T, but with positive
bitrage opportunities

6 Different payoffs

Most of the concrete examples of options considered so far have been the standard examples of calls and puts. Such options have liquid markets, their prices are fairly well determined and margins are competitive. Any option that is not one of these *vanilla* calls or puts is called an exotic option. Such options are introduced to extend a bank's product range or to meet hedging and speculative needs of clients. There are usually no markets in these options and they are bought and sold purely 'over the counter'. Although the principles of pricing and hedging exotics are exactly the same as for vanillas, risk management requires care. Not only are these exotic products much less liquid than standard options, but they often have discontinuous payoffs and so can have huge 'deltas' close to the expiry time making them difficult to hedge.

This chapter is devoted to examples of exotic options. The simplest exotics to price and hedge are *packages*, that is, options for which the payoff is a combination of our standard 'vanilla' options and the underlying asset. We already encountered such options in §1.1. We relegate their valuation to the exercises. The next simplest examples are European options, meaning options whose payoff is a function of the stock price at the maturity time. The payoffs considered in §6.1 are discontinuous and we discover potential hedging problems. In §6.2 we turn our attention to multistage options. Such options allow decisions to be made or stipulate conditions at intermediate dates during their lifetime. The rest of the chapter is devoted to path-dependent options. In §6.3 we use our work of §3.3 to price lookback and barrier options. Asian options, whose payoff depends on the average of the stock price over the lifetime of the option, are discussed briefly in §6.4 and finally §6.5 is a very swift introduction to pricing American options in continuous time.

6.1 European options with discontinuous payoffs

We work in the basic Black–Scholes framework. That is, our market consists of a riskless cash bond whose value at time t is $B_t = e^{rt}$ and a single risky asset whose price, $\{S_t\}_{t\geq 0}$, follows a geometric Brownian motion.

In §5.2 we established explicit formulae for both the price and the hedging portfolio for European options within this framework. Specifically, if the payoff of

the option at the maturity time T is $C_T = f(S_T)$ then for $0 \le t \le T$ the value of the option at time t is

$$\begin{aligned} V_t = F(t, S_t) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} f(S_T) \middle| \mathcal{F}_t \right] \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} f\left(S_t \exp\left(\left(r - \frac{\sigma^2}{2} \right)(T-t) + \sigma y \sqrt{T-t} \right) \right) \\ &\quad \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2} \right) dy, \end{aligned} \tag{6.1}$$

where $\mathbb{Q}$ is the martingale measure, and the claim $f(S_T)$ can be replicated by a portfolio consisting at time t of ϕ_t units of stock and $\psi_t = e^{-rt}(V_t - \phi_t S_t)$ cash bonds where

$$\phi_t = \frac{\partial F}{\partial x}(t, x)\bigg|_{x=S_t}. \tag{6.2}$$

Mathematically, other than the issue of actually *evaluating* the integrals, that would appear to be the end of the story. However, as we shall see, rather more careful consideration of our assumptions might lead us to doubt the usefulness of these formulae when the payoff is a discontinuous function of S_T.

Digitals and pin risk

Example 6.1.1 (Digital options) *The payoff of a* digital option, *also sometimes called a* binary option *or a* cash-or-nothing option, *is given by a Heaviside function. For example, a digital call option with strike price K at time T has payoff*

$$C_T = \begin{cases} 1 & \text{if } S_T \ge K, \\ 0 & \text{if } S_T < K \end{cases}$$

at maturity. Find the price and the hedge for such an option.

Solution: In order to implement the formula (6.1) we must establish the range of y for which

$$S_t \exp\left(\left(r - \frac{\sigma^2}{2} \right)(T-t) + \sigma y \sqrt{T-t} \right) > K.$$

Rearranging we see that this holds for $y > d$ where

$$d = \frac{1}{\sigma\sqrt{T-t}} \left(\log\left(\frac{K}{S_t} \right) - \left(r - \frac{\sigma^2}{2} \right)(T-t) \right).$$

Writing Φ for the normal distribution function and substituting in equation (6.1) we obtain

$$\begin{aligned} V_t &= e^{-r(T-t)} \int_d^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy = e^{-r(T-t)} \int_{-\infty}^{-d} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &= e^{-r(T-t)} \Phi(-d) = e^{-r(T-t)} \Phi(d_2), \end{aligned}$$

where

$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right),$$

as in Example 5.2.2.

Now we turn to the hedge. By (6.2), the stock holding in our replicating portfolio at time t is

$$\begin{aligned}\phi_t &= e^{-r(T-t)}\frac{1}{S_t}\frac{1}{\sqrt{2\pi(T-t)}\sigma} \\ &\quad\times \exp\left(-\frac{1}{2(T-t)\sigma^2}\left(\log\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)^2\right).\end{aligned}$$

Now as $t \uparrow T$, this converges to $1/K$ times the delta function concentrated on $S_T = K$. Consider what this means for the replicating portfolio as $t \uparrow T$. Away from $S_t = K$, ϕ_t is close to zero, but if S_t is close to K the stock holding in the portfolio will be very large. Now if near expiry the asset price is close to K, there is a high probability that its value will cross the value $S_t = K$ many times before expiry. But if the asset price oscillates around the strike price close to expiry our prescription for the hedging portfolio will tell us to rapidly buy and sell large numbers of the underlying asset. Since markets are not the perfect objects envisaged in our Black–Scholes model and we cannot instantaneously buy and sell, risk from small asset price changes (not to mention transaction costs) can easily outweigh the maximum liability that we are exposed to by having sold the digital. This is known as the *pin risk* associated with the option. □

If we can overcome our misgivings about the validity of the Black–Scholes price for digitals, then we can use them as building blocks for other exotics. Indeed, since the option with payoff $\mathbf{1}_{[K_1,K_2]}(S_T)$ at time T can be replicated by buying a digital with strike K_2 and maturity T and selling a digital with strike K_1 and maturity T, in theory we could price any European option by replicating it by (possibly infinite) linear combinations of digitals.

6.2 Multistage options

Some options either allow decisions to be made or stipulate conditions at intermediate dates during their lifetime. An example is the forward start option of Exercise 3 of Chapter 2. To illustrate the procedure for valuation of multistage options, we find the Black–Scholes price of a forward start.

Example 6.2.1 (Forward start option) *Recall that a forward start option is a contract in which the holder receives, at time T_0, at no extra cost, an option with expiry date $T_1 > T_0$ and strike price equal to S_{T_0}. If the risk-free rate is r find the Black–Scholes price, V_t, of such an option at times $t < T_1$.*

Solution: First suppose that $t \in [T_0, T_1]$. Then by time t we know S_{T_0} and so the value of the option is just that of a European call option with strike S_{T_0} and maturity T_1, namely

$$V_t = e^{-r(T_1-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(S_{T_1} - S_{T_0}\right)_+ \middle| \mathcal{F}_t\right],$$

where $\mathbb{Q}$ is a probability measure under which the discounted price of the underlying is a martingale. In particular, at time T_0, using Example 5.2.2,

$$V_{T_0} = S_{T_0}\Phi(d_1) - S_{T_0}e^{-r(T_1-T_0)}\Phi(d_2)$$

where

$$d_1 = \frac{\left(r + \frac{\sigma^2}{2}\right)(T_1 - T_0)}{\sigma\sqrt{T_1 - T_0}} \quad \text{and} \quad d_2 = \frac{\left(r - \frac{\sigma^2}{2}\right)(T_1 - T_0)}{\sigma\sqrt{T_1 - T_0}}.$$

In other words

$$\begin{aligned} V_{T_0} &= S_{T_0}\left\{\Phi\left(\left(r + \frac{\sigma^2}{2}\right)\frac{\sqrt{T_1 - T_0}}{\sigma}\right) - e^{-r(T_1-T_0)}\Phi\left(\left(r - \frac{\sigma^2}{2}\right)\frac{\sqrt{T_1 - T_0}}{\sigma}\right)\right\} \\ &= cS_{T_0} \end{aligned}$$

where $c = c(r, \sigma, T_0, T_1)$ is independent of the asset price.

To find the price at time $t < T_0$, observe that the portfolio consisting of c units of the underlying over the time interval $0 \le t \le T_0$ exactly replicates the option at time T_0. Thus for $t < T_0$, the price is given by cS_t. In particular, the time zero price of the option is

$$V_0 = S_0\left\{\Phi\left(\left(r + \frac{\sigma^2}{2}\right)\frac{\sqrt{T_1 - T_0}}{\sigma}\right) - e^{-r(T_1-T_0)}\Phi\left(\left(r - \frac{\sigma^2}{2}\right)\frac{\sqrt{T_1 - T_0}}{\sigma}\right)\right\}.$$

□

General strategy

Notice that, in order to price the forward start option, we worked our way back from time T_1. This reflects a general strategy. For a multistage option with maturity T_1 and conditions stipulated at an intermediate time T_0, we invoke the following procedure.

Valuing multistage options:

1 Find the payoff at time T_1.
2 Use Black–Scholes to value the option for $t \in [T_0, T_1]$.
3 Apply the contract conditions at time T_0.
4 Use Black–Scholes to value the option for $t \in [0, T_0]$.

We put this into action for two more examples.

Example 6.2.2 (Ratio derivative) *A ratio derivative can be described as follows. Two times $0 < T_0 < T_1$ are fixed. The derivative matures at time T_1 when its payoff is S_{T_1}/S_{T_0}. Find the value of the option at times $t < T_1$.*

Solution: First suppose that $t \in [T_0, T_1]$. At such times S_{T_0} is known and so

$$V_t = \frac{1}{S_{T_0}} \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T_1-t)} S_{T_1} \middle| \mathcal{F}_t \right]$$

where, under $\mathbb{Q}$, the discounted asset price is a martingale. Hence $V_t = S_t / S_{T_0}$. In particular, $V_{T_0} = 1$. Evidently the value of the option for $t < T_0$ is therefore $e^{-r(T_0-t)}$. □

Both forward start options and ratio derivatives, in which the strike price is set to be a function of the stock price at some intermediate time T_0, are examples of *cliquets*.

Compound options

A rather more complex class of examples is provided by the *compound options*. These are 'options on options', that is options in which the rôle of the underlying is itself played by an option. There are four basic types of compound option: call-on-call, call-on-put, put-on-call and put-on-put.

Example 6.2.3 (Call-on-call option) *To describe the call-on-call option we must specify two exercise prices, K_0 and K_1, and two maturity times $T_0 < T_1$. The 'underlying' option is a European call with strike price K_1 and maturity T_1. The call-on-call contract gives the holder the right to buy the underlying option for price K_0 at time T_0. Find the value of such an option for $t < T_0$.*

Solution: We know how to price the underlying call. Its value at time T_0 is given by the Black–Scholes formula as

$$\begin{aligned} C\left(S_{T_0}, T_0; K_1, T_1\right) &= S_{T_0} \Phi\left(d_1\left(S_{T_0}, T_1 - T_0, K_1\right)\right) \\ &\quad - K_0 e^{-r(T_1-T_0)} \Phi\left(d_2\left(S_{T_0}, T_1 - T_0, K_1\right)\right) \end{aligned}$$

where

$$d_1\left(S_{t_0}, T_1 - T_0, K_1\right) = \frac{\log\left(\frac{S_{T_0}}{K_1}\right) + \left(r + \frac{\sigma^2}{2}\right)(T_1 - T_0)}{\sigma\sqrt{T_1 - T_0}}$$

and $d_2\left(S_{t_0}, T_1 - T_0, K_1\right) = d_1\left(S_{t_0}, T_1 - T_0, K_1\right) - \sigma\sqrt{T_1 - T_0}$. The value of the compound option at time T_0 is then

$$V\left(T_0, S_{T_0}\right) = \left(C\left(S_{T_0}, T_0; K_1, T_1\right) - K_0\right)_+ .$$

Now we apply Black–Scholes again. The value of the option at times $t < T_0$ is

$$V(t, S_t) = e^{-r(T_0-t)} \mathbb{E}^{\mathbb{Q}} \left[\left(C(S_{T_0}, T_0, K_1, T_1) - K_0\right)_+ \middle| \mathcal{F}_t^S \right] \tag{6.3}$$

where the discounted asset price is a martingale under $\mathbb{Q}$. Using that

$$S_{T_0} = S_t \exp\left(\sigma Z\sqrt{T_0 - t} + \left(r - \frac{1}{2}\sigma^2\right)(T_0 - t)\right),$$

where, under $\mathbb{Q}$, $Z \sim N(0,1)$, equation (6.3) now gives an analytic expression for the value in terms of the cumulative distribution function of a bivariate normal random variable. We write

$$f(y) = S_0 \exp\left(\sigma y\sqrt{T_0 - t} + \left(r - \frac{1}{2}\sigma^2\right)(T_0 - t)\right)$$

and define x_0 implicitly by

$$x_0 = \inf\{y \in \mathbb{R} : C(f(y), T_0; K_1, T_1) \geq K_0\}.$$

Now

$$\log\left(\frac{f(y)}{K_1}\right) = \log\left(\frac{S_0}{K_1}\right) + \sigma y\sqrt{T_0 - t} + \left(r - \frac{1}{2}\sigma^2\right)(T_0 - t)$$

and so writing

$$\hat{d}_1(y) = \frac{\log(S_0/K_1) + \sigma y\sqrt{T_0 - t} + rT_1 - \sigma^2 T_0 + \frac{1}{2}\sigma^2 T_1}{\sigma\sqrt{T_1 - T_0}}$$

and

$$\hat{d}_2(y) = \frac{\log(S_0/K_1) + \sigma y\sqrt{T_0 - t} + rT_1 - \frac{1}{2}\sigma^2 T_1}{\sigma\sqrt{T_1 - T_0}}$$

we obtain

$$\begin{aligned} V(t, S_t) &= e^{-r(T_0-t)} \int_{x_0}^{\infty} \left(f(y)\Phi(\hat{d}_1(y)) - K_0 e^{-r(T_1-T_0)}\Phi(\hat{d}_2(y)) - K_0\right) \\ &\quad \times \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy. \end{aligned}$$

□

6.3 Lookbacks and barriers

We now turn to our first example of path-dependent options, that is options for which the history of the asset price over the duration of the contract determines the payout at expiry.

As usual we use $\{S_t\}_{0 \leq t \leq T}$ to denote the price of the underlying asset over the duration of the contract. In this section we shall consider options whose payoff at maturity depends on S_T and one or both of the maximum and minimum values taken by the asset price over $[0, T]$.

Notation: We write

$$S_*(t) = \min\{S_u : 0 \leq u \leq t\},$$

$$S^*(t) = \max\{S_u : 0 \leq u \leq t\}.$$

Definition 6.3.1 (Lookback call) A lookback call *gives the holder the right to buy a unit of stock at time T for a price equal to the minimum achieved by the stock up to time T. That is the payoff is*

$$C_T = S_T - S_*(T).$$

Definition 6.3.2 (Barrier options) A barrier option *is one that is activated or deactivated if the asset price crosses a preset barrier. There are two basic types:*

1 **knock-ins**

(a) *the barrier is* up-and-in *if the option is only active if the barrier is hit from below,*
(b) *the barrier is* down-and-in *if the option is only active if the barrier is hit from above;*

2 **knock-outs**

(a) *the barrier is* up-and-out *if the option is worthless if the barrier is hit from below,*
(b) *the barrier is* down-and-out *if the option is worthless if the barrier is hit from above.*

Example 6.3.3 A down-and-in call option *pays out $(S_T - K)_+$ only if the stock price fell below some preagreed level c some time before T, otherwise it is worthless. That is, the payoff is*

$$C_T = \mathbf{1}_{\{S_*(T) \le c\}} (S_T - K)_+.$$

As always we can express the value of such an option as a discounted expected value under the martingale measure $\mathbb{Q}$. Thus the value at time zero can be written as

$$V(0, S_0) = e^{-rT} \mathbb{E}^{\mathbb{Q}} [C_T] \tag{6.4}$$

where r is the riskless borrowing rate and the discounted stock price is a $\mathbb{Q}$-martingale. However, in order to actually evaluate the expectation in (6.4) for barrier options we need to know the joint distribution of $(S_T, S_*(T))$ and $(S_T, S^*(T))$ under the martingale measure $\mathbb{Q}$. Fortunately we did most of the work in Chapter 3.

Joint distribution of the stock price and its minimum

In Lemma 3.3.4 we found the joint distribution of Brownian motion and its maximum. Specifically, if $\{W_t\}_{t \ge 0}$ is a standard $\mathbb{P}$-Brownian motion, writing $M_t = \max_{0 \le s \le t} W_s$, for $a > 0$ and $x \le a$

$$\mathbb{P}[M_t \ge a, W_t \le x] = 1 - \Phi\left(\frac{2a - x}{\sqrt{t}}\right).$$

By symmetry, writing $m_t = \min_{0 \le s \le t} W_s$, for $a < 0$ and $x \ge a$,

$$\mathbb{P}[m_t \le a, W_t \ge x] = 1 - \Phi\left(\frac{-2a + x}{\sqrt{t}}\right),$$

or, differentiating, if $a < 0$ and $x \ge a$

$$\mathbb{P}[m_T \le a, W_T \in dx] = p_T(0, -2a + x)dx = p_T(2a, x)dx$$

where

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-|x-y|^2/2t\right).$$

Combining these results with (two applications of) the Girsanov Theorem will allow us to calculate the joint distribution of $(S_T, S^*(T))$ and of $(S_T, S_*(T))$ under the martingale measure $\mathbb{Q}$.

As usual, under the market measure $\mathbb{P}$,

$$S_t = S_0 \exp(\nu t + \sigma W_t)$$

where $\{W_t\}_{t \geq 0}$ is a $\mathbb{P}$-Brownian motion. Let us suppose, temporarily, that $\nu = 0$ so that $S_t = S_0 \exp(\sigma W_t)$ and moreover $S_*(t) = S_0 \exp(\sigma m_t)$ and $S^*(t) = S_0 \exp(\sigma M_t)$. In this special case then the joint distribution of the stock price and its minimum (resp. maximum) can be deduced from that of (W_t, m_t) (resp. (W_t, M_t)). Of course, in general, ν will not be zero either under the market measure $\mathbb{P}$ or under the martingale measure $\mathbb{Q}$. Our strategy will be to use the Girsanov Theorem not only to switch to the martingale measure but also to switch, temporarily, to an equivalent measure under which $S_t = S_0 \exp(\sigma W_t)$.

Lemma 6.3.4 *Let $\{Y_t\}_{t \geq 0}$ be given by $Y_t = bt + X_t$ where b is a constant and $\{X_t\}_{t \geq 0}$ is a $\mathbb{Q}$-Brownian motion. Writing $Y_*(t) = \min\{Y_u : 0 \leq u \leq t\}$,*

$$\mathbb{Q}\left[Y_*(T) \leq a, Y_T \in dx\right] = \begin{cases} p_T(bT, x)dx & \text{if } x < a, \\ e^{2ab} p_T(2a + bT, x)dx & \text{if } x \geq a, \end{cases}$$

where, as above, $p_t(x, y)$ is the Brownian transition density function.

Proof: By the Girsanov Theorem, there is a measure $\mathbb{P}$, equivalent to $\mathbb{Q}$, under which $\{Y_t\}_{t \geq 0}$ is a $\mathbb{P}$-Brownian motion and

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = \exp\left(-bX_T - \frac{1}{2}b^2 T\right).$$

Notice that this depends on $\{X_t\}_{0 \leq t \leq T}$ only through X_T. The $\mathbb{Q}$-probability of the event $\{Y_*(T) \leq a, Y_T \in dx\}$ will be the $\mathbb{P}$-probability of that event multiplied by $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T}$ evaluated at $Y_T = x$. Now

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(bX_T + \frac{1}{2}b^2 T\right) = \exp\left(bY_T - \frac{1}{2}b^2 T\right)$$

and so for $a < 0$ and $x \geq a$

$$\begin{aligned} \mathbb{Q}\left[Y_*(T) \leq a, Y_T \in dx\right] &= \mathbb{P}\left[Y_*(T) \leq a, Y_T \in dx\right] \exp\left(bx - \frac{1}{2}b^2 T\right) \\ &= p_T(2a, x) \exp\left(bx - \frac{1}{2}b^2 T\right) dx \\ &= e^{2ab} p_T(2a + bT, x)dx. \end{aligned} \tag{6.5}$$

Evidently for $x \leq a$, $\{Y_*(T) \leq a, Y_T \in dx\} = \{Y_T \in dx\}$ and so for $x \leq a$

$$\begin{aligned} \mathbb{Q}\left[Y_*(T) \leq a, Y_T \in dx\right] &= \mathbb{Q}\left[Y_T \in dx\right] \\ &= \mathbb{Q}\left[bT + X_T \in dx\right] \\ &= p_T(bT, x)dx \end{aligned}$$

and the proof is complete. □

Differentiating (6.5) with respect to a, we see that, in terms of joint densities, for $a < 0$

$$\mathbb{Q}\left[Y_*(T) \in da, Y_T \in dx\right] = \frac{2e^{2ab}}{T}|x - 2a| p_T(2a + bT, x)dx\, da \quad \text{for } x \geq a.$$

The joint density evidently vanishes if $x < a$ or $a > 0$. In Exercise 13 you are asked to find the joint distribution of Y_T and $Y^*(T)$ under $\mathbb{Q}$.

An expression for the price

From Chapter 5, under the martingale measure $\mathbb{Q}$, $S_t = S_0 \exp(\sigma Y_t)$ where

$$Y_t = \frac{(r - \frac{1}{2}\sigma^2)}{\sigma} t + X_t$$

and $\{X_t\}_{t \geq 0}$ is a $\mathbb{Q}$-Brownian motion. So by applying these results with $b = (r - \frac{1}{2}\sigma^2)/\sigma$ we can now evaluate the price of any option maturing at time T whose payoff depends just on the stock price at time T and its minimum (or maximum) value over the lifetime of the contract. If the payoff is $C_T = g(S_*(T), S_T)$ and r is the riskless borrowing rate then the value of the option at time zero is

$$\begin{aligned} V(0, S_0) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[g\left(S_*(T), S_T\right)\right] \\ &= e^{-rT} \int_{a=-\infty}^{0} \int_{x=a}^{\infty} g\left(S_0 e^{\sigma x}, S_0 e^{\sigma a}\right) \mathbb{Q}\left[Y_*(T) \in da, Y_T \in dx\right]. \end{aligned}$$

Example 6.3.5 (Down-and-in call option) *Find the time zero price of a down-and-in call option whose payoff at time T is*

$$C_T = \mathbf{1}_{\{S_*(T) \leq c\}}\left(S_T - K\right)_+$$

where c is a (positive) preagreed constant less than K.

Solution: Using $S_t = S_0 \exp(\sigma Y_t)$ we rewrite the payoff as

$$C_T = \mathbf{1}_{\{Y_*(T) \leq \frac{1}{\sigma}\log(c/S_0)\}}\left(S_0 e^{\sigma Y_T} - K\right)_+.$$

Writing $b = (r - \frac{1}{2}\sigma^2)/\sigma$, $a = \frac{1}{\sigma}\log(c/S_0)$ and $x_0 = \frac{1}{\sigma}\log(K/S_0)$ we obtain

$$V(0, S_0) = e^{-rT} \int_{x_0}^{\infty} \left(S_0 e^{\sigma x} - K\right) \mathbb{Q}\left(Y_*(T) \leq a, Y_T \in dx\right).$$

Using the expression for the joint distribution of $(Y_*(T), Y_T)$ obtained above yields

$$V(0, S_0) = e^{-rT} \int_{x_0}^{\infty} \left(S_0 e^{\sigma x} - K\right) e^{2ab} p_T(2a + bT, x)dx.$$

We have used the fact that, since $c < K$, $x_0 \geq a$. First observe that

$$\begin{aligned} e^{-rT} \int_{x_0}^{\infty} K e^{2ab} p_T(2a + bT, x)dx &= K e^{-rT} e^{2ab} \int_{(x_0 - 2a - bT)/\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &= K e^{-rT} e^{2ab} \int_{-\infty}^{(2a + bT - x_0)/\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &= K e^{-rT} \left(\frac{c}{S_0}\right)^{\frac{2r}{\sigma^2} - 1} \Phi\left(\frac{2a + bT - x_0}{\sqrt{T}}\right) \\ &= K e^{-rT} \left(\frac{c}{S_0}\right)^{\frac{2r}{\sigma^2} - 1} \Phi\left(\frac{\log(F/K) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \end{aligned}$$

where $F = e^{rT} c^2 / S_0$.

Similarly,

$$\begin{aligned} & e^{-rT} \int_{x_0}^{\infty} S_0 e^{\sigma x} e^{2ab} p_T(2a + bT, x)dx \\ &= S_0 e^{-rT} e^{2ab} \int_{x_0}^{\infty} \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{(x - (2a + bT))^2 - 2\sigma x T}{2T}\right) dx \\ &= S_0 e^{-rT} e^{2ab} \int_{(x_0 - (2a + bT) - \sigma T)/\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \\ &\quad \times \exp\left(\frac{1}{2}\sigma^2 T + 2a\sigma + b\sigma T\right) \\ &= e^{-rT} \left(\frac{c}{S_0}\right)^{\frac{2r}{\sigma^2} - 1} F \Phi\left(\frac{\log(F/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Comparing this with Example 5.2.2

$$V(0, S_0) = \left(\frac{c}{S_0}\right)^{\frac{2r}{\sigma^2} - 1} C\left(\frac{c^2}{S_0}, 0; K, T\right),$$

where $C(x, t; K, T)$ is the price at time t of a European call option with strike K and maturity T if the stock price at time t is x. □

The price of a barrier option can also be expressed as the solution of a partial differential equation.

Example 6.3.6 (Down-and-out call) *A down-and-out call has the same payoff as a European call option,* $(S_T - K)_+$, *unless during the lifetime of the contract the price of the underlying asset has fallen below some preagreed barrier,* c, *in which case the option is 'knocked out' worthless.*

Writing $V(t,x)$ for the value of such an option at time t if $S_t = x$ and assuming that $K > c$, $V(t,x)$ solves the Black–Scholes equation for $(t,x) \in [0,T] \times [c,\infty)$ subject to the boundary conditions

$$\begin{aligned} V(T, S_T) &= (S_T - K)_+, \\ V(t, c) &= 0, \quad t \in [0, T], \\ \frac{V(t,x)}{x} &\to 1, \quad \text{as } x \to \infty. \end{aligned}$$

The last boundary condition follows since as $S_t \to \infty$, the probability of the asset price hitting level c before time T tends to zero.

Exercise 16 provides a method for solving the Black–Scholes partial differential equation with these boundary conditions.

Of course more and more complicated barrier options can be dreamt up. For example, a *double knock-out option* is worthless if the stock price leaves some interval $[c_1, c_2]$ during the lifetime of the contract. The probabilistic pricing formula for such a contract then requires the joint distribution of the triple $(S_T, S_*(T), S^*(T))$. As in the case of a single barrier, the trick is to use Girsanov's Theorem to deduce the joint distribution from that of (W_T, m_T, M_T) where $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion and $\{m_t\}_{t\geq 0}$, $\{M_t\}_{t\geq 0}$ are its running minimum and maximum respectively. This in turn is given by

$$\mathbb{P}\left[W_T \in dy, a < m_T, M_T < b\right] = \sum_{n\in\mathbb{Z}} \Big\{ p_T(2n(a-b), y) - p(2n(b-a), y-2a) \Big\} dy;$$

see Freedman (1971) for a proof. An explicit pricing formula will then be in the form of an infinite sum. In Exercise 20 you obtain the pricing formula by directly solving the Black–Scholes differential equation.

Probability or pde?

As we have seen in Exercise 7 of Chapter 5 and we see again in the exercises at the end of this chapter, the Black–Scholes partial differential equation can be solved by first transforming it to the heat equation (with appropriate boundary conditions). This is entirely parallel to our probabilistic technique of transforming the expectation price to an expectation of a function of Brownian motion.

6.4 Asian options

The payoff of an Asian option is a function of the average of the asset price over the lifetime of the contract. For example, the payoff of an *Asian call* with strike price K and maturity time T is

$$C_T = \left(\frac{1}{T} \int_0^T S_t dt - K \right)_+ .$$

Evidently $C_T \in \mathcal{F}_T$ and so our Black–Scholes analysis of Chapter 5 gives the value of such an option at time zero as

$$V_0 = \mathbb{E}^{\mathbb{Q}} \left[e^{-rT} \left(\frac{1}{T} \int_0^T S_t dt - K \right)_+ \right]. \tag{6.6}$$

However, evaluation of this integral is a highly non-trivial matter and we do not obtain the nice explicit formulae of the previous sections.

There are many variants on this theme. For example, we might want to value a claim with payoff

$$C_T = f\left(S_T, \frac{1}{T}\int_0^T S_t dt\right).$$

In §7.2 we shall develop the technology to express the price of such claims (and indeed slightly more complex claims) as solutions to a multidimensional version of the Black–Scholes equation. Moreover (see Exercise 12 of Chapter 7) one can also find an explicit expression for the hedging portfolio in terms of the solution to this equation. However, multidimensional versions of the Black–Scholes equation are much harder to solve than their one-dimensional counterpart and generally one must resort to numerical techniques.

The main difficulty with evaluating (6.6) directly is that, although there are explicit formulae for all the moments of the average process $\frac{1}{T}\int_0^T S_t dt$, in contrast to the lognormal distribution of S_T, we do not have an expression for the distribution function. A number of approaches have been suggested to overcome this, including simply approximating the distribution of the average process by a lognormal distribution with suitably chosen parameters.

A very natural approach is to replace the continuous average by a discrete analogue obtained by sampling the price of the process at agreed times $t_1, \dots, t_n$ and averaging the result. This also makes sense from a practical point of view as calculating the continuous average for a real asset can be a difficult process. Many contracts actually specify that the average be calculated from such a discrete sample – for example from daily closing prices. Mathematically, the continuous average $\frac{1}{T}\int_0^T S_t dt$ is replaced by $\frac{1}{n}\sum_{i=1}^n S_{t_i}$. Options based on a discrete sample can be treated in the same way as multistage options, although evaluation of the price rapidly becomes impractical (see Exercise 21).

A further approximation is to replace the arithmetic average by a *geometric* average. That is, in place of $\frac{1}{n}\sum_{i=1}^n S_{t_i}$ we consider $\left(\prod_{i=1}^n S_{t_i}\right)^{1/n}$. This quantity has a lognormal distribution (Exercise 22) and so the corresponding approximate pricing formula for the Asian option can be evaluated exactly. (You are asked to find the pricing formula for an Asian call option based on a continuous version of the geometric average in Exercise 23.) Of course the arithmetic mean of a collection of positive numbers always dominates their geometric mean and so it is no surprise that this approximation consistently under-prices the Asian call option.

6.5 American options

A full treatment of American options is beyond our scope here. Explicit formulae for the prices of American options only exist in a few special cases and so one must employ numerical techniques. One approach is to use our discrete (binomial tree) models of Chapter 2. An alternative is to reformulate the price as a solution to a

partial differential equation. We do not give a rigorous derivation of this equation, but instead we use the results of Chapter 2 to give a heuristic explanation of its form.

The discrete case

As we saw in Chapter 2, the price of an American call option on non-dividend-paying stock is the same as that of a European call and so we concentrate on the *American put*. This option gives the holder the right to buy one unit of stock for price K at any time before the maturity time T.

As we illustrated in §2.2, in our discrete time model, if $V(n, S_n)$ is the value of the option at time $n\delta t$ given that the asset price at time $n\delta t$ is S_n then

$$V(n, S_n) = \max\left\{(K - S_n)_+, \mathbb{E}^{\mathbb{Q}}\left[e^{-r\delta t}V(n+1, S_{n+1})\middle|\mathcal{F}_n\right]\right\},$$

where $\mathbb{Q}$ is the martingale measure. In particular, $V(n, S_n) \geq (K - S_n)_+$ everywhere. We saw that for each fixed n the possible values of S_n are separated into two ranges by a boundary value that we shall denote by $S_f(n)$: if $S_n > S_f(n)$ then it is optimal to hold the option whereas if $S_n \leq S_f(n)$ it is optimal to exercise. We call $\{S_f(n)\}_{0\leq n\leq N}$ the *exercise boundary*.

In Example 2.4.7 we found a characterisation of the exercise boundary. We showed that the discounted option price can be written as $\tilde{V}_n = \tilde{M}_n - \tilde{A}_n$ where $\{\tilde{M}_n\}_{0\leq n\leq N}$ is a $\mathbb{Q}$-martingale and $\{\tilde{A}_n\}_{0\leq n\leq N}$ is a non-decreasing predictable process. The option is exercised at the first time $n\delta t$ when $\tilde{A}_{n+1} \neq 0$. In summary, within the exercise region $\tilde{A}_{n+1} \neq 0$ and $V_n = (K - S_n)_+$, whereas away from the exercise region, that is when $S_n > S_f(n)$, $V(n, S_n) = M_n$.

The strategy of exercising the option at the first time when $\tilde{A}_{n+1} \neq 0$ is *optimal* in the sense that if we write $\mathcal{T}_N$ for the set of all possible stopping times taking values in $\{0, 1, \ldots, N\}$ then

$$V(0, S_0) = \sup_{\tau\in\mathcal{T}_N} \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}(K - S_\tau)_+\middle|\mathcal{F}_0\right].$$

Since the exercise time of any permissible strategy must be a stopping time, this says that as holder of the option one can't do better by choosing any other exercise strategy. That this optimality characterises the fair price follows from a now familiar arbitrage argument that you are asked to provide in Exercise 24.

Continuous time

Now suppose that we formally pass to the continuous limit as in §2.6. We expect that in the limit too $V(t, S_t) \geq (K - S_t)_+$ everywhere and that for each t we can define $S_f(t)$ so that if $S_t > S_f(t)$ it is optimal to hold on to the option, whereas if $S_t \leq S_f(t)$ it is optimal to exercise. In the exercise region $V(t, S_t) = (K - S_t)_+$ whereas away from the exercise region $V(t, S_t) = M_t$ where the discounted process $\{\tilde{M}_t\}_{0\leq t\leq T}$ is a $\mathbb{Q}$-martingale and $\mathbb{Q}$ is the measure, equivalent to $\mathbb{P}$, under which the discounted stock price is a martingale. Since $\{\tilde{M}_t\}_{0\leq t\leq T}$ can be thought of as the discounted value of a European option, this tells us that away from the exercise region, $V(t, x)$ must satisfy the Black–Scholes differential equation.

We guess then that for $\{(t, x) : x > S_f(t)\}$ the price $V(t, x)$ must satisfy the Black–Scholes equation whereas outside this region $V(t, x) = (K - x)_+$. This

can be extended to a characterisation of $V(t, x)$ if we specify appropriate boundary conditions on S_f. This is complicated by the fact that $S_f(t)$ is a *free boundary* – we don't know its location a priori.

An arbitrage argument (Exercise 25) says that the price of an American put option should be continuous. We have checked already that $V(t, S_f(t)) = (K - S_f(t))_+$. Since it is clearly not optimal to exercise at a time $t < T$ if the value of the option is zero, in fact we have $V(t, S_f(t)) = K - S_f(t)$. Let us suppose now that $V(t, x)$ is continuously differentiable with respect to x as we cross the exercise boundary (we shall omit the proof of this). Then, since

$$\begin{aligned} V(t,x) &= (K-x) \quad \text{for } x \le S_f \text{ and} \\ V(t,x) &\ge (K-x) \quad \text{for } x > S_f, \end{aligned}$$

we must have that at the exercise boundary $\frac{\partial V}{\partial x} \ge -1$. Suppose that $\frac{\partial V}{\partial x} > -1$ at some point of the exercise boundary. Then by reducing the value of the stock price at which we choose to exercise from S_f to S_f^* we can actually *increase* the value of the option at $(t, S_f(t))$. This contradicts the optimality of our exercise strategy. It must be that $\frac{\partial V}{\partial x} = -1$ at the exercise boundary.

We can now fully characterise $V(t, x)$ as a solution to a free boundary value problem:

Proposition 6.5.1 (The value of an American put) *We write $V(t, x)$ for the value of an American put option with strike price K and maturity time T and r for the riskless borrowing rate. $V(t, x)$ can be characterised as follows. For each time $t \in [0, T]$ there is a number $S_f(t) \in (0, \infty)$ such that for $0 \le x \le S_f(t)$ and $0 \le t \le T$,*

$$V(t,x) = K - x \quad \textit{and} \quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} - rV < 0.$$

For $t \in [0, T]$ and $S_f(t) < x < \infty$

$$V(t,x) > (K-x)_+ \quad \textit{and} \quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} - rV = 0.$$

The boundary conditions at $x = S_f(t)$ are that the option price process is continuously differentiable with respect to x, is continuous in time and

$$V(t, S_f(t)) = (K - S_f(t))_+, \quad \frac{\partial V}{\partial x}(t, S_f(t)) = -1.$$

In addition, V satisfies the terminal condition

$$V(T, S_T) = (K - S_T)_+.$$

The free boundary problem of Proposition 6.5.1 is easier to analyse as a *linear complementarity problem*. If we use the notation

$$\mathcal{L}_{BS} f = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + rx\frac{\partial f}{\partial x} - rf,$$

then the free boundary value problem can be restated as

$$\mathcal{L}_{BS}V(t,x)\,(V(t,x)-(K-x)_+)=0,$$

subject to $\mathcal{L}_{BS}V(t,x)\le 0$, $V(t,x)-(K-x)_+\ge 0$, $V(T,x)=(K-x)_+$, $V(t,x)\to\infty$ as $x\to\infty$ and $V(t,x)$, $\frac{\partial V}{\partial x}(t,x)$ are continuous.

Notice that this reformulation has removed explicit dependence on the free boundary. Variational techniques can be applied to solve the problem and then the boundary is recovered from that solution. This is beyond our scope here. See Wilmott, Howison & Dewynne (1995) for more detail.

An explicit solution

We finish this chapter with one of the rare examples of an American option for which the price can be obtained explicitly.

Example 6.5.2 (Perpetual American put) *Find the value of a perpetual American put option on non-dividend-paying stock, that is a contract that the holder can choose to exercise at any time t in which case the payoff is $(K-S_t)_+$.*

Solution(s): We sketch *two* possible solutions to this problem, first via the free boundary problem of Proposition 6.5.1 and second via the expectation price.

Since the time to expiry of the contract is always infinite, $V(t,x)$ is a function of x alone and the exercise boundary must be of the form $S_f(t)=\alpha$ for *all* $t>0$ and some constant α. The option will be exercised as soon as $S_t\le\alpha$. The Black–Scholes equation reduces to an *ordinary* differential equation:

$$\frac{1}{2}\sigma^2x^2\frac{d^2V}{dx^2}+rx\frac{dV}{dx}-rV=0,\quad \text{for all } x\in(\alpha,\infty). \tag{6.7}$$

The general solution to equation (6.7) is of the form $v(x)=c_1x^{d_1}+c_2x^{d_2}$ for some constants c_1, c_2, d_1 and d_2. Fitting the boundary conditions

$$V(\alpha)=K-\alpha,\quad \lim_{x\downarrow\alpha}\frac{dV}{dx}=-1 \quad\text{and}\quad \lim_{x\to\infty}V(x)=0$$

gives

$$V(x)=\begin{cases}(K-\alpha)\left(\frac{\alpha}{x}\right)^{2r\sigma^{-2}}, & x\in(\alpha,\infty),\\ (K-x), & x\in[0,\alpha],\end{cases}$$

where

$$\alpha=\frac{2r\sigma^{-2}K}{2r\sigma^{-2}+1}.$$

An alternative approach to this problem would be to apply the results of §3.3. As we argued above, the option will be exercised when the stock price first hits level α for some $\alpha>0$. This means that the value will be of the form

$$V(0,S_0)=\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau_\alpha}(K-\alpha)_+\right],$$

where $\tau_\alpha = \inf\{t > 0 : S_t \leq \alpha\}$. We rewrite this stopping time in terms of the time that it takes a $\mathbb{Q}$-Brownian motion to hit a sloping line. Since

$$S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma X_t\right)$$

where $\{X_t\}_{t\geq 0}$ is a standard Brownian motion under the martingale measure $\mathbb{Q}$, the event $\{S_t \leq \alpha\}$ is the same as the event

$$\left\{-\sigma X_t - \left(r - \frac{1}{2}\sigma^2\right)t \geq \log\left(\frac{S_0}{\alpha}\right)\right\}.$$

The process $\{-X_t\}_{t\geq 0}$ is also a standard $\mathbb{Q}$-Brownian motion and so, in the notation of §3.3, the time τ_α is given by $T_{a,b}$ with

$$a = \frac{1}{\sigma}\log\left(\frac{S_0}{\alpha}\right), \quad b = \frac{r - \frac{1}{2}\sigma^2}{\sigma}.$$

We can then read off $\mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau_\alpha}\right]$ from Proposition 3.3.5 and maximise over α to yield the result. □

Exercises

1 Let K_1 and K_2 be fixed real numbers with $0 < K_1 < K_2$. A *collar option* has payoff

$$C_T = \min\{\max\{S_T, K_1\}, K_2\}.$$

Find the Black–Scholes price for such an option.

2 What is the maximum potential loss associated with taking the long position in a forward contract? And with taking the short position?
Consider the derivative whose payoff at expiry to the holder of the long position is

$$C_T = \min\{S_T, F\} - K,$$

where F is the standard forward price for the underlying stock and K is a constant. Such a contract is constructed so as to have zero value at the time at which it is struck. Find an expression for the value of K that should be written into such a contract. What is the maximum potential loss for the holder of the long or short position now?

3 The *digital put option* with strike K at time T has payoff

$$C_T = \begin{cases} 0, & S_T \geq K, \\ 1, & S_T < K. \end{cases}$$

Find the Black–Scholes price for a digital put. What is the put–call parity for digital options?

4 *Digital call option* In Example 6.1.1 we calculated the price of a digital call. Here is an alternative approach:

(a) Use the Feynman–Kac stochastic representation to find the partial differential equation satisfied by the value of a digital call with strike K and maturity T.
(b) Show that the delta of a standard European call option solves the partial differential equation that you have found in (a).
(c) Hence or otherwise solve the equation in (a) to find the value of the digital.

5 An *asset-or-nothing* call option with strike K and maturity T has payoff

$$C_T = \begin{cases} S_T, & S_T \geq K, \\ 0, & S_T < K. \end{cases}$$

Find the Black–Scholes price and hedge for such an option. What happens to the stock holding in the replicating portfolio if the asset price is near K at times close to T? Comment.

6 Construct a portfolio consisting entirely of cash-or-nothing and asset-or-nothing options whose value at time T is exactly that of a European call option with strike K at maturity T.

7 In §6.1 we have seen that for certain options with discontinuous payoffs at maturity, the stock holding in the replicating portfolio can oscillate wildly close to maturity. Do you see this phenomenon if the payoff is continuous?

8 *Pay-later option* This option, also known as a *contingent premium option*, is a standard European option except that the buyer pays the premium only at maturity of the option and then only if the option is in the money. The premium is chosen so that the value of the option at time zero is zero. This option is equivalent to a portfolio consisting of one standard European call option with strike K and maturity T and $-V$ digital call options with maturity T where V is the premium for the option.

(a) What is the value of holding such a portfolio at time zero?
(b) Find an expression for V.
(c) If a speculator enters such a contract, what does this suggest about her market view?

9 *Ratchet option* A two-leg ratchet call option can be described as follows. At time zero an initial strike price K is set. At time $T_0 > 0$ the strike is *reset* to S_{T_0}, the value of the underlying at time T_0. At the maturity time $T_1 > T_0$ the holder receives the payoff of the call with strike S_{T_0} plus $S_{T_1} - S_{T_0}$ if this is positive. That is, the payoff is $(S_{T_1} - S_{T_0})_+ + (S_{T_0} - K)_+$.
If $(S_{T_0} - K)$ is positive, then the intermediate profit $(S_{T_0} - K)_+$ is said to be 'locked in'. Why? Value this option for $0 < t < T_1$.

10 *Chooser option* A chooser option is specified by two strike prices, K_0 and K_1, and two maturity dates, $T_0 < T_1$. At time T_0 the holder has the right to buy, for price K_0, *either* a call *or* a put with strike K_1 and maturity T_1.

What is the value of the option at time T_0? In the special case $K_0 = 0$ use put–call parity to express this as the sum of the value of a call and a put with suitably chosen strike prices and maturity dates and hence find the value of the option at time zero.

11 *Options on futures* In our simple model where the riskless rate of borrowing is deterministic, forward and futures prices coincide. A European call option with strike price K and maturity T_0 written on an underlying futures contract with delivery date $T_1 > T_0$ delivers to the holder, at time T_0, a long position in the futures contract and an amount of money $(F(T_0, T_1) - K)_+$, where $F(T_0, T_1)$ is the value of the futures contract at time T_0. Find the value of such an option at time zero.

12 Use the method of Example 6.2.3 to find the value of a put-on-put option.

By considering the portfolio obtained by buying one call-on-put and selling one put-on-put (with the same strikes and maturities) obtain a put–call parity relation for compound options. Hence write down prices for all four classes of compound option.

13 Let $\{Y_t\}_{t\geq 0}$ be given by $Y_t = bt + X_t$ where b is a constant and $\{X_t\}_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion. Writing $Y^*(t) = \max\{Y_u : 0 \leq u \leq t\}$, find the joint distribution of $(Y_T, Y^*(T))$ under $\mathbb{Q}$.

14 What is the value of a portfolio consisting of one down-and-in call and one down-and-out call with the same strike price and maturity?

15 Find the value of a down-and-out call with barrier c and strike K at maturity T if $c > K$.

16 One approach to finding the value of the down-and-out call of Example 6.3.6 is to express it as an expectation under the martingale measure and exploit our knowledge of the joint distribution of Brownian motion and its minimum. Alternatively one can solve the partial differential equation directly and that is the purpose of this exercise.

(a) Use the method of Exercise 7 of Chapter 5 to transform the equation for the price into the heat equation. What are the boundary conditions for this heat equation?

(b) Solve the heat equation that you have obtained using, for example, the 'method of images'. (If you are unfamiliar with this technique, then try Wilmott, Howison & Dewynne (1995).)

(c) Undo the transformation to obtain the solution to the partial differential equation.

17 An *American cash-or-nothing call option* can be exercised at any time $t \in [0, T]$. If exercised at time t its payoff is

$$\begin{array}{ll} 1 & \text{if } S_t \geq K, \\ 0 & \text{if } S_t < K. \end{array}$$

When will such an option be exercised? Find its value.

18 Suppose that the down-and-in call option of Example 6.3.5 is modified so that if the option is never activated, that is the stock price never crosses the barrier, then the holder receives a rebate of Z. Find the price of this modified option.

19 A *perpetual option* is one with no expiry time. For example, a perpetual American cash-or-nothing call option can be exercised at any time. If exercised at time t, its payoff is 1 if $S_t \geq K$ and 0 if $S_t < K$. What is the probability that such an option is never exercised?

20 Formulate the price of a double knock-out call option as a solution to a partial differential equation with suitably chosen boundary conditions. Mimic your approach in Exercise 16 to see that this too leads to an expression for the price as an infinite sum.

21 Calculate the value of an Asian call option, with strike price K, in which the average of the stock price is calculated on the basis of just two sampling times, 0 and T, where T is the maturity time of the contract.
Find an expression for the value of the corresponding contract when there are three sampling times, 0, $T/2$ and T.

22 Suppose that $\{S_t\}_{t\geq 0}$ is a geometric Brownian motion under $\mathbb{P}$. Let $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ be fixed times and define

$$G_n = \left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n}.$$

Show that G_n has a lognormal distribution under $\mathbb{P}$.

23 An asset price $\{S_t\}_{t\geq 0}$ is a geometric Brownian motion under the market measure $\mathbb{P}$. Define

$$Y_T = \exp\left(\frac{1}{T}\int_0^T \log S_t \, dt\right).$$

Suppose that an Asian call option has payoff $(Y_T - K)_+$ at time T. Find an explicit formula for the price of such an option at time zero.

24 Use an arbitrage argument to show that if $V(0, S_0)$ is the fair price of an American put option on non-divident-paying stock with strike price K and maturity T, then writing $\mathcal{T}_T$ for the set of all possible stopping times taking values in $[0, T]$

$$V(0, S_0) = \sup_{\tau \in \mathcal{T}_T} \mathbb{E}^{\mathbb{Q}}\left[e^{-r\tau}(K - S_\tau)_+ \middle| \mathcal{F}_0\right].$$

25 Consider the value of an American put on non-dividend-paying stock. Show that if there were a discontinuity in the option value (as a function of stock price) that persisted for more than an infinitesimal time then a portfolio consisting entirely of options would offer an arbitrage opportunity.

Remark: This does not mean that *all* option prices are continuous. If there is an instantaneous change in the conditions of a contract (as in multistage options) then discontinuities certainly can occur.

26 Find the value of a perpetual American call option on non-dividend-paying stock.