite-dimensional space. Unless we restrict the possible forms of these processes, this presents a major obstacle to implementation. In §7.4 we examine the effect of model misspecification on pricing and hedging strategies. Now, however, we set this worry aside and repeat the Black–Scholes analysis for our general class of market models.

A martingale measure

We must mimic the three steps to replication that we followed in the classical setting. The first of these is to find an equivalent probability measure, $\mathbb{Q}$, under which the discounted stock price, $\{\tilde{S}_t\}_{t\geq 0}$, is a martingale.

Exactly as before, we use the Girsanov Theorem to find a measure, $\mathbb{Q}$, under which the process $\{\tilde{W}_t\}_{t\geq 0}$ defined by

$$\tilde{W}_t = W_t + \int_0^t \gamma_s ds$$

is a standard Brownian motion. The discounted stock price, $\{\tilde{S}_t\}_{t\geq 0}$ defined as $\tilde{S}_t = S_t/B_t$, is governed by the stochastic differential equation

$$\begin{aligned} d\tilde{S}_t &= (\mu_t - r_t)\,\tilde{S}_t dt + \sigma_t \tilde{S}_t dW_t \\ &= (\mu_t - r_t - \sigma_t\gamma_t)\,\tilde{S}_t dt + \sigma_t \tilde{S}_t d\tilde{W}_t, \end{aligned}$$

and so we choose $\gamma_t = (\mu_t - r_t)/\sigma_t$. To ensure that $\{\tilde{S}_t\}_{t\geq 0}$ really is a $\mathbb{Q}$-martingale we make two further boundedness assumptions. First, in order to apply the Girsanov Theorem, we insist that

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\int_0^T \frac{1}{2}\gamma_t^2 dt\right)\right] < \infty.$$

Second we require that $\{\tilde{S}_t\}_{t\geq 0}$ is a $\mathbb{Q}$-martingale (not just a local martingale) and so we assume a second Novikov condition:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^T \frac{1}{2}\sigma_t^2 dt\right)\right] < \infty.$$

Under these extra boundedness assumptions $\{\tilde{S}_t\}_{t\geq 0}$ then is a martingale under the measure $\mathbb{Q}$ defined by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t^{(\gamma)} = \exp\left(-\int_0^t \gamma_s dW_s - \int_0^t \frac{1}{2}\gamma_s^2 ds\right).$$

Second step to replication

That completes the first step in our replication strategy. The second is to form the $(\mathbb{Q}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale $\{M_t\}_{t\geq 0}$ given by

$$M_t = \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} C_T \,\middle|\, \mathcal{F}_t\right].$$

Replicating a claim

The third step is to show that our market is complete, that is any claim C_T can be replicated. First we invoke the martingale representation theorem to write

$$M_t = M_0 + \int_0^t \theta_u d\tilde{W}_u$$

and consequently, provided that σ_t never vanishes,

$$M_t = M_0 + \int_0^t \phi_s d\tilde{S}_s,$$

where $\{\phi_t\}_{t\geq 0}$ is $\{\mathcal{F}_t\}_{t\geq 0}$-predictable.

Guided by our previous work we guess that a replicating portfolio should consist of ϕ_t units of stock and $\psi_t = M_t - \phi_t S_t$ units of cash bond at time t. In Exercise 1 it is checked that such a portfolio is self-financing. Its value at time t is

$$V_t = \phi_t S_t + \psi_t B_t = B_t M_t.$$

In particular, at time T, $V_T = B_T M_T = C_T$, and so we have a self-financing, replicating portfolio. The usual arbitrage argument tells us that the fair value of the claim at time t is V_t, that is the arbitrage price of the option at time t is

$$V_t = B_t \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} C_T \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u du} C_T \middle| \mathcal{F}_t\right].$$

The generalised Black–Scholes equation

In general such an expectation must be evaluated numerically. If r_t, μ_t and σ_t depend only on (t, S_t) then one approach to this is first to express the price as the solution to a generalised Black–Scholes partial differential equation. This is achieved with the Feynman–Kac Stochastic Representation Theorem. Specifically, using Example 4.8.6, $V_t = F(t, S_t)$ where $F(t, x)$ solves

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2(t,x)x^2\frac{\partial^2 F}{\partial x^2}(t,x) + r(t,x)x\frac{\partial F}{\partial x}(t,x) - r(t,x)F(t,x) = 0,$$

subject to the terminal condition corresponding to the claim C_T, at least provided

$$\int_0^T \mathbb{E}^{\mathbb{Q}}\left[\left(\sigma(t,x)\frac{\partial F}{\partial x}(t,x)\right)^2\right] ds < \infty.$$

For vanilla options, in the special case when r, μ and σ are functions of t alone, the partial differential equation can be solved explicitly. As is shown in Exercise 3 the procedure is exactly that used to solve the usual Black–Scholes equation. The price can be found from the classical Black–Scholes price via the following simple rule: for the value of the option at time t replace r and σ^2 by

$$\frac{1}{T-t}\int_t^T r(s)ds \quad \text{and} \quad \frac{1}{T-t}\int_t^T \sigma^2(s)ds$$

respectively.

7.2 Multiple stock models

So far we have assumed that the market consists of a riskless cash bond and a single 'risky' asset. However, the need to model whole portfolios of options or more complex equity products leads us to seek models describing several securities simultaneously. Such models must encode the *interdependence* between different security prices.

Correlated security prices

Suppose that we are modelling the evolution of n risky assets and, as ever, a single risk-free cash bond. We assume that it is not possible to exactly replicate one of the assets by a portfolio composed entirely of the others. In the most natural extension of the classical Black–Scholes model, considered individually the price of each risky asset follows a geometric Brownian motion, and interdependence of different asset prices is achieved by taking the driving Brownian motions to be correlated. Equivalently, we take a set of n independent Brownian motions and drive the asset prices by linear combinations of these; see Exercise 2. This suggests the following market model.

Multiple asset model: Our market consists of a cash bond $\{B_t\}_{0\le t\le T}$ and n different securities with prices $\{S_t^1, S_t^2, \dots, S_t^n\}_{0\le t\le T}$, governed by the system of stochastic differential equations

$$dB_t = rB_t dt,$$

$$dS_t^i = S_t^i\left(\sum_{j=1}^{n}\sigma_{ij}(t)dW_t^j + \mu_i(t)dt\right), \quad i = 1, 2, \dots, n, \tag{7.3}$$

where $\{W_t^j\}_{t\ge 0}$, $j = 1, \dots, n$, are independent Brownian motions. We assume that the matrix $\sigma = (\sigma_{ij})$ is invertible.

Remarks:

1 This model is called an n-factor model as there are n sources of randomness. If there are fewer sources of randomness than stocks then there is redundancy in the model as we can replicate one of the stocks by a portfolio composed of the others. On the other hand, if we are to be able to hedge any claim in the market, then, roughly speaking, we need as many 'independent' stocks as sources of randomness. This mirrors Proposition 1.6.5.

2 Notice that the volatility of each stock in this model is really a *vector*. Since the Brownian motions $\{W_t^j\}_{t\ge 0}$, $j = 1, \dots, n$, are independent, the total volatility of the process $\{S_t^i\}_{t\ge 0}$ is $\left\{\sqrt{\sum_{j=1}^{n}\sigma_{ij}^2(t)}\right\}_{t\ge 0}$. □

Of course we haven't checked that this model really makes sense. That is, we need to know that the system of stochastic differential equations (7.3) has a solution. In order to verify this and to analyse such multifactor market models we need multidimensional analogues of the key results of Chapter 4.

Multifactor Itô formula

The most basic tool will be an n-factor version of the Itô formula. In the same way as we used the one-factor Itô formula to find a description (in the form of a stochastic differential equation) of models constructed as functions of Brownian motion, here we shall build new multifactor models from old. Our basic building blocks will be solutions to systems of stochastic differential equations of the form

$$dX_t^i = \mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_t^j, \quad i = 1, \dots, n, \tag{7.4}$$

where $\{W_t^j\}_{t\geq 0}$, $j = 1, \dots, n$, are independent Brownian motions. We write $\{\mathcal{F}_t\}_{t\geq 0}$ for the σ-algebra generated by $\{W_t^j\}_{t\geq 0}$, $j = 1, \dots, n$. Our work of Chapter 4 gives a rigorous meaning to (the integrated version of) the system (7.4) provided $\{\mu_i(t)\}_{t\geq 0}$ and $\{\sigma_{ij}(t)\}_{t\geq 0}$, $1 \leq i \leq n$, $1 \leq j \leq n$, are $\{\mathcal{F}_t\}_{t\geq 0}$-predictable processes with

$$\mathbb{E}\left[\int_0^t \left(\sum_{j=1}^{n} \left(\sigma_{ij}(s)\right)^2 + |\mu_i(s)|\right) ds\right] < \infty, \quad t > 0, i = 1, \dots, n.$$

Let us write $\{X_t\}_{t\geq 0}$ for the vector of processes $\{X_t^1, X_t^2, \dots, X_t^n\}_{t\geq 0}$ and define a new stochastic process by $Z_t = f(t, X_t)$. Here we suppose that $f(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is sufficiently smooth that we can apply Taylor's Theorem, just as in §4.3, to find the stochastic differential equation governing $\{Z_t\}_{t\geq 0}$. Writing $x = (x_1, \dots, x_n)$, we obtain

$$dZ_t = \frac{\partial f}{\partial t}(t, X_t)dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_t)dX_t^i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)dX_t^i dX_t^j + \cdots. \tag{7.5}$$

Since the Brownian motions $\{W_t^i\}_{t\geq 0}$ are *independent* we have the multiplication table

$$\begin{array}{c|ccc} \times & dW_t^i & dW_t^j & dt \\ \hline dW_t^i & dt & 0 & 0 \\ dW_t^j & 0 & dt & 0 \\ dt & 0 & 0 & 0 \end{array} \qquad \text{for } i \neq j \tag{7.6}$$

and this gives $dX_t^i dX_t^j = \sum_{k=1}^{n} \sigma_{ik}\sigma_{jk}dt$. The same multiplication table tells us that $dX_t^i dX_t^j dX_t^k$ is $o(dt)$ and so substituting into equation (7.5) we have provided a heuristic justification of the following result.

Theorem 7.2.1 (Multifactor Itô formula) *Let* $\{X_t\}_{t\geq 0} = \{X_t^1, X_t^2, \dots, X_t^n\}_{t\geq 0}$ *solve*

$$dX_t^i = \mu_i(t)dt + \sum_{j=1}^n \sigma_{ij}(t)dW_t^j, \qquad i = 1, 2, \dots, n,$$

where $\{W_t^i\}_{t\geq 0}, i = 1, \dots, n,$ *are independent* $\mathbb{P}$*-Brownian motions. Further suppose that the real-valued function* $f(t, x)$ *on* $\mathbb{R}_+ \times \mathbb{R}^n$ *is continuously differentiable with respect to* t *and twice continuously differentiable in the* x*-variables. Then defining* $Z_t = f(t, X_t)$ *we have*

$$dZ_t = \frac{\partial f}{\partial t}(t, X_t)dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t, X_t)dX_t^i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)C_{ij}(t)dt$$

where $C_{ij}(t) = \sum_{k=1}^n \sigma_{ik}(t)\sigma_{jk}(t)$.

Remark: Notice that if we write σ for the matrix (σ_{ij}) then $C_{ij} = \left(\sigma\sigma^t\right)_{ij}$ where σ^t is the transpose of σ. □

We can now check that there *is* a solution to the system of equations (7.3).

Example 7.2.2 (Multiple asset model) *Let* $\{W_t^i\}_{t\geq 0}, i = 1, \dots, n,$ *be independent Brownian motions. Define* $\{S_t^1, S_t^2, \dots, S_t^n\}_{t\geq 0}$ *by*

$$S_t^i = S_0^i \exp\left(\int_0^t \left(\mu_i(s) - \frac{1}{2}\sum_{k=1}^n \sigma_{ik}^2(s)\right) ds + \int_0^t \sum_{j=1}^n \sigma_{ij}(s)dW_s^j\right);$$

then $\{S_t^1, S_t^2, \dots, S_t^n\}_{t\geq 0}$ *solves the system (7.3).*

Justification: Defining the processes $\{X_t^i\}_{t\geq 0}$ for $i = 1, 2, \dots, n$ by

$$dX_t^i = \left(\mu_i(t) - \frac{1}{2}\sum_{k=1}^n \sigma_{ik}^2(t)\right) dt + \sum_{j=1}^n \sigma_{ij}(t)dW_t^j$$

we see that $S_t^i = f^i(t, X_t)$ where, writing $x = (x_1, \dots, x_n)$, $f^i(t, x) \triangleq S_0^i e^{x_i}$. Applying Theorem 7.2.1 gives

$$\begin{aligned}
dS_t^i &= S_0^i \exp(X_t^i)dX_t^i + \frac{1}{2}S_0^i \exp(X_t^i)C_{ii}(t)dt \\
&= S_t^i\left\{\left(\mu_i(t) - \frac{1}{2}\sum_{k=1}^n \sigma_{ik}^2(t)\right)dt + \sum_{j=1}^n \sigma_{ij}(t)dW_t^j + \frac{1}{2}\sum_{k=1}^n \sigma_{ik}(t)\sigma_{ik}(t)dt\right\} \\
&= S_t^i\left\{\mu_i(t)dt + \sum_{j=1}^n \sigma_{ij}(t)dW_t^j\right\}
\end{aligned}$$

as required. □

Remark: Exactly as in the single factor models, although we can write down arbitrarily complicated systems of stochastic differential equations, existence and uniqueness of solutions are far from guaranteed. If the coefficients are bounded and uniformly Lipschitz then a unique solution does exist, but such results are beyond our scope here. Instead, once again, we refer to Chung & Williams (1990) or Ikeda & Watanabe (1989). □

Integration by parts

We can also use the multiplication table (7.6) to write down an n-factor version of the integration by parts formula.

Lemma 7.2.3 *If*

$$dX_t = \mu(t, X_t)dt + \sum_{i=1}^{n} \sigma_i(t, X_t)dW_t^i$$

and

$$dY_t = \nu(t, Y_t)dt + \sum_{i=1}^{n} \rho_i(t, Y_t)dW_t^i$$

then

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + \sum_{i=1}^{n} \sigma_i(t, X_t)\rho_i(t, Y_t)dt.$$

Change of measure

Pricing and hedging in the multiple stock model will follow a familiar pattern. First we find an equivalent probability measure under which *all* of the discounted stock prices $\{\tilde{S}_t^i\}_{t\geq 0}, i = 1, \ldots, n$, given by $\tilde{S}_t^i = e^{-rt}S_t^i$, are martingales. We then use a multifactor version of the Martingale Representation Theorem to construct a replicating portfolio.

Construction of the martingale measure is, of course, via a multifactor version of the Girsanov Theorem.

Theorem 7.2.4 (Multifactor Girsanov Theorem) *Let $\{W_t^i\}_{t\geq 0}, i = 1, \ldots, n$, be independent Brownian motions under the measure $\mathbb{P}$ generating the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and let $\{\theta_i(t)\}_{t\geq 0}, i = 1, \ldots, n$, be $\{\mathcal{F}_t\}_{t\geq 0}$-previsible processes such that*

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \sum_{i=1}^{n} \theta_i^2(s)ds\right)\right] ds < \infty. \tag{7.7}$$

Define

$$L_t = \exp\left(-\sum_{i=1}^{n}\left(\int_0^T \theta_i(s)dW_s^i + \frac{1}{2}\int_0^T \theta_i^2(s)ds\right)\right)$$

and let $\mathbb{P}^{(L)}$ be the probability measure defined by

$$\left.\frac{d\mathbb{P}^{(L)}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t.$$

Then under $\mathbb{P}^{(L)}$ the processes $\{X_t^i\}_{t\geq 0}$, $i = 1, \dots, n$, defined by

$$X_t^i = W_t^i + \int_0^t \theta_i(s)ds$$

are all martingales.

Sketch of proof: The proof mimics that in the one-factor case. It is convenient to write $L_t = \prod_{i=1}^n L_t^i$ where

$$L_t^i = \exp\left(-\int_0^t \theta_i(s)dW_s^i - \frac{1}{2}\int_0^t \theta_i^2(s)ds\right).$$

That $\{L_t\}_{t\geq 0}$ defines a martingale follows from (7.7) and the independence of the Brownian motions $\{W_t^i\}_{t\geq 0}$, $i = 1, \dots, n$.

To check that $\{X_t^i\}_{t\geq 0}$ is a (local) $\mathbb{P}^{(L)}$-martingale we find the stochastic differential equation satisfied by $\{X_t^i L_t\}_{t\geq 0}$. Since

$$dL_t^i = -\theta_i(t)L_t^i dW_t^i,$$

repeated application of our product rule gives

$$dL_t = -L_t \sum_{i=1}^n \theta_i(t)dW_t^i.$$

Moreover,

$$dX_t^i = dW_t^i + \theta_i(t)dt,$$

and so another application of our product rule gives

$$\begin{aligned} d(X_t^i L_t) &= X_t^i dL_t + L_t dW_t^i + L_t\theta_i(t)dt - L_t\theta_i(t)dt \\ &= -X_t^i L_t \sum_{i=1}^n \theta_i(t)dW_t^i + L_t dW_t^i. \end{aligned}$$

Combined with the boundedness condition (7.7), this proves that $\{X_t^i L_t\}_{t\geq 0}$ is a $\mathbb{P}$-martingale and hence $\{X_t^i\}_{t\geq 0}$ is a $\mathbb{P}^{(L)}$-martingale. $\mathbb{P}^{(L)}$ is equivalent to $\mathbb{P}$ so $\{X_t^i\}_{t\geq 0}$ has quadratic variation $[X^i]_t = t$ with $\mathbb{P}^{(L)}$-probability one and once again Lévy's characterisation of Brownian motion confirms that $\{X_t^i\}_{t\geq 0}$ is a $\mathbb{P}^{(L)}$-Brownian motion as required. □

A martingale measure

As promised we now use this to find a measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which the discounted stock price processes $\{\tilde{S}_t^i\}_{t\geq 0}$, $i = 1, \dots, n$, are all martingales. The measure $\mathbb{Q}$ will be one of the measures $\mathbb{P}^{(L)}$ of Theorem 7.2.4. We just need to identify the appropriate drifts $\{\theta_i\}_{t\geq 0}$.

The discounted stock price $\{\tilde{S}_t^i\}_{t\geq 0}$, defined by $\tilde{S}_t^i = B_t^{-1}S_t^i$, is governed by the stochastic differential equation

$$\begin{aligned} d\tilde{S}_t^i &= \tilde{S}_t^i(\mu_i(t) - r)dt + \tilde{S}_t^i \sum_{j=1}^n \sigma_{ij}(t)dW_t^j \\ &= \tilde{S}_t^i\left(\mu_i(t) - r - \sum_{j=1}^n \theta_j(t)\sigma_{ij}(t)\right)dt + \tilde{S}_t^i \sum_{j=1}^n \sigma_{ij}(t)dX_t^j, \end{aligned}$$

where as in Theorem 7.2.4

$$dX_t^j = \theta_j(t)dt + dW_t^j.$$

The discounted stock price processes will (simultaneously) be (local) martingales under $\mathbb{Q} = \mathbb{P}^{(L)}$ if we can make all the drift terms vanish. That is, if we can find $\{\theta_j(t)\}_{t\geq 0}$, $j = 1, \ldots, n$, such that

$$\mu_i(t) - r - \sum_{j=1}^{n} \theta_j(t)\sigma_{ij}(t) = 0 \quad \text{for all } i = 1, \ldots, n.$$

Dropping the dependence on t in our notation and writing

$$\mu = (\mu_1, \ldots, \mu_n), \quad \theta = (\theta_1, \ldots, \theta_n), \quad \mathbf{1} = (1, \ldots, 1) \quad \text{and } \sigma = \left(\sigma_{ij}\right),$$

this becomes

$$\mu - r\mathbf{1} = \theta\sigma. \tag{7.8}$$

A solution certainly exists if the matrix σ is invertible, an assumption that we made in setting up our multiple asset model.

In order to guarantee that the discounted price processes are martingales, not just local martingales, once again we impose a Novikov condition:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_0^t \frac{1}{2}\sum_{j=1}^{n} \sigma_{ij}^2(t)dt\right)\right] < \infty \quad \text{for each } i.$$

Replicating the claim

At this point we guess, correctly, that the value of a claim $C_T \in \mathcal{F}_T$ at time $t < T$ is its discounted expected value under the measure $\mathbb{Q}$. To prove this we show that there is a self-financing replicating portfolio and this we infer from a multifactor version of the Martingale Representation Theorem.

Theorem 7.2.5 (Multifactor Martingale Representation Theorem) *Let*

$$\{W_t^i\}_{t\geq 0}, \qquad i = 1, \ldots, n,$$

be independent $\mathbb{P}$-Brownian motions generating the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let $\{M_t^1, \ldots, M_t^n\}_{t\geq 0}$ be given by

$$dM_t^i = \sum_{j=1}^{n} \sigma_{ij}(t)dW_t^j,$$

where

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \sum_{j=1}^{n} \sigma_{ij}(t)^2 dt\right)\right] < \infty.$$

Suppose further that the volatility matrix $\left(\sigma_{ij}(t)\right)$ is non-singular (with probability one). Then if $\{N_t\}_{t\geq 0}$ is any one-dimensional $\left(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}\right)$-martingale there exists an n-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-previsible process $\{\phi_t\}_{t\geq 0} = \{\phi_t^1, \ldots, \phi_t^n\}_{t\geq 0}$ such that

$$N_t = N_0 + \sum_{j=1}^{n} \int_0^t \phi_s^j dM_s^j.$$

A proof of this result is beyond our scope here. It can be found, for example, in Protter (1990). Notice that the non-singularity of the matrix σ reflects our remark about non-vanishing quadratic variation after the proof of Theorem 4.6.2.

We are now in a position to verify that our guess was correct: the value of a claim in the multifactor world is its discounted expected value under the martingale measure $\mathbb{Q}$.

Let $C_T \in \mathcal{F}_T$ be a claim at time T and let $\mathbb{Q}$ be the martingale measure obtained above. We write

$$M_t = \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} C_T \middle| \mathcal{F}_t \right].$$

Since, by assumption, the matrix $\sigma = (\sigma_{ij})$ is invertible, the n-factor Martingale Representation Theorem tells us that there is an $\{\mathcal{F}_t\}_{t\geq 0}$-previsible process $\{\phi_t^1, \dots, \phi_t^n\}_{t\geq 0}$ such that

$$M_t = M_0 + \sum_{j=1}^{n} \int_0^t \phi_s^j d\tilde{S}_s^j.$$

Our hedging strategy will be to hold ϕ_t^i units of the ith stock at time t for each $i = 1, \dots, n$, and to hold ψ_t units of bond where

$$\psi_t = M_t - \sum_{j=1}^{n} \phi_t^j \tilde{S}_t^j.$$

The value of the portfolio is then $V_t = B_t M_t$, which at time T is exactly the value of the claim, and the portfolio is self-financing in that

$$dV_t = \sum_{j=1}^{n} \phi_t^j dS_t^j + \psi_t dB_t.$$

In the absence of arbitrage the value of the derivative at time t is

$$V_t = B_t \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} C_T \middle| \mathcal{F}_t \right] = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[C_T \middle| \mathcal{F}_t \right]$$

as predicted.

Remark: The multifactor market that we have constructed is complete and arbitrage-free. We have simplified the exposition by insisting that the number of sources of noise in our market is exactly matched by the number of risky tradable assets that we are modelling. More generally, we could model k risky assets driven by d sources of noise. Existence of a martingale measure corresponds to existence of a solution to (7.8). It is *uniqueness* of the martingale measure that provides us with the Martingale Representation Theorem and hence the ability to replicate any claim. For a complete arbitrage-free market we then require that $d \leq k$ and that σ has full rank. That is, the number of independent sources of randomness should exactly match the number of 'independent' risky assets trading in our market. □

The multi-dimensional Black–Scholes equation

In Exercise 7 you are asked to use a delta-hedging argument to obtain this price as the solution to the multidimensional Black–Scholes equation. This partial differential equation can also be obtained directly from the expectation price and a multidimensional version of the Feynman–Kac stochastic representation. We quote the appropriate version of this useful result here.

Theorem 7.2.6 (Multidimensional Feynman–Kac stochastic representation) *Let $\sigma(t,x) = (\sigma_{ij}(t,x))$ be a real symmetric $n \times n$ matrix, $\Phi : \mathbb{R}^n \to \mathbb{R}$ and $\mu_i : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$, be real-valued functions and r be a constant. We suppose that the function $F(t,x)$, defined for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, solves the boundary value problem*

$$\frac{\partial F}{\partial t}(t,x) + \sum_{i=1}^{n} \mu_i(t,x)\frac{\partial F}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(t,x)\frac{\partial^2 F}{\partial x_i \partial x_j}(t,x) - rF(t,x) = 0,$$

$$F(T,x) = \Phi(x),$$

where $C_{ij}(t,x) = \sum_{k=1}^{n} \sigma_{ik}(t,x)\sigma_{jk}(t,x)$.

Assume further that for each $i = 1, \dots, n$, the process $\{X_t^i\}_{t \geq 0}$ solves the stochastic differential equation

$$dX_t^i = \mu_i(t, X_t)dt + \sum_{j=1}^{n} \sigma_{ij}(t, X_t)dW_t^j$$

where $X_t = \{X_t^1, \dots, X_t^n\}$. Finally, suppose that

$$\int_0^T \mathbb{E}\left[\sum_{j=1}^{n}\left(\sigma_{ij}(s, X_s)\frac{\partial F}{\partial x_i}(s, X_s)\right)^2\right] ds < \infty, \quad i = 1, \dots, n.$$

Then

$$F(t,x) = e^{-r(T-t)}\mathbb{E}\left[\Phi(X_T) \mid X_t = x\right].$$

Corollary 7.2.7 *Let $S_t = \{S_t^1, \dots, S_t^n\}$ be as above and $C_T = \Phi(S_T)$ be a claim at time T. Then the price of the claim at time $t < T$,*

$$V_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T) \mid \mathcal{F}_t\right] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T) \mid S_t = x\right] \triangleq F(t,x)$$

satisfies

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} C_{ij}(t,x)x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j}(t,x) + r\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}(t,x) - rF(t,x) = 0,$$

$$F(T,x) = \Phi(x).$$

Proof: The process $\{S_t\}_{t \geq 0}$ is governed by

$$dS_t^i = rS_t^i dt + \sum_{j=1}^{n} \sigma_{ij}(t, S_t)S_t^j dX_t^j,$$

where $\{X_t^j\}_{t \geq 0}$, $j = 1, \dots, n$, are $\mathbb{Q}$-Brownian motions, so the result follows from an application of Theorem 7.2.6.

□

Numeraires The more assets there are in our market, the more freedom we have in choosing our 'numeraire' or 'reference asset'. Usually it is chosen to be a cash bond, but in fact it could be any of the tradable assets available. In the context of foreign exchange we checked that we could use as reference the riskless bond in either currency and always obtain the same value for a claim. Here we consider two numeraires in the *same* market, but they may have non-zero volatility.

Suppose that our market consists of $n+2$ tradable assets whose prices we denote by $\{B_t^1, B_t^2, S_t^1, \dots, S_t^n\}_{t\geq 0}$. We compare the prices obtained for a derivative by two traders, one of whom chooses $\{B_t^1\}_{t\geq 0}$ as numeraire and the other of whom chooses $\{B_t^2\}_{t\geq 0}$. We always assume our multidimensional geometric Brownian motion model for the evolution of prices, but now neither of the processes $\{B_t^i\}_{t\geq 0}$ necessarily has finite variation.

If we choose $\{B_t^1\}_{t\geq 0}$ as numeraire then we first find an equivalent measure, $\mathbb{Q}^1$, under which the asset prices discounted by $\{B_t^1\}_{t\geq 0}$, that is

$$\left\{\frac{B_t^2}{B_t^1}, \frac{S_t^1}{B_t^1}, \dots, \frac{S_t^n}{B_t^1}\right\}_{t\geq 0},$$

are all $\mathbb{Q}^1$-martingales. The value that we obtain for a derivative with payoff C_T at time T is then

$$V_t^1 = B_t^1 \mathbb{E}^{\mathbb{Q}^1}\left[\left.\frac{C_T}{B_T^1}\right|\mathcal{F}_t\right]$$

(see Exercise 7).

If instead we had chosen $\{B_t^2\}_{t\geq 0}$ as our numeraire then the price would have been

$$V_t^2 = B_t^2 \mathbb{E}^{\mathbb{Q}^2}\left[\left.\frac{C_T}{B_T^2}\right|\mathcal{F}_t\right]$$

where $\mathbb{Q}^2$ is an equivalent probability measure under which

$$\left\{\frac{B_t^1}{B_t^2}, \frac{S_t^1}{B_t^2}, \dots, \frac{S_t^n}{B_t^2}\right\}_{t\geq 0}$$

are all martingales. We have not proved that such a measure $\mathbb{Q}^2$ is unique, but if a claim can be replicated we obtain the same price for any measure $\mathbb{Q}^2$ with this property.

Suppose that we choose $\mathbb{Q}^2$ so that its Radon–Nikodym derivative with respect to $\mathbb{Q}^1$ is given by

$$\left.\frac{d\mathbb{Q}^2}{d\mathbb{Q}^1}\right|_{\mathcal{F}_t} = \frac{B_t^2}{B_t^1}.$$

Notice that since $\mathbb{Q}^1$ is a martingale measure for an investor choosing $\{B_t^1\}_{t\geq 0}$ as numeraire, we know that $\{B_t^2/B_t^1\}_{t\geq 0}$ is a $\mathbb{Q}^1$-martingale. Recall that if

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \zeta_t, \quad \text{for all } t > 0,$$

then, for $0 \le s \le t$,

$$\mathbb{E}^{\mathbb{Q}}[X_t | \mathcal{F}_s] = \mathbb{E}^{\mathbb{P}}\left[\left.\frac{\zeta_t}{\zeta_s}X_t\right| \mathcal{F}_s\right].$$

We first apply this to check that $\{S_t^i/B_t^2\}_{t\ge 0}$ is a $\mathbb{Q}^2$-martingale for each $i = 1, \dots, n$.

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^2}\left[\left.\frac{S_t^i}{B_t^2}\right|\mathcal{F}_s\right] &= \mathbb{E}^{\mathbb{Q}^1}\left[\left.\frac{B_t^2}{B_t^1}\frac{B_s^1}{B_s^2}\frac{S_t^i}{B_t^2}\right|\mathcal{F}_s\right]\\
&= \mathbb{E}^{\mathbb{Q}^1}\left[\left.\frac{B_s^1}{B_s^2}\frac{S_t^i}{B_t^1}\right|\mathcal{F}_s\right]\\
&= \frac{B_s^1}{B_s^2}\frac{S_s^i}{B_s^1} = \frac{S_s^i}{B_s^2},
\end{aligned}$$

where the last line follows since B_s^1 and B_s^2 are $\mathcal{F}_s$-measurable and $\{S_t^i/B_t^1\}_{t\ge 0}$ is a $\mathbb{Q}^1$-martingale. In other words, $\{S_t^i/B_t^2\}_{t\ge 0}$ is a $\mathbb{Q}^2$-martingale as required. That $\{B_t^1/B_t^2\}_{t\ge 0}$ is a $\mathbb{Q}^2$-martingale follows in the same way.

The price for our derivative given that we chose $\{B_t^2\}_{t\ge 0}$ as numeraire is then

$$\begin{aligned}
V_t^2 &= \mathbb{E}^{\mathbb{Q}^2}\left[\left.\frac{B_t^2}{B_T^2}C_T\right|\mathcal{F}_t\right]\\
&= \mathbb{E}^{\mathbb{Q}^1}\left[\left.\frac{B_T^2}{B_T^1}\frac{B_t^1}{B_t^2}\frac{B_t^2}{B_T^2}C_T\right|\mathcal{F}_t\right]\\
&= \mathbb{E}^{\mathbb{Q}^1}\left[\left.\frac{B_t^1}{B_T^1}C_T\right|\mathcal{F}_t\right] = V_t^1.
\end{aligned}$$

In other words, the choice of numeraire is unimportant – we always arrive at the same price.

Quantos

We now apply our multifactor technology in an example. We are going to price a *quanto forward contract.*

Definition 7.2.8 *A financial asset is called a* quanto product *if it is denominated in a currency other than that in which it is traded.*

A quanto forward contract is also known as a *guaranteed exchange rate forward.* It is most easily explained through an example.

Example 7.2.9 *BP, a UK company, has a Sterling denominated stock price that we denote by $\{S_t\}_{t\ge 0}$. For a dollar investor, a quanto forward contract on BP stock with maturity T has payoff $(S_T - K)$ converted into dollars according to some prearranged exchange rate. That is the payout will be \$$E(S_T - K)$ for some preagreed E, where S_T is the* Sterling *asset price at time T.*

As in our foreign exchange market of §5.3 we shall assume that there is a riskless cash bond in each of the dollar and Sterling markets, but now we have two random

processes to model, the stock price, $\{S_t\}_{t\geq 0}$ and the exchange rate, that is the value of one pound in dollars which we denote by $\{E_t\}_{t\geq 0}$. This will then require a *two-factor* model.

> **Black–Scholes quanto model:** We write $\{B_t\}_{t\geq 0}$ for the dollar cash bond and $\{D_t\}_{t\geq 0}$ for its Sterling counterpart. Writing E_t for the dollar worth of one pound at time t and S_t for the Sterling asset price at time t, our model is
>
> $$\begin{aligned} \text{Dollar bond} \quad & B_t = e^{rt}, \\ \text{Sterling bond} \quad & D_t = e^{ut}, \\ \text{Sterling asset price} \quad & S_t = S_0 \exp\left(\nu t + \sigma_1 W_t^1\right), \\ \text{Exchange rate} \quad & E_t = E_0 \exp\left(\lambda t + \rho\sigma_2 W_t^1 + \sqrt{1-\rho^2}\sigma_2 W_t^2\right), \end{aligned}$$
>
> where $\{W_t^1\}_{t\geq 0}$ and $\{W_t^2\}_{t\geq 0}$ are independent $\mathbb{P}$-Brownian motions and r, u, ν, λ, σ_1, σ_2 and ρ are constants.

In this model the volatilities of $\{S_t\}_{t\geq 0}$ and $\{E_t\}_{t\geq 0}$ are σ_1 and σ_2 respectively and $\{W_t^1, \rho W_t^1 + \sqrt{1-\rho^2} W_t^2\}_{t\geq 0}$ is a pair of correlated Brownian motions with correlation coefficient ρ. There is no extra generality in replacing the expressions for S_t and E_t by

$$S_t = S_0 \exp\left(\nu t + \sigma_{11}\tilde{W}_t^1 + \sigma_{12}\tilde{W}_t^2\right),$$
$$E_t = E_0 \exp\left(\lambda t + \sigma_{21}\tilde{W}_t^1 + \sigma_{22}\tilde{W}_t^2\right),$$

for independent Brownian motions $\{\tilde{W}_t^1, \tilde{W}_t^2\}_{t\geq 0}$.

Pricing a quanto forward contract

What is the value of K that makes the value at time zero of the quanto forward contract zero?

As in our discussion of foreign exchange, the first step is to reformulate the problem in terms of the dollar tradables. We now have three dollar tradables: the dollar worth of the Sterling bond, $E_t D_t$; the dollar worth of the stock, $E_t S_t$; and the dollar cash bond, B_t. Choosing the dollar cash bond as numeraire, we first find the stochastic differential equations governing the discounted values of the other two dollar tradables. We write $Y_t = B_t^{-1} E_t D_t$ and $Z_t = B_t^{-1} E_t S_t$. Since

$$dE_t = \left(\lambda + \frac{1}{2}\sigma_2^2\right) E_t dt + \rho\sigma_2 E_t dW_t^1 + \sqrt{1-\rho^2}\sigma_2 E_t dW_t^2,$$

application of our multifactor integration by parts formula gives

$$d(E_t D_t) = uE_t D_t dt + \left(\lambda + \frac{1}{2}\sigma_2^2\right) E_t D_t dt + \rho\sigma_2 E_t D_t dW_t^1 + \sqrt{1-\rho^2}\,\sigma_2 E_t D_t dW_t^2$$

and

$$dY_t = \left(\lambda + \frac{1}{2}\sigma_2^2 + u - r\right) Y_t dt + Y_t\left(\rho\sigma_2 dW_t^1 + \sqrt{1-\rho^2}\sigma_2 dW_t^2\right).$$

Similarly, since

$$dS_t = \left(\nu + \frac{1}{2}\sigma_1^2\right) S_t dt + \sigma_1 S_t dW_t^1,$$

$$\begin{aligned} d(E_t S_t) &= \left(\nu + \frac{1}{2}\sigma_1^2\right) E_t S_t dt + \sigma_1 E_t S_t dW_t^1 \\ &\quad + \left(\lambda + \frac{1}{2}\sigma_2^2\right) S_t E_t dt + \rho\sigma_2 S_t E_t dW_t^1 \\ &\quad + \sqrt{1-\rho^2}\sigma_2 S_t E_t dW_t^2 + \rho\sigma_1\sigma_2 S_t E_t dt \end{aligned}$$

and so

$$\begin{aligned} dZ_t &= \left(\nu + \frac{1}{2}\sigma_1^2 + \lambda + \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 - r\right) Z_t dt \\ &\quad + (\sigma_1 + \rho\sigma_2) Z_t dW_t^1 + \sqrt{1-\rho^2}\sigma_2 Z_t dW_t^2. \end{aligned}$$

Now we seek a change of measure to make these two processes martingales. Our calculations after the proof of Theorem 7.2.4 reduce this to finding θ_1, θ_2 such that

$$\lambda + \frac{1}{2}\sigma_2^2 + u - r - \theta_1\rho\sigma_2 - \theta_2\sqrt{1-\rho^2}\sigma_2 = 0$$

and

$$\nu + \frac{1}{2}\sigma_1^2 + \lambda + \frac{1}{2}\sigma_2^2 + \rho\sigma_1\sigma_2 - r - \theta_1(\sigma_1 + \rho\sigma_2) - \theta_2\sqrt{1-\rho^2}\sigma_2 = 0.$$

Solving this pair of simultaneous equations gives

$$\theta_1 = \frac{\nu + \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 - u}{\sigma_1}$$

and

$$\theta_2 = \frac{\lambda + \frac{1}{2}\sigma_2^2 + u - r - \rho\sigma_2\theta_1}{\sqrt{1-\rho^2}\sigma_2}.$$

Under the martingale measure, $\mathbb{Q}$, $\{X_t^1\}_{t\geq 0}$ and $\{X_t^2\}_{t\geq 0}$ defined by $X_t^1 = W_t^1 + \theta_1 t$ and $X_t^2 = W_t^2 + \theta_2 t$ are independent Brownian motions. We have

$$S_t = S_0 \exp\left(\left(u - \rho\sigma_1\sigma_2 - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 X_t^1\right).$$

In particular,

$$S_T = \exp(-\rho\sigma_1\sigma_2 T)\, S_0 e^{uT} \exp\left(\sigma_1 X_T^1 - \frac{1}{2}\sigma_1^2 T\right)$$

and we are finally in a position to price the forward. Since $\{X_t^1\}_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion,

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sigma_1 X_T^1 - \frac{1}{2}\sigma_1^2 T\right)\right] = 1,$$

so

$$\begin{aligned} V_0 &= e^{-rT} E\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)\right] \\ &= e^{-rT} E\left(\exp\left(-\rho\sigma_1\sigma_2 T\right) S_0 e^{uT} - K\right). \end{aligned}$$

Writing $F = S_0 e^{uT}$ for the forward price in the Sterling market and setting $V_0 = 0$ we see that we should take

$$K = F\exp\left(-\rho\sigma_1\sigma_2 T\right).$$

Remark: The exchange rate is given by

$$E_t = E_0 \exp\left(\left(r - u - \frac{1}{2}\sigma_2^2\right)t + \rho\sigma_2 X_t^1 + \sqrt{1-\rho^2}\sigma_2 X_t^2\right).$$

It is reassuring to observe that $\rho X_t^1 + \sqrt{1-\rho^2} X_t^2$ is a $\mathbb{Q}$-Brownian motion with variance one so that this expression for $\{E_t\}_{t\geq 0}$ is precisely that obtained in §5.3. Notice also that the discounted stock price process $e^{-rt}S_t$ is *not* a martingale; there is an extra term, reflecting the fact that the *Sterling* price is *not* a dollar tradable. □

7.3 Asset prices with jumps

The Black–Scholes framework is highly flexible. The critical assumptions are continuous time trading and that the dynamics of the asset price are continuous. Indeed, provided this second condition is satisfied, the Black–Scholes price can be justified as an asymptotic approximation to the arbitrage price under discrete trading, as the trading interval goes to zero. But are asset prices continuous?

So far, we have always assumed that any contracts written will be honoured. In particular, if a government or company issues a bond, we have ignored the possibility that they might default on that contract at maturity. But defaults do happen. This has been dramatically illustrated in recent years by credit crises in Asia, Latin America and Russia. If a company A holds a substantial quantity of company B's debt securities, then a default by B might be expected to have the knock-on effect of a sudden drop in company A's share price. How can we incorporate these market 'shocks' into our model?

A Poisson process of jumps

By their very nature, defaults are unpredictable. If we assume that we have absolutely no information to help us predict the default times or other market shocks, then we should model them by a Poisson random variable. That is the time between shocks is exponentially distributed and the number of shocks by time t, denoted by N_t, is a Poisson random variable with parameter λt for some $\lambda > 0$. Between shocks we assume that an asset price follows our familiar geometric Brownian motion model.

A typical model for the evolution of the price of a risky asset with jumps is

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t - \delta dN_t, \tag{7.9}$$

where $\{W_t\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$ are independent. To make sense of equation (7.9) we write it in integrated form, but then we must define the stochastic integral with respect to $\{N_t\}_{t\geq 0}$. Writing τ_i for the time of the ith jump of the Poisson process, we define

$$\int_0^t f(u, S_u) dN_u = \sum_{i=1}^{N_t} f\left(\tau(i)-, S_{\tau(i)-}\right).$$

For the model (7.9), if there is a shock, then the asset price is decreased by a factor of $(1-\delta)$. This observation tells us that the solution to (7.9) is

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)(1-\delta)^{N_t}.$$

To deal with more general models we must extend our theory of stochastic calculus to incorporate processes with jumps. As usual, the first step is to find an (extended) Itô formula.

> **Assumption:** We assume that asset price processes are càdlàg, that is they are right continuous with left limits.

Theorem 7.3.1 (Itô's formula with jumps) *Suppose*

$$dY_t = \mu_t dt + \sigma_t dW_t + \nu_t dN_t$$

where, under $\mathbb{P}$, $\{W_t\}_{t\geq 0}$ is a standard Brownian motion and $\{N_t\}_{t\geq 0}$ is a Poisson process with intensity λ. If f is a twice continuously differentiable function on $\mathbb{R}$ then

$$\begin{aligned} f(Y_t) &= f(Y_0) + \int_0^t f'(Y_{s-}) dY_s + \frac{1}{2}\int_0^t f''(Y_{s-})\sigma_s^2 ds \\ &\quad - \sum_{i=1}^{N_t} f'(T_{\tau_i-})\left(Y_{\tau_i} - Y_{\tau_i-}\right) + \sum_{i=1}^{N_t}\left(f(Y_{\tau_i}) - f(Y_{\tau_i-})\right), \end{aligned} \tag{7.10}$$

where $\{\tau_i\}$ are the times of the jumps of the Poisson process.

We don't prove this here, but heuristically it is not difficult to see that this should be the correct result. The first three terms are exactly what we'd expect if the process $\{Y_t\}_{t\geq 0}$ were continuous, but now, because of the discontinuities, we must distinguish Y_{s-} from Y_s. In between jumps of $\{N_t\}_{t\geq 0}$, precisely this equation should apply, but we must compensate for changes at jump times. In the first three terms we have included a term of the form $\sum_{i=1}^{N_t} f'(Y_{\tau_i-})\left(Y_{\tau_i} - Y_{\tau_i-}\right)$ and the first sum in equation (7.10) corrects for this. Since N_t is finite, we do not have to correct the term involving f''. Now we add in the *actual* contribution from the jump times and this is the second sum.

Compensation

As usual a key rôle will be played by martingales. Evidently a Poisson process, $\{N_t\}_{t\geq 0}$ with intensity λ under $\mathbb{P}$ is not a $\mathbb{P}$-martingale – it is monotone increasing. But we can write it as a martingale plus a drift. In Exercise 13 it is shown that the process $\{M_t\}_{t\geq 0}$ defined by $M_t = N_t - \lambda t$ *is* a $\mathbb{P}$-martingale.

More generally we can consider time-inhomogeneous Poisson processes. For such processes the intensity $\{\lambda_t\}_{t\geq 0}$ is a function of time. The probability of a jump in the time interval $[t, t+\delta t)$ is $\lambda_t \delta t + o(\delta t)$. Thus, for example, the probability that there is no jump in the interval $[s,t]$ is $\exp\left(-\int_s^t \lambda_u du\right)$. The corresponding *Poisson martingale* is $M_t = N_t - \int_0^t \lambda_s ds$. The process $\{\Lambda_t\}_{t\geq 0}$ given by $\Lambda_t = \int_0^t \lambda_s ds$ is the *compensator* of $\{N_t\}_{t\geq 0}$.

In Exercise 14 it is shown that just as integration with respect to Brownian martingales gives rise to (local) martingales, so integration with respect to Poisson martingales gives rise to martingales.

Poisson exponential martingales

Example 7.3.2 *Let $\{N_t\}_{t\geq 0}$ be a Poisson process with intensity $\{\lambda_t\}_{t\geq 0}$ under $\mathbb{P}$ where for each $t > 0$, $\int_0^t \lambda_s ds < \infty$. For a given bounded deterministic function $\{\alpha_t\}_{t\geq 0}$, let*

$$L_t = \exp\left(\int_0^t \alpha_s dM_s + \int_0^t \left(1 + \alpha_s - e^{\alpha_s}\right) \lambda_s ds\right) \tag{7.11}$$

where $dM_s = dN_s - \lambda_s ds$. Find the stochastic differential equation satisfied by $\{L_t\}_{t\geq 0}$ and deduce that $\{L_t\}_{t\geq 0}$ is a $\mathbb{P}$-martingale.

Solution: First write

$$Z_t = \int_0^t \alpha_s dM_s + \int_0^t \left(1 + \alpha_s - e^{\alpha_s}\right) \lambda_s ds$$

so that $L_t = e^{Z_t}$. Then

$$dZ_t = \alpha_t dN_t - \alpha_t \lambda_t dt + \left(1 + \alpha_t - e^{\alpha_t}\right) \lambda_t dt$$

and by our generalised Itô formula

$$dL_t = L_{t-} dZ_t + \left(-e^{Z_{t-}} \alpha_t + e^{Z_{t-} + \alpha_t} - e^{Z_{t-}}\right) dN_t,$$

where we have used the fact that if a jump in $\{Z_t\}_{t\geq 0}$ takes place at time t, then that jump is of size α_{t-}. Substituting and rearranging give

$$\begin{aligned} dL_t &= L_{t-}\alpha_t dM_t + L_{t-}\left(1 + \alpha_t - e^{\alpha_t}\right)\lambda_t dt - L_{t-}\left(1 + \alpha_t - e^{\alpha_t}\right) dN_t \\ &= L_{t-}\left(e^{\alpha_t} - 1\right) dM_t. \end{aligned}$$

By Exercise 14, $\{L_t\}_{t\geq 0}$ is a $\mathbb{P}$-martingale. □

Definition 7.3.3 *Processes of the form of $\{L_t\}_{t\geq 0}$ defined by (7.11) will be called* Poisson exponential martingales.

Our Poisson exponential martingales and Brownian exponential martingales are examples of Doléans–Dade exponentials.

Definition 7.3.4 *For a semimartingale $\{X_t\}_{t\geq 0}$ with $X_0 = 0$, the* Doléans–Dade exponential *of $\{X_t\}_{t\geq 0}$ is the unique semimartingale solution $\{Z_t\}_{t\geq 0}$ to*

$$Z_t = 1 + \int_0^t Z_{s-} dX_s.$$

Change of measure

In the same way as we used Brownian exponential martingales to change measure and thus 'transform drift' in the continuous world, so we shall combine Brownian and Poisson exponential martingales in our discontinuous asset pricing models. A change of drift for a Poisson martingale will correspond to a change of intensity for the Poisson process $\{N_t\}_{t\geq 0}$. More precisely, we have the following version of the Girsanov Theorem.

Theorem 7.3.5 (Girsanov Theorem for asset prices with jumps) *Let $\{W_t\}_{t\geq 0}$ be a standard $\mathbb{P}$-Brownian motion and $\{N_t\}_{t\geq 0}$ a (possibly time-inhomogeneous) Poisson process with intensity $\{\lambda_t\}_{t\geq 0}$ under $\mathbb{P}$. That is*

$$M_t = N_t - \int_0^t \lambda_u du$$

is a $\mathbb{P}$-martingale. We write $\mathcal{F}_t$ for the σ-field generated by $\mathcal{F}_t^W \cup \mathcal{F}_t^N$. Suppose that $\{\theta_t\}_{t\geq 0}$ and $\{\phi_t\}_{t\geq 0}$ are $\{\mathcal{F}_t\}_{t\geq 0}$-previsible processes with ϕ_t positive for each t, such that

$$\int_0^t \|\theta_s\|^2 ds < \infty \quad \text{and} \quad \int_0^t \phi_s \lambda_s ds < \infty.$$

Then under the measure $\mathbb{Q}$ whose Radon–Nikodym derivative with respect to $\mathbb{P}$ is given by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t$$

where $L_0 = 1$ and

$$\frac{dL_t}{L_{t-}} = \theta_t dW_t - (1 - \phi_t)\, dM_t,$$

the process $\{X_t\}_{t\geq 0}$ defined by $X_t = W_t - \int_0^t \theta_s ds$ is a Brownian motion and $\{N_t\}_{t\geq 0}$ has intensity $\{\phi_t \lambda_t\}_{t\geq 0}$.

In Exercise 16 it is shown that $\{L_t\}_{t\geq 0}$ is actually the product of a Brownian exponential martingale and a Poisson exponential martingale.

The proof of Theorem 7.3.5 is once again beyond our scope, but to check that the processes $\{X_t\}_{t\geq 0}$ and $\left\{N_t - \int_0^t \phi_s \lambda_s ds\right\}_{t\geq 0}$ are both local martingales under $\mathbb{Q}$ is an exercise based on the Itô formula.

Heuristics: An informal justification of the result is based on the extended multiplication table:

$\times$	dW_t	dN_t	dt
dW_t	dt	0	0
dN_t	0	dN_t	0
dt	0	0	0

Thus, for example,

$$
\begin{aligned}
d\left(L_t\left(N_t - \int_0^t \phi_s\lambda_s ds\right)\right) &= \left(N_t - \int_0^t \phi_s\lambda_s ds\right) dL_t + L_t\left(dN_t - \phi_t\lambda_t dt\right) \\
&\quad - L_t(1-\phi_t)(dN_t)^2 \\
&= \left(N_t - \int_0^t \phi_s\lambda_s ds\right) dL_t + L_t\left(dM_t + \lambda_t dt\right) \\
&\quad - L_t\phi_t\lambda_t dt - L_t(1-\phi_t)\left(dM_t + \lambda_t dt\right) \\
&= \left(N_t - \int_0^t \phi_s\lambda_s ds\right) dL_t + L_t\phi_t dM_t.
\end{aligned}
$$

Since $\{M_t\}_{t\geq 0}$ and $\{L_t\}_{t\geq 0}$ are $\mathbb{P}$-martingales, subject to appropriate boundedness assumptions, $\left\{L_t\left(N_t - \int_0^t \phi_s\lambda_s ds\right)\right\}_{t\geq 0}$ should be a $\mathbb{P}$-martingale and consequently $\left\{\left(N_t - \int_0^t \phi_s\lambda_s ds\right)\right\}_{t\geq 0}$ should be a $\mathbb{Q}$-martingale. $\square$

Our instinct is to use the extended Girsanov Theorem to find an equivalent probability measure under which the discounted asset price is a martingale.

Suppose then that

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t - \delta dN_t.$$

Evidently the discounted asset price satisfies

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = (\mu - r)\, dt + \sigma dW_t - \delta dN_t.$$

But now we see that there are *many* choices of $\{\theta_t\}_{t\geq 0}$ and $\{\phi_t\}_{t\geq 0}$ in Theorem 7.3.5 that lead to a martingale measure. The difficulty of course is that our market is not *complete*, so that although for any replicable claim we can use any of the martingale measures and arrive at the same answer, there are claims that cannot be hedged. There are two independent sources of risk, the Brownian motion and the Poisson point process, and so if we are to be able to hedge arbitrary claims $C_T \in \mathcal{F}_T$, we need two tradable risky assets subject to the same two noises.

Market price of risk

So if there *are* enough assets available to hedge claims, can we find a measure under which once discounted they are *all* martingales? Remember that otherwise there will be arbitrage opportunities in our market.

If the asset price has no jumps, we can write

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \mu dt + \sigma dW_t \\
&= (r + \gamma\sigma)\, dt + \sigma dW_t,
\end{aligned}
$$

where $\gamma = (\mu - r)/\sigma$ is the market price of risk. We saw in Chapter 5 that in the absence of arbitrage (so when there *is* an equivalent martingale measure for our market), γ will be the same for *all* assets driven by $\{W_t\}_{t\geq 0}$.

If the asset price has jumps, then investors will expect to be compensated for the additional risk associated with the possibility of downward jumps, even if we have

'compensated' the jumps (replaced dN_t by dM_t) so that their mean is zero. The price of such an asset is governed by

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu dt + \sigma dW_t + \nu dM_t \\ &= (r + \gamma\sigma + \eta\lambda\nu)\, dt + \sigma dW_t + \nu dM_t \end{aligned}$$

where ν measures the sensitivity of the asset price to the market shock and η is the excess rate of return per unit of jump risk. Again if there is to be a martingale measure under which *all* the discounted asset prices are martingales, then σ and η should be the same for all assets whose prices are driven by $\{W_t\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$. The martingale measure, $\mathbb{Q}$, will then be the measure $\mathbb{Q}$ of Theorem 7.3.5 under which

$$W_t + \int_0^t \frac{\mu - r}{\sigma} ds \quad \text{and} \quad M_t - \int_0^t \eta\lambda ds$$

are martingales. That is we take $\theta = \gamma$ and $\phi = -\eta$.

Multiple noises

The same ideas can be extended to assets driven by larger numbers of independent noises. For example, we might have n assets with dynamics

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sum_{\alpha=1}^{n} \sigma_{i\alpha} dW_t^\alpha + \sum_{\beta=1}^{m} \nu_{i\beta} dM_t^\beta$$

where, under $\mathbb{P}$, $\{W_t^\alpha\}_{t\geq 0}$, $\alpha = 1, \dots, n$, are independent Brownian motions and $\{M_t^\beta\}_{t\geq 0}$, $\beta = 1, \dots, m$, are independent Poisson martingales.

There will be an equivalent martingale measure under which *all* the discounted asset prices are martingales if we can associate a unique market price of risk with each source of noise. In this case we can write

$$\mu_i = r + \sum_{\alpha=1}^{n} \gamma_\alpha \sigma_{i\alpha} + \sum_{\beta=1}^{m} \eta_\beta \lambda_\beta \nu_{i\beta}.$$

All discounted asset prices will be martingales under the measure $\mathbb{Q}$ for which

$$\tilde{W}_t^\alpha = W_t^\alpha + \gamma_\alpha t$$

is a martingale for each α and

$$\tilde{M}_t^\beta = M_t^\beta + \eta_\beta \lambda_\beta t$$

is a martingale for each β.

As always it is *replication* that drives the theory. Note that in order to be able to hedge arbitrary $C_T \in \mathcal{F}_T$ we'll require $n + m$ 'independent' tradable risky assets driven by these sources of noise. With fewer assets at our disposal there will be claims C_T that we cannot hedge.

All this is little changed if we take the coefficients μ, σ, λ to be adapted to the filtration generated by $\{W_t^i\}_{t\geq 0}$, $i = 1, \dots, n$; see Exercise 15. Since we are not introducing any extra sources of noise, the same number of assets will be needed for market completeness. These ideas form the basis of Jarrow–Madan theory.

7.4 Model error

Even in the absence of jumps (or between jumps) we have given only a very vague justification for the Samuelson model

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{7.12}$$

Moreover, although we have shown that under this model the pricing and hedging of derivatives are dictated by the single parameter σ, we have said nothing about how actually to estimate this number from market data. So what is market practice?

Implied volatility

Vanilla options are generally traded on exchanges, so if a trader wants to know the price of, say, a European call option, then she can read it from her trading screen. However, for an over-the-counter derivative, the price is not quoted on an exchange and so one needs a pricing model. The normal practice is to build a Black–Scholes model and then *calibrate* it to the market – that is estimate σ from the market. But it is *not* usual to estimate σ directly from data for the stock price. Instead one uses the quoted price for exchange-traded options written on the same stock. The procedure is simple: for given strike price and maturity, we can think of the Black–Scholes pricing formula for a European option as a mapping from volatility, σ, to price V. In Exercise 17, it is shown that for vanilla options this mapping is strictly monotone and so can be inverted to infer σ from the price. In other words, given the option price one can recover the corresponding value of σ in the Black–Scholes formula. This number is the so-called *implied volatility*.

If the markets really did follow our Black–Scholes model, then this procedure would give the same value of σ, irrespective of the strike price and maturity of the exchange-traded option chosen. Sadly, this is far from what we observe in reality: not only is there dependence on the strike price for a fixed maturity, giving rise to the famous volatility smile, but also implied volatility tends to increase with time to maturity (Figure 7.1). Market practice is to choose as volatility parameter for pricing an over-the-counter option the implied volatility obtained from 'comparable' exchange-traded options.

Hedging error

This procedure can be expected to lead to a *consistent* price for exchange-traded and over-the-counter options and model error is not a serious problem. The difficulties arise in *hedging*. Even for exchange-traded options a model is required to determine the replicating portfolio. We follow Davis (2001).

Suppose that the true stock price process follows

$$dS_t = \alpha_t S_t dS_t + \beta_t S_t dW_t$$

where $\{\alpha_t\}_{t\geq 0}$ and $\{\beta_t\}_{t\geq 0}$ are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes, but we price *and hedge* an option with payoff $\Phi(S_T)$ at time T as though $\{S_t\}_{t\geq 0}$ followed equation (7.12) for some parameter σ.

Our estimate for the value of the option at time $t < T$ will be $V(t, S_t)$ where

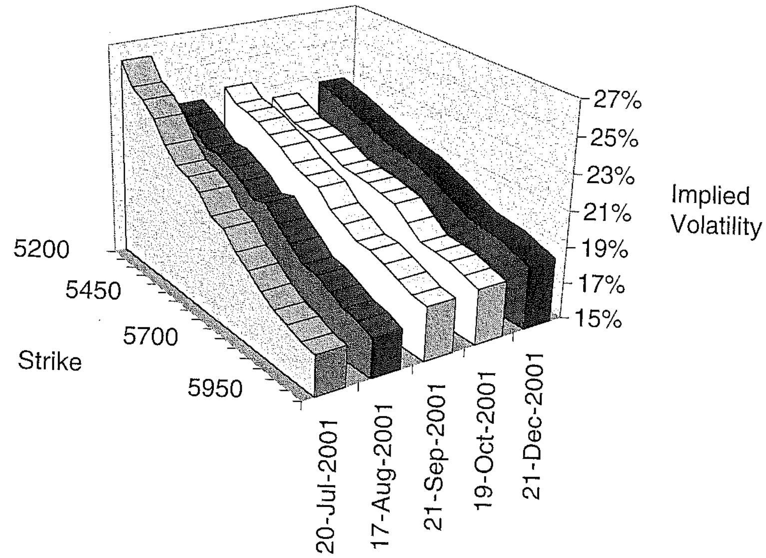

Figure 7.1 Implied volatility as a function of strike price and maturity for European call options based on the FTSE stock index.

$V(t,x)$ satisfies the Black–Scholes partial differential equation

$$\frac{\partial V}{\partial t}(t,x) + rx\frac{\partial V}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2}(t,x) - rV(t,x) = 0,$$
$$V(T,x) = \Phi(x).$$

Our hedging portfolio consists at time t of $\phi_t = \frac{\partial V}{\partial x}(t, S_t)$ units of stock and cash bonds with total value $\psi_t e^{rt} \triangleq V(t, S_t) - \phi_t S_t$.

Our first worry is that because of model misspecification, the portfolio is not self-financing. So what is the cost of following such a strategy? Since the cost of purchasing the 'hedging' portfolio at time t is $V(t, S_t)$, the incremental cost of the strategy over an infinitesimal time interval $[t, t+\delta t)$ is

$$\frac{\partial V}{\partial x}(t, S_t)(S_{t+\delta t} - S_t) + \left(V(t, S_t) - \frac{\partial V}{\partial x}(t, S_t)S_t\right)(e^{r\delta t} - 1)$$
$$- V(t+\delta t, S_{t+\delta t}) + V(t, S_t).$$

In other words, writing Z_t for our net position at time t, we have

$$dZ_t = \frac{\partial V}{\partial x}(t, S_t)dS_t + \left(V(t, S_t) - \frac{\partial V}{\partial x}(t, S_t)S_t\right)rdt - dV(t, S_t).$$

Since $V(t,x)$ solves the Black–Scholes partial differential equation, applying Itô's

formula gives

$$\begin{aligned} dZ_t &= \frac{\partial V}{\partial x}(t, S_t)dS_t + \left(V(t, S_t) - \frac{\partial V}{\partial x}(t, S_t)S_t\right) r dt \\ &\quad - \frac{\partial V}{\partial t}(t, S_t)dt - \frac{\partial V}{\partial x}(t, S_t)dS_t - \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(t, S_t)\beta_t^2 S_t^2 dt \\ &= \frac{1}{2}S_t^2 \frac{\partial^2 V}{\partial x^2}\left(\sigma^2 - \beta_t^2\right) dt. \end{aligned}$$

Irrespective of the model, $V(T, S_T) = \Phi(S_T)$ precisely matches the claim against us at time T, so our net position at time T (having honoured the claim $\Phi(S_T)$ against us) is

$$Z_T = \int_0^T \frac{1}{2}S_t^2 \frac{\partial^2 V}{\partial x^2}(t, S_t)\left(\sigma^2 - \beta_t^2\right) dt.$$

For European call and put options $\frac{\partial^2 V}{\partial x^2} > 0$ (see Exercise 18) and so if $\sigma^2 > \beta_t^2$ for all $t \in [0, T]$ our hedging strategy makes a profit. This means that regardless of the price dynamics, we make a profit if the parameter σ in our Black–Scholes model dominates the true diffusion coefficient β. This is key to successful hedging. Our calculation won't work if the price process has jumps, although by choosing σ large enough one can still arrange for Z_T to have positive expectation.

The choice of σ is still a tricky matter. If we are too cautious no one will buy the option, too optimistic and we are exposed to the risk associated with changes in volatility and we should try to hedge that risk. Such hedging is known as *vega hedging*, the Greek *vega* of an option being the sensitivity of its Black–Scholes price to changes in σ. The idea is the same as that of delta hedging (Exercise 5 of Chapter 5). For example, if we buy an over-the-counter option for which $\frac{\partial V}{\partial \sigma} = v$, then we also sell a number v/v' of a comparable exchange traded option whose value is V' and for which $\frac{\partial V'}{\partial \sigma} = v'$. The resulting portfolio is said to be *vega-neutral*.

Stochastic volatility and implied volatility

Since we cannot observe the volatility directly, it is natural to try to model it as a random process. A huge amount of effort has gone into developing so-called *stochastic volatility models*. Fat-tailed returns distributions observed in data can be modelled in this framework and sometimes 'jumps' in the asset price can be best modelled by jumps in the volatility. For example if jumps occur according to a Poisson process with constant rate and at the time, τ, of a jump, $S_\tau/S_{\tau-}$ has a lognormal distribution, then the distribution of S_t will be lognormal but with variance parameter given by a multiple of a Poisson random variable (Exercise 19). Stochastic volatility can also be used to model the 'smile' in the implied volatility curve and we end this chapter by finding the correspondence between the choice of a stochastic volatility model and of an implied volatility model. Once again we follow Davis (2001). A typical stochastic volatility model takes the form

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dW_t^1, \\ d\sigma_t &= a(S_t, \sigma_t)dt + b(S_t, \sigma_t)\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right), \end{aligned}$$

where $\{W_t^1\}_{t\geq 0}$, $\{W_t^2\}_{t\geq 0}$ are independent $\mathbb{P}$-Brownian motions, ρ is a constant in $(0, 1)$ and the coefficients $a(x, \sigma)$ and $b(x, \sigma)$ define the volatility model.

As usual we'd like to find a martingale measure. If $\mathbb{Q}$ is equivalent to $\mathbb{P}$, then its Radon–Nikodym derivative with respect to $\mathbb{P}$ takes the form

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t \hat{\theta}_s dW_s^1 - \frac{1}{2}\int_0^t \hat{\theta}_s^2 ds - \int_0^t \theta_s dW_s^2 - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

for some integrands $\{\hat{\theta}_t\}_{t\geq 0}$ and $\{\theta_t\}_{t\geq 0}$. In order for the discounted asset price $\{\tilde{S}_t\}_{t\geq 0}$ to be a $\mathbb{Q}$-martingale, we choose

$$\hat{\theta}_t = \frac{\mu - r}{\sigma_t}.$$

The choice of $\{\theta_t\}_{t\geq 0}$ however is arbitrary as $\{\sigma_t\}_{t\geq 0}$ is not a tradable and so no arbitrage argument can be brought to bear to dictate its drift. Under $\mathbb{Q}$,

$$X_t^1 = W_t^1 + \int_0^t \hat{\theta}_s ds$$

and

$$X_t^2 = W_t^2 + \int_0^t \theta_s ds$$

are independent Brownian motions. The dynamics of $\{S_t\}_{t\geq 0}$ and $\{\sigma_t\}_{t\geq 0}$ are then most conveniently written as

$$dS_t = rS_t dt + \sigma_t S_t dX_t^1$$

and

$$d\sigma_t = \tilde{a}(S_t, \sigma_t)dt + b(S_t, \sigma_t)\left(\rho dX_t^1 + \sqrt{1-\rho^2}dX_t^2\right)$$

where

$$\tilde{a}(S_t, \sigma_t) = a(S_t, \sigma_t) - b(S_t, \sigma_t)\left(\rho\hat{\theta}_t + \sqrt{1-\rho^2}\theta_t\right).$$

We now *introduce* a second tradable asset. Suppose that we have an option written on $\{S_t\}_{t\geq 0}$ whose exercise value at time T is $\Phi(S_T)$. We *define* its value at times $t < T$ to be the discounted value of $\Phi(S_T)$ under the measure $\mathbb{Q}$. That is

$$V(t, S_t, \sigma_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\Phi(S_T)\middle|\mathcal{F}_t\right].$$

Our multidimensional Feynman–Kac Stochastic Representation Theorem (combined with the usual product rule) tells us that the function $V(t, x, \sigma)$ solves the partial differential equation

$$\begin{aligned}\frac{\partial V}{\partial t}(t, x, \sigma) &+ rx\frac{\partial V}{\partial x}(t, x, \sigma) + \tilde{a}(t, x, \sigma)\frac{\partial V}{\partial \sigma}(t, x, \sigma) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 V}{\partial x^2}(t, x, \sigma)\\ &+ \frac{1}{2}b(t, x, \sigma)^2\frac{\partial^2 V}{\partial \sigma^2}(t, x, \sigma) + \rho\sigma x b(t, x, \sigma)\frac{\partial^2 V}{\partial x \partial \sigma}(t, x, \sigma) - rV(t, x, \sigma) = 0.\end{aligned}$$

Writing $Y_t = V(t, S_t, \sigma_t)$ and suppressing the dependence of V, $\tilde{a}$ and b on (t, S_t, σ_t) in our notation, an application of Itô's formula tells us that

$$\begin{aligned}
dY_t &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dS_t + \frac{\partial V}{\partial \sigma}d\sigma_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}\sigma_t^2 S_t^2 dt \\
&\quad + \frac{\partial^2 V}{\partial x \partial \sigma}\rho b \sigma_t S_t dt + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2} b^2 dt \\
&= \left(rV - rS_t\frac{\partial V}{\partial x} - \tilde{a}\frac{\partial V}{\partial \sigma} - \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 V}{\partial x^2} - \frac{1}{2}b^2\frac{\partial^2 V}{\partial \sigma^2} - \rho\sigma_t S_t b\frac{\partial^2 V}{\partial x \partial \sigma}\right)dt \\
&\quad + rS_t\frac{\partial V}{\partial x}dt + \sigma_t S_t\frac{\partial V}{\partial x}dX_t^1 + \tilde{a}\frac{\partial V}{\partial \sigma}dt + b\rho\frac{\partial V}{\partial \sigma}dX_t^1 + b\sqrt{1-\rho^2}\frac{\partial V}{\partial \sigma}dX_t^2 \\
&\quad + \frac{1}{2}\sigma_t^2 S_t^2\frac{\partial^2 V}{\partial x^2}dt + \rho b \sigma_t S_t \frac{\partial^2 V}{\partial x \partial \sigma}dt + \frac{1}{2}b^2\frac{\partial^2 V}{\partial \sigma^2}dt \\
&= rY_t dt + \sigma_t S_t \frac{\partial V}{\partial x}dX_t^1 + b\rho\frac{\partial V}{\partial \sigma}dX_t^1 + b\sqrt{1-\rho^2}\frac{\partial V}{\partial \sigma}dX_t^2.
\end{aligned}$$

If the mapping $\sigma \mapsto y = V(t, x, \sigma)$ is invertible so that $\sigma = D(t, x, y)$ for some nice function D, then

$$dY_t = rY_t dt + c(t, S_t, Y_t)dX_t^1 + d(t, S_t, Y_t)dX_t^2$$

for some functions c and d.

We have now created a complete market model with tradables $\{S_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ for which $\mathbb{Q}$ is the unique martingale measure. Of course, we have actually created one such market for each choice of $\{\theta_t\}_{t\geq 0}$ and it is the choice of $\{\theta_t\}_{t\geq 0}$ that specifies the functions c and d and it is precisely these functions that tell us how to hedge.

So what model for implied volatility corresponds to this stochastic volatility model? The implied volatility, $\hat{\sigma}(t)$, will be such that Y_t is the Black–Scholes price evaluated at (t, S_t) if the volatility in equation (7.12) is taken to be $\hat{\sigma}(t)$. In this way each choice of $\{\theta_t\}_{t\geq 0}$, or equivalently model for $\{Y_t\}_{t\geq 0}$, provides a model for the implied volatility.

There is a huge literature on stochastic volatility. A good starting point is Fouque, Papanicolau and Sircar (2000).

Exercises

1 Check that the replicating portfolio defined in §7.1 is self-financing.

2 Suppose that $\{W_t^1\}_{t\geq 0}$ and $\{W_t^2\}_{t\geq 0}$ are independent Brownian motions under $\mathbb{P}$ and let ρ be a constant with $0 < \rho < 1$. Find coefficients $\{\alpha_{ij}\}_{i,j=1,2}$ such that

$$\tilde{W}_t^1 = \alpha_{11}W_t^1 + \alpha_{12}W_t^2$$

and

$$\tilde{W}_t^2 = \alpha_{21}W_t^1 + \alpha_{22}W_t^2$$

define two standard Brownian motions under $\mathbb{P}$ with $\mathbb{E}\left[\tilde{W}_t^1\tilde{W}_t^2\right] = \rho t$. Is your solution unique?

3 Suppose that $F(t,x)$ solves the time-inhomogeneous Black–Scholes partial differential equation

$$\frac{\partial F}{\partial t}(t,x)+\frac{1}{2}\sigma^2(t)x^2\frac{\partial^2 F}{\partial x^2}(t,x)+r(t)x\frac{\partial F}{\partial x}(t,x)-r(t)F(t,x)=0, \qquad (7.13)$$

subject to the boundary conditions appropriate to pricing a European call option. Substitute

$$y=xe^{\alpha(t)}, \quad v=Fe^{\beta(t)}, \quad \tau=\gamma(t)$$

and choose $\alpha(t)$ and $\beta(t)$ to eliminate the coefficients of v and $\frac{\partial v}{\partial y}$ in the resulting equation and $\gamma(t)$ to remove the remaining time dependence so that the equation becomes

$$\frac{\partial v}{\partial \tau}(\tau,y)=\frac{1}{2}y^2\frac{\partial^2 v}{\partial y^2}(\tau,y).$$

Notice that the coefficients in this equation are independent of time and there is no reference to r or σ. Deduce that the solution to equation (7.13) can be obtained by making appropriate substitutions in the classical Black–Scholes formula.

4 Let $\{W_t^i\}_{t\geq 0}$, $i=1,\dots,n$, be independent Brownian motions. Show that $\{R_t\}_{t\geq 0}$ defined by

$$R_t=\sqrt{\sum_{i=1}^{n}(W_t^i)^2}$$

satisfies a stochastic differential equation. The process $\{R_t\}_{t\geq 0}$ is the radial part of Brownian motion in $\mathbb{R}^n$ and is known as the *n-dimensional Bessel process*.

5 Recall that we define two-dimensional Brownian motion, $\{X_t\}_{t\geq 0}$, by $X_t=(W_t^1,W_t^2)$, where $\{W_t^1\}_{t\geq 0}$ and $\{W_t^2\}_{t\geq 0}$ are independent (one-dimensional) standard Brownian motions. Find the Kolmogorov backward equation for $\{X_t\}_{t\geq 0}$. Repeat your calculation if $\{W_t^1\}_{t\geq 0}$ and $\{W_t^2\}_{t\geq 0}$ are replaced by *correlated* Brownian motions, $\{\tilde{W}_t^1\}_{t\geq 0}$ and $\{\tilde{W}_t^1\}_{t\geq 0}$ with $\mathbb{E}[d\tilde{W}_t^1 d\tilde{W}_t^2]=\rho dt$ for some $-1<\rho<1$.

6 Use a delta-hedging argument to obtain the result of Corollary 7.2.7.

7 Repeat the Black–Scholes analysis of §7.2 in the case when the chosen numeraire, $\{B_t\}_{t\geq 0}$, has non-zero volatility and check that the fair price of a derivative with payoff C_T at time T is once again

$$V_t=B_t\mathbb{E}^{\mathbb{Q}}\left[\frac{C_T}{B_T}\,\middle|\,\mathcal{F}_t\right]$$

for a suitable choice of $\mathbb{Q}$ (which you should specify).

8 Two traders, operating in the same complete arbitrage-free Black–Scholes market of §7.2, sell identical options, but make different choices of numeraire. How will their hedging strategies differ?

9 Find a portfolio that replicates the quanto forward contract of Example 7.2.9.

10 A *quanto digital contract* written on the BP stock of Example 7.2.9 pays \$1 at time T if the BP Sterling stock price, S_T, is larger than K. Assuming the Black–Scholes quanto model of §7.2, find the time zero price of such an option and the replicating portfolio.

11 A *quanto call option* written on the BP stock of Example 7.2.9 is worth $E(S_T - K)_+$ *dollars* at time T, where S_T is the *Sterling* stock price. Assuming the Black–Scholes quanto model of §7.2, find the time zero price of the option and the replicating portfolio.

12 *Asian options* Suppose that our market, consisting of a riskless cash bond, $\{B_t\}_{t\geq 0}$, and a single risky asset with price $\{S_t\}_{t\geq 0}$, is governed by

$$dB_t = rB_t dt, \quad B_0 = 1$$

and

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion.
An option is written with payoff $C_T = \Phi(S_T, Z_T)$ at time T where

$$Z_t = \int_0^t g(u, S_u)du$$

for some (deterministic) real-valued function g on $\mathbb{R}_+ \times \mathbb{R}$.
From our general theory we know that the value of such an option at time t satisfies

$$V_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T, Z_T)|\,\mathcal{F}_t\right]$$

where $\mathbb{Q}$ is the measure under which $\{S_t/B_t\}_{t\geq 0}$ is a martingale.
Show that $V_t = F(t, S_t, Z_t)$ where the real-valued function $F(t, x, z)$ on $\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}$ solves

$$\begin{aligned}\frac{\partial F}{\partial t} + rx\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 F}{\partial x^2} + g\frac{\partial F}{\partial z} - rF &= 0,\\ F(T, x, z) &= \Phi(x, z).\end{aligned}$$

Show further that the claim C_T can be hedged by a self-financing portfolio consisting at time t of

$$\phi_t = \frac{\partial F}{\partial x}(t, S_t, Z_t)$$

units of stock and

$$\psi_t = e^{-rt}\left(F(t, S_t, Z_t) - S_t\frac{\partial F}{\partial x}(t, S_t, Z_t)\right)$$

cash bonds.

13 Suppose that $\{N_t\}_{t\geq 0}$ is a Poisson process whose intensity under $\mathbb{P}$ is $\{\lambda_t\}_{t\geq 0}$. Show that $\{M_t\}_{t\geq 0}$ defined by

$$M_t = N_t - \int_0^t \lambda_s ds$$

is a $\mathbb{P}$-martingale with respect to the σ-field generated by $\{N_t\}_{t\geq 0}$.

14 Suppose that $\{N_t\}_{t\geq 0}$ is a Poisson process under $\mathbb{P}$ with intensity $\{\lambda_t\}_{t\geq 0}$ and $\{M_t\}_{t\geq 0}$ is the corresponding Poisson martingale. Check that for an $\{\mathcal{F}_t^M\}_{t\geq 0}$-predictable process $\{f_t\}_{t\geq 0}$,

$$\int_0^t f_s dM_s$$

is a $\mathbb{P}$-martingale.

15 Show that our analysis of §7.3 is still valid if we allow the coefficients in the stochastic differential equations driving the asset prices to be $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes, provided we make some boundedness assumptions that you should specify.

16 Show that the process $\{L_t\}_{t\geq 0}$ in Theorem 7.3.5 is the product of a Poisson exponential martingale and a Brownian exponential martingale and hence prove that it is a martingale.

17 Show that in the classical Black–Scholes model the *vega* for a European call (or put) option is strictly positive. Deduce that for vanilla options we can infer the volatility parameter of the Black–Scholes model from the price.

18 Suppose that $V(t, x)$ is the Black–Scholes price of a European call (or put) option at time t given that the stock price at time t is x. Prove that $\frac{\partial^2 V}{\partial x^2} \geq 0$.

19 Suppose that an asset price $\{S_t\}_{t\geq 0}$ follows a geometric Brownian motion with jumps occurring according to a Poisson process with constant intensity λ. At the time, τ, of each jump, independently, $S_\tau / S_{\tau-}$ has a lognormal distribution. Show that, for each fixed t, S_t has a lognormal distribution with the variance parameter σ^2 given by a multiple of a Poisson random variable.

Bibliography

Background reading:

- *Probability, an Introduction,* Geoffrey Grimmett and Dominic Welsh, Oxford University Press (1986).
- *Options, Futures and Other Derivative Securities,* John Hull, Prentice-Hall (Second edition 1993).

Grimmett & Welsh contains all the concepts that we assume from probability theory. Hull is popular with practitioners. It explains the operation of markets in some detail before turning to modelling.

Supplementary textbooks:

- *Arbitrage Theory in Continuous Time,* Tomas Björk, Oxford University Press (1998).
- *Dynamic Asset Pricing Theory,* Darrell Duffie, Princeton University Press (1992).
- *Introduction to Stochastic Calculus Applied to Finance,* Damien Lamberton and Bernard Lapeyre, translated by Nicolas Rabeau and François Mantion, Chapman and Hall (1996).
- *The Mathematics of Financial Derivatives,* Paul Wilmott, Sam Howison and Jeff Dewynne, Cambridge University Press (1995).

These all represent useful supplementary reading. The first three employ a variety of techniques while Wilmott, Howison & Dewynne is devoted exclusively to the partial differential equations approach.

Further topics in financial mathematics:

- *Financial Calculus: an Introduction to Derivatives Pricing,* Martin Baxter and Andrew Rennie, Cambridge University Press (1996).
- *Derivatives in Financial Markets with Stochastic Volatility,* Jean-Pierre Fouque, George Papanicolau and Ronnie Sircar, Cambridge University Press (2000).
- *Continuous Time Finance,* Robert Merton, Blackwell (1990).
- *Martingale Methods in Financial Modelling,* Marek Musiela and Marek Rutkowski, Springer-Verlag (1998).

Although aimed at practitioners rather than university courses, Chapter 5 of Baxter & Rennie provides a good starting point for the study of interest rates. Fouque, Papanicolau & Sircar is a highly accessible text that would provide an excellent basis for a special topic in a *second* course in financial mathematics. Merton is a synthesis of the remarkable research contributions of its Nobel-prize-winning author. Musiela & Rutkowski provides an encyclopaedic reference.

Brownian motion, martingales and stochastic calculus:

- *Introduction to Stochastic Integration,* Kai Lai Chung and Ruth Williams, Birkhäuser (Second edition 1990).
- *Stochastic Differential Equations and Diffusion Processes,* Nobuyuki Ikeda and Shinzo Watanabe, North-Holland (Second edition 1989).
- *Brownian Motion and Stochastic Calculus,* Ioannis Karatzas and Steven Shreve, Springer-Verlag (Second edition 1991).
- *Probability with Martingales,* David Williams, Cambridge University Press (1991).

Williams is an excellent introduction to discrete parameter martingales and much more (integration, conditional expectation, measure, ...). The others all deal with the continuous world. Chung & Williams is short enough that it can simply be read cover to cover.
A further useful reference is *Handbook of Brownian Motion: Facts and Formulae,* Andrei Borodin and Paavo Salminen, Birkhäuser (1996).

Additional references from the text:

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- M Davis, Mathematics of financial markets, in *Mathematics Unlimited – 2001 and Beyond,* Bjorn Engquist and Wilfried Schmid (eds), Springer-Verlag (2001).
- D Freedman, *Brownian Motion and Diffusion,* Holden-Day (1971).
- J M Harrison and D M Kreps, Martingales and arbitrage in multiperiod securities markets. *J Econ Theory* **20** (1979), 381–408
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- P Protter, *Stochastic Integration and Differential Equations,* Springer-Verlag (1990).
- D Revuz and M Yor, *Continuous Martingales and Brownian Motion,* Springer-Verlag (Third edition 1998).
- P A Samuelson, Proof that properly anticipated prices fluctuate randomly, *Industrial Management Review* **6**, (1965), 41–50.

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Notation

Financial instruments and the Black–Scholes model

T, maturity time.
C_T, value of claim at time T.
$\{S_n\}_{n\geq 0}$, $\{S_t\}_{t\geq 0}$, value of the underlying stock.
K, the strike price in a vanilla option.
$(S_T - K)_+ = \max\{(S_T - K), 0\}$.
r, continuously compounded interest rate.
σ, volatility.
$\mathbb{P}$, a probability measure, usually the market measure.
$\mathbb{Q}$, a martingale measure equivalent to the market measure.
$\mathbb{E}^{\mathbb{Q}}$, the expectation under $\mathbb{Q}$.
$\frac{d\mathbb{Q}}{d\mathbb{P}}$ the Radon–Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$.
$\{\tilde{S}_t\}_{t\geq 0}$, the *discounted* value of the underlying stock. In general, for a process $\{Y_t\}_{t\geq 0}$, $\tilde{Y}_t = Y_t/B_t$ where $\{B_t\}_{t\geq 0}$ is the value of the riskless cash bond at time t.
$V(t, x)$, the value of a portfolio at time t if the stock price $S_t = x$. Also the Black–Scholes price of an option.

General probability

$(\Omega, \mathcal{F}, \mathbb{P})$, probability triple.
$\mathbb{P}[A|B]$, conditional probability of A given B.
Φ, standard normal distribution function.
$p(t, x, y)$, transition density of Brownian motion.
$X \stackrel{\mathcal{D}}{=} Y$, the random variables X and Y have the same distribution.
$Z \sim N(0, 1)$, the random variable Z has a standard normal distribution.
$\mathbb{E}[X; A]$, see Definition 2.3.4.

Martingales and other stochastic processes

$\{M_t\}_{t\geq 0}$, a martingale under some specified probability measure.
$\{[M]_t\}_{t\geq 0}$, the quadratic variation of $\{M_t\}_{t\geq 0}$.
$\{\mathcal{F}_n\}_{n\geq 0}$, $\{\mathcal{F}_t\}_{t\geq 0}$, filtration.

$\{\mathcal{F}_n^X\}_{n\geq 0}$ (resp. $\{\mathcal{F}_t^X\}_{t\geq 0}$), filtration generated by the process $\{X_n\}_{n\geq 0}$ (resp. $\{X_t\}_{t\geq 0}$).
$\mathbb{E}[X|\mathcal{F}]$, $\mathbb{E}\left[X_{n+1}\middle| X_n\right]$, conditional expectation; see pages 30ff.
$\{W_t\}_{t\geq 0}$, Brownian motion under a specified measure, usually the market measure.
$X^*(t)$, $X_*(t)$, maximum and minimum processes corresponding to $\{X_t\}_{t\geq 0}$.

Miscellaneous

$\triangleq$, defined equal to.
$\delta(\pi)$, the mesh of the partition π.
$f\big|_x$, the function f evaluated at x.
θ^t (for a vector or matrix θ), the transpose of θ.
$x > 0$, $x \gg 0$ for a vector $x \in \mathbb{R}^n$, see page 11.

Index