5 The Black–Scholes model

Summary

We now, finally, have all the tools that we need for pricing and hedging in the continuous time world of Black and Scholes. We shall begin with the most basic setting, in which our market has just two securities: a cash bond and a risky asset whose price is modelled by a geometric Brownian motion.

In §5.1 we prove the Fundamental Theorem of Asset Pricing in this framework. In line with our analysis in the discrete world, this provides an explicit formula for the price of a derivative as the discounted expected payoff under the martingale measure. Just as in the discrete setting, we shall see that there are three steps to replication. In §5.2 we put this into action for European options. For simple calls and puts, the expectation that gives the price of the claim can be evaluated. We also obtain an explicit expression for the stock and bond holding in the replicating portfolio, via an application of the Feynman–Kac representation.

The rest of the book consists of increasing the complexity of the derivative contracts and of the market models. Before embarking on this programme, we relax the financial assumptions that we have made within the basic Black–Scholes framework. The risky asset that we have specified has a very simplistic financial side. We have assumed that it can be held without additional cost or benefit and that it can be freely traded at the quoted price. Even leaving aside the issues of transaction costs and illiquidity, not much of the financial market is like that. Foreign exchange involves two assets that pay interest, equities pay dividends and bonds pay coupons. In §5.3–§5.5 we see how to apply the Black–Scholes technology in these more sophisticated financial settings. Finally, in §5.6, we characterise tradable assets within a given market and we define the market price of risk.

5.1 The basic Black–Scholes model

In this section we provide a rigorous derivation of the Black–Scholes pricing formula obtained in §2.6. As in Chapter 2, our market consists of just two securities. The first is our old friend the cash bond, $\{B_t\}_{t\geq 0}$. We retain (for now) our assumption that

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the risk-free interest rate is constant, so that if $B_0 = 1$ then $B_t = e^{rt}$. The second security in our market is a risky asset whose price at time t we denote by S_t. In this, our basic reference model, we suppose that $\{S_t\}_{t\geq 0}$ is a geometric Brownian motion, that is it solves

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

for some constants μ and σ, where $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion. Notice that this corresponds to taking $\nu = \mu - \frac{1}{2}\sigma^2$ in our calculations of §2.6. We call $\mathbb{P}$ the *market measure*. Exactly as in the discrete world, the market measure tells us which market events have positive probability, but we shall reweight the probabilities for the purposes of pricing and hedging.

Self-financing strategies

As in the discrete world, our starting point is that the market does not admit any arbitrage opportunities. Our strategy also parallels the work of Chapter 2: to obtain the time zero price of a claim, C_T, against us at time T, we seek a self-financing portfolio whose value at time T is exactly C_T. In the absence of arbitrage, the value of the claim must be the same as the cost of constructing the replicating portfolio. Of course for this argument to work, the trading strategy for this portfolio must be *previsible*. Moreover, because we are now allowed to rebalance the portfolio as often as we like, rather than just at the 'ticks' of a clock, to avoid obvious arbitrage opportunities we must introduce a further restriction on admissible trading strategies for our portfolio. We illustrate with an example.

Example 5.1.1 (The doubling strategy) *Consider the following strategy for betting on successive (independent) flips of a coin that comes up heads with probability $p > 0$. We bet \$$K$ that the first flip comes up heads. If it* does *come up heads then we stop, having won \$$K$. If it* does not *come up heads, then we bet \$$2K$ that the second flip comes up heads. If it does, then our net gain is \$$K$ and we stop. Otherwise we have lost \$$3K$ and we bet \$$4K$ that the next flip is a head. And so on. If the first $n-1$ flips all come up tails, then we have lost \$$\sum_{j=0}^{n-1} 2^j K$ = \$$(2^n - 1)K$ and we bet \$$2^n K$ on the nth flip. Since with probability one the coin will* eventually *come up heads, we are guaranteed to win \$$K$. Of course, this relies on our having infinite credit. If we only have limited funds, then the apparent arbitrage opportunity disappears.*

With this example in mind, we make the following definition.

Definition 5.1.2 *A* self-financing strategy *is defined by a pair of predictable processes $\{\psi_t\}_{0\leq t\leq T}$, $\{\phi_t\}_{0\leq t\leq T}$, denoting the quantities of riskless and risky asset respectively held in the portfolio at time t, satisfying*

$$\int_0^T |\psi_t|\, dt + \int_0^T |\phi_t|^2\, dt < \infty$$

(with probability one), and

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$$\psi_t B_t + \phi_t S_t = \psi_0 B_0 + \phi_0 S_0 + \int_0^t \psi_u dB_u + \int_0^t \phi_u dS_u$$

(with probability one) for all $t \in [0, T]$.

Remarks: Condition 1 ensures that the integrals in condition 2 make sense. Moreover, $\int_0^t \phi_u dW_u$ will be a $\mathbb{P}$-martingale.

In differential form, condition 2 says that the value, $V_t(\psi, \phi) = \psi_t B_t + \phi_t S_t$, of the portfolio at time t satisfies

$$dV_t(\psi, \phi) = \psi_t dB_t + \phi_t dS_t,$$

that is, changes of value of the portfolio over an infinitesimal time interval are due entirely to changes in value of the assets and not to injection (or removal) of wealth from outside. □

As in the discrete setting, the key will be to work with a probability measure, $\mathbb{Q}$, equivalent to the 'market measure' $\mathbb{P}$ and under which the discounted stock price, $\{\tilde{S}_t\}_{t\geq 0}$, is a $\mathbb{Q}$-martingale. This means that it is convenient to think of the *discounted* asset price process as the object of central interest. With this in mind we prove the following continuous analogue of equation (2.5).

Lemma 5.1.3 *Let* $\{\psi_t\}_{0\leq t\leq T}$ *and* $\{\phi_t\}_{t\geq 0}$ *be predictable processes satisfying*

$$\int_0^T |\psi_t|\, dt + \int_0^T |\phi_t|^2\, dt < \infty$$

(with probability one). Set

$$V_t(\psi, \phi) = \psi_t B_t + \phi_t S_t, \qquad \tilde{V}_t(\psi, \phi) = e^{-rt} V_t(\psi, \phi).$$

Then $\{\psi_t, \phi_t\}_{0\leq t\leq T}$ *defines a self-financing strategy if and only if*

$$\tilde{V}_t(\psi, \phi) = \tilde{V}_0(\psi, \phi) + \int_0^t \phi_u d\tilde{S}_u$$

with probability one for all $t \in [0, T]$.

Proof: Suppose first that the portfolio $\{\psi_t, \phi_t\}_{0\leq t\leq T}$ is self-financing. Then

$$\begin{aligned}
d\tilde{V}_t(\psi, \phi) &= -re^{-rt} V_t(\psi, \phi)dt + e^{-rt} dV_t(\psi, \phi) \\
&= -re^{-rt}\left(\psi_t e^{rt} + \phi_t S_t\right) dt + e^{-rt}\psi_t d(e^{rt}) + e^{-rt}\phi_t dS_t \\
&= \phi_t \left(-re^{-rt} S_t dt + e^{-rt} dS_t\right) \\
&= \phi_t d\tilde{S}_t
\end{aligned}$$

as required.

The other direction is similar and is left as an exercise. □

A strategy for pricing

Before going further, we outline our strategy. We write C_T for the claim at time T that we are trying to replicate. It may depend on $\{S_t\}_{0\le t\le T}$ in more complex ways than just through S_T. Suppose that *somehow* we can find a predictable process $\{\phi_t\}_{0\le t\le T}$ such that the claim C_T, discounted, satisfies

$$\tilde{C_T} \triangleq e^{-rT}C_T = \phi_0 + \int_0^T \phi_u d\tilde{S}_u.$$

Then we can replicate the claim by a portfolio in which we hold ϕ_t units of stock and ψ_t units of cash bond at time t, where ψ_t is chosen so that

$$\tilde{V}_t(\psi,\phi) = \phi_t\tilde{S}_t + \psi_t e^{-rt} = \phi_0 + \int_0^t \phi_u d\tilde{S}_u.$$

By Lemma 5.1.3, the portfolio is then self-financing, and, moreover, $V_T = C_T$. The fair price of the claim at time zero is then $V_0 = \phi_0$.

This is fine if we know ϕ_0, but there is a quick and easy way to find the right price without explicitly finding the strategy $\{\psi_t,\phi_t\}_{t\ge0}$. Suppose instead that we can find a probability measure, $\mathbb{Q}$, under which the discounted stock price is a martingale. Then, at least provided $\int_0^T \phi_u^2 du < \infty$,

$$\int_0^t \phi_u d\tilde{S}_u$$

will be a mean zero $\mathbb{Q}$-martingale and so

$$\mathbb{E}^{\mathbb{Q}}\big[\tilde{V}_T(\psi,\phi)\big] = \phi_0 + \mathbb{E}^{\mathbb{Q}}\left[\int_0^t \phi_u d\tilde{S}_u\right] = \phi_0.$$

So $\phi_0 = \mathbb{E}^{\mathbb{Q}}\big[\tilde{C_T}\big]$ is the fair price.

This then is entirely analogous to the pricing formula of Theorem 2.3.13. If there is a probability measure, $\mathbb{Q}$, equivalent to $\mathbb{P}$ and under which the discounted stock price is a martingale, then, provided a replicating portfolio exists, the fair time zero price of the claim is $\mathbb{E}^{\mathbb{Q}}[\tilde{C_T}]$, the discounted expected value of the claim under this measure.

We have assumed that the process $\{\phi_t\}_{t\ge0}$ exists. We prove this (for this basic Black–Scholes market model) in Theorem 5.1.5 via an application of the Martingale Representation Theorem. First, if our pricing formula is to be of any use, we should find the *equivalent martingale measure* $\mathbb{Q}$.

An equivalent martingale measure

Lemma 5.1.4 (A probability measure under which $\{\tilde{S}_t\}_{t\ge0}$ is a martingale) *There is a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, under which the discounted stock price $\{\tilde{S}_t\}_{t\ge0}$ is a martingale. Moreover, the Radon–Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ is given by*

$$L_t^{(\theta)} \triangleq \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\theta W_t - \frac{1}{2}\theta^2 t\right),$$

where $\theta = (\mu - r)/\sigma$.

Proof: Recall that

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

so

$$d\tilde{S}_t = \tilde{S}_t\left(-rdt + \mu dt + \sigma dW_t\right).$$

Consequently, if we set $X_t = W_t + (\mu - r)t/\sigma$,

$$d\tilde{S}_t = \tilde{S}_t \sigma dX_t.$$

Now from Theorem 4.5.1, $\{X_t\}_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion and so $\{\tilde{S}_t\}_{t\geq 0}$ is a $\mathbb{Q}$ martingale. Moreover,

$$\tilde{S}_t = \tilde{S}_0 \exp\left(\sigma X_t - \sigma^2 t/2\right).$$

□

The Fundamental Theorem of Asset Pricing

We can now prove the Fundamental Theorem of Asset Pricing in the Black–Schole framework.

Theorem 5.1.5 *Let $\mathbb{Q}$ be the measure given by Lemma 5.1.4. Suppose that a clair at time T is given by the non-negative random variable $C_T \in \mathcal{F}_T$. If*

$$\mathbb{E}^{\mathbb{Q}}[C_T^2] < \infty,$$

then the claim is replicable and the value at time t of any replicating portfolio i given by

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}C_T \middle| \mathcal{F}_t\right].$$

In particular, the fair price at time zero for the option is

$$V_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rT}C_T] = \mathbb{E}^{\mathbb{Q}}[\tilde{C}_T].$$

Proof: In the argument that followed Lemma 5.1.3 we showed that if we could fin a process $\{\phi_t\}_{0\leq t\leq T}$ such that

$$\tilde{C}_T = \phi_0 + \int_0^T \phi_u d\tilde{S}_u,$$

then we could construct a replicating portfolio whose value at time t satisfies

$$\tilde{V}_t(\psi, \phi) = \phi_0 + \int_0^t \phi_u d\tilde{S}_u, \tag{5.1}$$

which, by the martingale property of the stochastic integral is precisely

$$\tilde{V}_t(\psi, \phi) = \mathbb{E}^{\mathbb{Q}}\left[\phi_0 + \int_0^T \phi_u d\tilde{S}_u \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\tilde{C}_T \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}C_T \middle| \mathcal{F}_t\right].$$

Undoing the discounting on $[0, t]$ gives

$$V_t(\psi, \phi) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} C_T \middle| \mathcal{F}_t \right].$$

Let's just reassure ourselves that such a price is unique. Evidently any other replicating portfolio, $\{\hat{\psi}_t, \hat{\phi}_t\}_{0\leq t\leq T}$, has $V_T(\hat{\psi}, \hat{\phi}) = C_T$ and if it is self-financing (by Lemma 5.1.3) it satisfies an equation of the form (5.1). Repeating the argument above we see that we obtain the same value for any self-financing replicating portfolio.

The proof of the theorem will be complete if we can show that there *is* a predictable process $\{\phi_t\}_{0\leq t\leq T}$ such that

$$\tilde{C}_T = \phi_0 + \int_0^T \phi_u d\tilde{S}_u.$$

Now, by Exercise 2,

$$M_t \triangleq \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} C_T \middle| \mathcal{F}_t \right]$$

is a square-integrable $\mathbb{Q}$-martingale. The natural filtration of our original Brownian motion is the same as that for the process $\{X_t\}_{t\geq 0}$ defined in Lemma 5.1.4. That is, $\{M_t\}_{t\geq 0}$ is a square-integrable 'Brownian martingale' and by the Brownian Martingale Representation Theorem 4.6.2 there exists an $\{\mathcal{F}_t\}_{0\leq t\leq T}$-predictable process $\{\theta_t\}_{0\leq t\leq T}$ such that

$$M_t = M_0 + \int_0^t \theta_s dX_s.$$

Since $d\tilde{S}_s = \sigma \tilde{S}_s dX_s$, we set

$$\phi_t = \frac{\theta_t}{\sigma \tilde{S}_t} \quad \text{and} \quad \psi_t = M_t - \phi_t \tilde{S}_t.$$

Condition 1 of Definition 5.1.2 is easily seen to be satisfied and so the strategy corresponding to $\{\psi_t, \phi_t\}_{0\leq t\leq T}$ defines a self-financing replicating portfolio as required. □

Remark: The theorem that we have just proved is *very* general. Subject to our mild boundedness condition, the claim C_T could be almost arbitrarily complex provided it depends only on the path of the stock price up to time T. The price of the claim at time zero is $\mathbb{E}^{\mathbb{Q}}\left[e^{-rT} C_T\right]$ and this can be evaluated, at least numerically, even for complex claims C_T.

We have proved that not only does there exist a fair price, but moreover, we *can hedge* the claim. Its shortcoming is that although we have asserted the existence of a hedging strategy, we have not obtained an explicit expression for it. We shall find such an expression for European options, that is claims that depend only on the stock price *at maturity*, in the next section. □

Just as in the discrete world, we have identified a procedure for valuing and replicating a claim.

Three steps to replication:

1 Find a measure $\mathbb{Q}$ under which the discounted asset price $\{\tilde{S}_t\}_{t\geq 0}$ is a martingale.

2 Form the process $M_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}C_T \middle| \mathcal{F}_t\right]$.

3 Find a predictable process $\{\phi_t\}_{t\geq 0}$ such that $dM_t = \phi_t d\tilde{S}_t$.

5.2 Black–Scholes price and hedge for European options

In the case of European options, that is options whose payoff depends only on the price of the underlying at the time of maturity, both the price of the option and the hedging portfolio can be obtained explicitly.

First we evaluate the price of the claim. Our assumptions are exactly those of §5.1.

Proposition 5.2.1 *The value at time t of a European option whose payoff at maturity is $C_T = f(S_T)$ is $V_t = F(t, S_t)$, where*

$$\begin{aligned} F(t,x) &= e^{-r(T-t)}\int_{-\infty}^{\infty} f\left(x\exp\left((r-\sigma^2/2)(T-t)+\sigma y\sqrt{T-t}\right)\right) \\ &\qquad \times \frac{\exp(-y^2/2)}{\sqrt{2\pi}}dy. \end{aligned}$$

Proof: From Theorem 5.1.5 we know that the value at time t is

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}f(S_T)\middle| \mathcal{F}_t\right], \tag{5.2}$$

where $\mathbb{Q}$ is the martingale measure obtained in Lemma 5.1.4. Under this measure $X_t = W_t + (\mu - r)t/\sigma$ is a Brownian motion and

$$d\tilde{S}_t = \sigma \tilde{S}_t dX_t.$$

Solving this equation,

$$\tilde{S}_T = \tilde{S}_t \exp\left(\sigma(X_T - X_t) - \frac{1}{2}\sigma^2(T-t)\right).$$

We can now substitute into (5.2) to obtain

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}f\left(S_t e^{r(T-t)}\exp\left(\sigma(X_T-X_t)-\frac{1}{2}\sigma^2(T-t)\right)\right)\middle| \mathcal{F}_t\right].$$

Since under $\mathbb{Q}$, conditional on $\mathcal{F}_t$, $X_T - X_t$ is a normally distributed random variable

with mean zero and variance $(T-t)$, we can evaluate this as

$$\begin{aligned} V_t &= F(t, S_t) \\ &= \int_{-\infty}^{\infty} e^{-r(T-t)} f\left(S_t e^{r(T-t)} \exp\left(\sigma z - \frac{1}{2}\sigma^2(T-t)\right)\right) \\ &\qquad \times \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{z^2}{2(T-t)}\right) dz \\ &= e^{-r(T-t)} \int_{-\infty}^{\infty} f\left(S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma y \sqrt{T-t}\right)\right) \\ &\qquad \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy, \end{aligned}$$

as required. □

Pricing calls and puts

For European calls and puts, the function F of Proposition 5.2.1 can be calculated explicitly.

Example 5.2.2 (European call) *In the notation of Proposition 5.2.1, suppose that $f(S_T) = (S_T - K)_+$. Then, writing $\theta = (T-t)$,*

$$F(t, x) = x\Phi(d_1) - Ke^{-r\theta}\Phi(d_2), \tag{5.3}$$

where $\Phi(\cdot)$ is the standard normal distribution function, given by

$$\begin{aligned} \Phi(y) &= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy, \\ d_1 &= \frac{\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} \end{aligned}$$

and $d_2 = d_1 - \sigma\sqrt{\theta}$.

Proof: Substituting for f and x in the last line of the proof of Proposition 5.2.1 we have that

$$F(t, x) = \mathbb{E}\left[\left(xe^{\sigma\sqrt{\theta}Z - \sigma^2\theta/2} - Ke^{-r\theta}\right)_+\right], \tag{5.4}$$

where $Z \sim N(0, 1)$. First we establish for what range of values of Z the integrand is non-zero. Rearranging,

$$xe^{\sigma\sqrt{\theta}Z - \sigma^2\theta/2} > Ke^{-r\theta}$$

is equivalent to

$$Z > \frac{\log\left(\frac{K}{x}\right) + \frac{\sigma^2}{2}\theta - r\theta}{\sigma\sqrt{\theta}}.$$

Thus the integrand in (5.4) is non-zero if $Z + d_2 \geq 0$. Using this notation

$$\begin{aligned} F(t,x) &= \mathbb{E}\left[\left(xe^{\sigma\sqrt{\theta}Z-\sigma^2\theta/2} - Ke^{-r\theta}\right)\mathbf{1}_{Z+d_2\geq 0}\right] \\ &= \int_{-d_2}^{\infty}\left(xe^{\sigma\sqrt{\theta}y-\sigma^2\theta/2} - Ke^{-r\theta}\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy \\ &= \int_{-\infty}^{d_2}\left(xe^{-\sigma\sqrt{\theta}y-\sigma^2\theta/2} - Ke^{-r\theta}\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy \\ &= x\int_{-\infty}^{d_2} e^{-\sigma\sqrt{\theta}y-\sigma^2\theta/2}\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy - Ke^{-r\theta}\Phi(d_2). \end{aligned}$$

Substituting $z = y + \sigma\sqrt{\theta}$ in the first integral in the last line we finally obtain

$$F(t,x) = x\Phi(d_1) - Ke^{-r\theta}\Phi(d_2).$$

□

Equation (5.3) is known as the *Black–Scholes pricing formula* for a European call option. The corresponding formula for a European *put* option can be found in Exercise 3.

Remarks:

1 Bachelier actually obtained a formula that looks very like this for the price of a European call option, except that the geometric Brownian motion is replaced by Brownian motion. This, however, was a fluke. Bachelier was using expectation pricing and did not have the notion of dynamic hedging.

2 Notice that the pricing formula depends on just one unknown parameter, σ, called the *volatility* by practitioners. The same will be true of our hedging portfolio. The problem that then arises is how to estimate σ from market data. The commonest approach is to use the *implied volatility*. Some options are quoted on organised markets. The price of European call and put options is an increasing function of volatility and so we can invert the Black–Scholes formula and associate an implied volatility with each option. Unfortunately, the estimate of σ obtained in this way usually depends on strike price and time to maturity. We briefly discuss the implications of this in §7.4. □

Hedging calls and puts

We now turn to the problem of *hedging* European options. That is, how should we construct a portfolio that replicates the claim against us?

The Martingale Representation Theorem tells us that since the discounted option price and the discounted stock price are martingales under the same measure, one is locally just a scaled version of the other. It is this local scaling that we should like an expression for. In our discrete world of §2.5 we found ϕ_{i+1} as the ratio of the change in value of the option to that of the stock over the $(i+1)$st tick of the clock. It is reasonable to guess then that in the continuous world ϕ_t should be the partial

derivative of the option value with respect to the stock price and this is what we now prove.

Proposition 5.2.3 *In the notation of Proposition 5.2.1, the process $\{\phi_t\}_{0\le t\le T}$ that determines the stock holding in the replicating portfolio of Theorem 5.1.5 is given by*

$$\phi_t = \frac{\partial F}{\partial x}(t,x)\bigg|_{x=S_t}.$$

Proof: In this notation, the result of Theorem 5.1.5 becomes

$$F(t,x) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} f(S_T)\Big|\, S_t = x\right]$$

where, under $\mathbb{Q}$,

$$d\tilde{S}_t = \sigma \tilde{S}_t dX_t$$

and $\{X_t\}_{0\le t\le T}$ is a Brownian motion. Evidently

$$dS_t = rS_t dt + \sigma S_t dX_t.$$

Combining the Feynman–Kac representation and the usual product rule of differentiation, $F(t,x)$ satisfies

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) + rx\frac{\partial F}{\partial x}(t,x) = 0, \qquad 0 \le t \le T.$$

This is the *Black–Scholes equation*. You are asked to verify that $F(t,x)$ satisfies this equation via a different route in Exercise 4.

Define the function $\tilde{F}(t,x) = e^{-rt}F(t, xe^{rt})$, then $\tilde{V}_t = \tilde{F}(t, \tilde{S}_t)$. Observing that

$$\frac{\partial \tilde{F}}{\partial t}(t,x) = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \tilde{F}}{\partial x^2}(t,x)$$

and applying Itô's formula to $\tilde{F}(u, \tilde{S}_u)$ for $0 \le u \le T$,

$$\begin{aligned} \tilde{F}(T, \tilde{S}_T) &= F(0, S_0) + \int_0^T \sigma \tilde{S}_s \frac{\partial \tilde{F}}{\partial x}(s, \tilde{S}_s) dX_s \\ &= F(0, S_0) + \int_0^T \frac{\partial \tilde{F}}{\partial x}(s, \tilde{S}_s) d\tilde{S}_s. \end{aligned}$$

This gives

$$\phi_t = \frac{\partial \tilde{F}}{\partial x}(t, \tilde{S}_t) = \frac{\partial F}{\partial x}(t, S_t)$$

as required. □

Example 5.2.4 (Hedging a European call) *Using the notation of Example 5.2.2, for a European call option we obtain*

$$\frac{\partial F}{\partial x}(t,x) = \Phi(d_1).$$

Proof: Using the same notation as in Example 5.2.2 we have

$$F(t,x) = \mathbb{E}\left[\left(x \exp\left(\sigma\sqrt{\theta}Z - \sigma^2\theta/2\right) - K\right)_+\right],$$

where $Z \sim N(0,1)$ and $\theta = (T-t)$. Differentiating the integrand with respect to x gives $\exp\left(\sigma\sqrt{\theta}Z - \sigma^2/2\right)$ if the integrand is strictly positive and zero otherwise. Then, again using the notation of Example 5.2.2,

$$\begin{aligned}\frac{\partial F}{\partial x}(t,x) &= \mathbb{E}\left[\exp\left(\sigma\sqrt{\theta}Z - \sigma^2\theta/2\right)\mathbf{1}_{Z+d_2\geq 0}\right]\\ &= \int_{-d_2}^{\infty}\exp\left(\sigma\sqrt{\theta}y - \sigma^2\theta/2 - y^2/2\right)\frac{1}{\sqrt{2\pi}}dy.\end{aligned}$$

Substituting first $u = -y$ and then $z = u + \sigma\sqrt{\theta}$ as before this reduces to $\Phi(d_1)$. So

$$\frac{\partial F}{\partial x}(t,x) = \Phi(d_1).$$

For the European put one calculates

$$\frac{\partial F}{\partial x}(t,x) = -\Phi(-d_1).$$

□

Remark: (The Greeks) The quantity $\partial F/\partial x$ is often called the *delta* of the option by practitioners. For a portfolio π of assets and derivatives, the sensitivities of the price to the parameters of the market are determined by the *Greeks*. If we write $\pi(t,x)$ for the value of the portfolio if the asset price at time t is x, then in addition to the delta, given by $\frac{\partial \pi}{\partial x}$, we have the gamma, the theta and the vega:

$$\Gamma = \frac{\partial^2\pi}{\partial x^2}, \quad \Theta = \frac{\partial \pi}{\partial t}, \quad \mathcal{V} = \frac{\partial \pi}{\partial \sigma}.$$

□

5.3 Foreign exchange

In this section we begin our programme of increasing the financial sophistication in our models by looking at the foreign exchange market. Holding currency is a risky business, and with this risk comes a demand for derivatives. To operate in this market we should like to be able to value claims based on the future value of one unit of currency in terms of another.

The pricing problem for an exchange rate forward was solved in Exercise 13 of Chapter 1. In contrast to the pricing problem for a forward contract based on an underlying stock that pays no dividends, which we solved in §1.2, for an exchange rate forward we needed to take into account interest rates in *both* currencies. Similarly, in valuing a European call option based on the exchange rate between

Sterling and US dollars in Example 1.6.6 we needed a model for both the Sterling and the US dollar cash bond. Our Black–Scholes model for foreign exchange markets too must incorporate cash bonds in both currencies. For definiteness, we suppose that the two currencies are US dollars and pounds Sterling.

> **Black–Scholes currency model:** We write $\{B_t\}_{t\geq 0}$ for the dollar cash bond and $\{D_t\}_{t\geq 0}$ for its Sterling counterpart. Writing E_t for the dollar worth of one pound at time t, our model is
>
> $$\begin{array}{rl} \text{Dollar bond} & B_t = e^{rt}, \\ \text{Sterling bond} & D_t = e^{ut}, \\ \text{Exchange rate} & E_t = E_0 \exp\left(\nu t + \sigma W_t\right), \end{array}$$
>
> where $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion and r, u, ν and σ are constants.

We now encounter exactly the problem that we had to overcome in the discrete world: the exchange rate is *not tradable*. We must confine ourselves to operating within a single market. Let us work first from the point of view of the dollar investor. In the dollar markets, neither the Sterling cash bond nor the exchange rate is tradable. However, the product of the two, $S_t = E_t D_t$, can be thought of as a dollar tradable. The dollar investor can hold Sterling cash bonds and their dollar value is precisely S_t at time t. Moreover, any claim based on E_T can be thought of as a claim based on S_T.

We now have a set-up that precisely mirrors the basic Black–Scholes model of §5.1. From the point of view of the dollar trader there are really two processes, the dollar cash bond, $\{B_t\}_{t\geq 0}$, and the dollar value of the Sterling cash bond, $\{S_t\}_{t\geq 0}$. We can now apply the Black–Scholes methodology in this setting. Let C_T denote the claim value (in dollars) at time T.

> **Three steps to replication (foreign exchange):**
> 1 Find a measure $\mathbb{Q}$ under which the (dollar bond) discounted process $\{\tilde{S}_t = B_t^{-1} S_t\}_{t\geq 0}$ is a martingale.
> 2 Form the process $M_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} C_T \middle| \mathcal{F}_t\right]$.
> 3 Find an adapted process $\{\phi_t\}_{0\leq t\leq T}$ such that $dM_t = \phi_t d\tilde{S}_t$.

Since $S_t = E_t D_t = \exp\left((\nu + u)\, t + \sigma W_t\right)$, the process $\{S_t\}_{t\geq 0}$ is just a geometric Brownian motion and so our work of §5.1 ensures that we can indeed follow these steps.

First we apply Itô's formula to obtain the stochastic differential equation satisfied

by $\{S_t\}_{t\geq 0}$:

$$dS_t = \left(\nu + u + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t.$$

Now applying Lemma 5.1.4, we find that the Radon–Nikodym derivative with respect to $\mathbb{P}$ of the measure $\mathbb{Q}$ under which the dollar-discounted process $\{\tilde{S}_t\}_{t\geq 0}$ is a martingale is

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t^{(\theta)} \triangleq \exp\left(-\theta W_t - \frac{1}{2}\theta^2 t\right),$$

where $\theta = \left(\nu + u + \frac{1}{2}\sigma^2 - r\right)/\sigma$. Moreover,

$$X_t \triangleq W_t + \frac{\left(\nu + u + \frac{1}{2}\sigma^2 - r\right)}{\sigma} t$$

is a $\mathbb{Q}$-Brownian motion.

We follow the rest of the procedure in a special case (see also Exercise 11).

Example 5.3.1 (Forward contract) *At what price should we agree to trade Sterling at a future date T?*

Solution: Of course we already solved this problem in Exercise 13 of Chapter 1, but now rather than guessing the hedging portfolio, we follow our three steps to replication.

We have already found the measure $\mathbb{Q}$. Now if we agree to buy a unit of Sterling for K dollars at time T, then the payoff of the contract will be

$$C_T = E_T - K.$$

The value of the contract at time t is then

$$\begin{aligned} V_t &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} C_T \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} (E_T - K) \middle| \mathcal{F}_t\right]. \end{aligned}$$

A forward contract costs nothing at time zero and so we must choose K so that $V_0 = 0$. In other words, $K = \mathbb{E}^{\mathbb{Q}}[E_T]$. Expressing E_T as a function of X_T gives

$$E_T = E_0 \exp\left(\sigma X_T - \frac{1}{2}\sigma^2 T + (r-u)T\right),$$

and so using that $\{X_t\}_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion gives the fair value of K as

$$K = \mathbb{E}^{\mathbb{Q}}[E_T] = e^{(r-u)T} E_0.$$

Finally we find the hedging portfolio. With this choice of strike price, the value of the contract at time t is

$$V_t = \mathbb{E}\left[e^{-r(T-t)} \left(E_T - E_0 e^{(r-u)T}\right) \middle| \mathcal{F}_t\right].$$

Since $E_T = D_T^{-1} B_T \tilde{S}_T$, under $\mathbb{Q}$ we have

$$\mathbb{E}[E_T | \mathcal{F}_t] = e^{(r-u)(T-t)} D_t^{-1} B_t \mathbb{E}\left[\tilde{S}_T \middle| \mathcal{F}_t\right] = e^{(r-u)(T-t)} D_t^{-1} B_t \tilde{S}_t = e^{(r-u)(T-t)} E_t,$$

so, substituting,

$$V_t = e^{-u(T-t)} E_t - e^{rt-uT} E_0 = e^{-uT}\left(e^{ut} E_t - e^{rt} E_0\right).$$

The dollar-discounted portfolio value is

$$M_t = e^{-rt} V_t = e^{-uT} e^{-(r-u)t} E_t - e^{-uT} E_0 = e^{-uT} \tilde{S}_t - e^{-uT} E_0.$$

The required hedging portfolio is constant, consisting of $\phi_t = e^{-uT}$ Sterling bonds and $\psi_t = -e^{-uT} E_0$ (dollar) cash bonds. □

The Sterling investor

We now turn to the Sterling investor. From her point of view, tradables are quoted in pounds Sterling. Once again, in effect there are two Sterling tradables. The first is the Sterling cash bond. The second is the Sterling value of the dollar bond given by $Z_t = E_t^{-1} B_t$.

Once again we can follow our three-step replication programme. The Sterling-discounted value of the dollar bond is

$$\tilde{Z}_t = D_t^{-1} E_t^{-1} B_t = E_0^{-1} \exp\left(-\sigma W_t - (\nu + u - r)t\right).$$

We use Lemma 5.1.4 to see that under the measure $\mathbb{Q}^{£}$ given by

$$\left.\frac{d\mathbb{Q}^{£}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t^{(\lambda)} \triangleq \exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right),$$

with $\lambda = \left(\nu + u - r - \sigma^2/2\right)/\sigma$, $\{\tilde{Z}_t\}_{t\geq 0}$ is a martingale and

$$X'_t = W_t + \frac{\left(\nu + u - r - \sigma^2/2\right)}{\sigma} t$$

is a $\mathbb{Q}^{£}$-Brownian motion. For the Sterling investor then the option price is

$$U_t = D_t \mathbb{E}^{\mathbb{Q}^{£}}\left[D_T^{-1} E_T^{-1} C_T \middle| \mathcal{F}_t\right].$$

Change of numeraire

We now have the same worry that we encountered in Exercise 15 of Chapter 1. The risk-neutral measures $\mathbb{Q}$ and $\mathbb{Q}^{£}$ can be thought of as defining a probability measure on the paths followed by $\{E_t\}_{t\geq 0}$ – the only truly random part of our model – and the two measures are *different*. So do they give the same price?

To put our minds at rest we find the dollar worth of the Sterling investor's valuation, that is

$$E_t U_t = E_t D_t \mathbb{E}^{\mathbb{Q}^{£}}\left[D_T^{-1} E_T^{-1} C_T \middle| \mathcal{F}_t\right].$$

To compare this with our expression for V_t we express the expectation as a $\mathbb{Q}$-expectation, again using Girsanov's Theorem. Now

$$X'_t = W_t + \frac{\left(\nu + u - r - \frac{1}{2}\sigma^2\right)}{\sigma} t = X_t - \sigma t$$

and so

$$\zeta_t \triangleq \left.\frac{d\mathbb{Q}^{\pounds}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \exp\left(\sigma X_t - \frac{1}{2}\sigma^2 t\right).$$

Now

$$E_t = E_0 \exp\left(\sigma X_t - \frac{1}{2}\sigma^2 t + (r-u)t\right),$$

so we can write $\zeta_t = B_t^{-1} D_t E_t$. Substituting gives

$$\begin{aligned} E_t U_t &= E_t D_t \mathbb{E}^{\mathbb{Q}^{\pounds}}\left[D_T^{-1} E_T^{-1} C_T \middle| \mathcal{F}_t\right] \\ &= E_t D_t \zeta_t^{-1} \mathbb{E}^{\mathbb{Q}}\left[D_T^{-1} E_T^{-1} \zeta_T C_T \middle| \mathcal{F}_t\right] \\ &= B_t \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} C_T \middle| \mathcal{F}_t\right]. \end{aligned}$$

In other words the dollar value obtained by the Sterling investor is precisely V_t. The difference in the measures is merely an artefact of the different choice of 'reference asset' or *numeraire*.

5.4 Dividends

Our assumption so far has been that there is no value in simply holding a stock. We should now like to relax that assumption to allow the pricing and hedging of options based on equities – stocks that make periodic cash payments.

Continuous payments

It is simplest to begin with a dividend that is paid continuously. Assume as before that the stock price follows a geometric Brownian motion given by

$$S_t = S_0 \exp(\nu t + \sigma W_t),$$

but now in the infinitesimal time interval $[t, t+dt)$ the holder of the stock receives a dividend payment of $\delta S_t dt$ where δ is a constant. As always, we also assume that the market contains a riskless cash bond, $\{B_t\}_{t\geq 0}$, and we denote the continuously compounded interest rate by r.

The difficulty that we now face is that $\{S_t\}_{t\geq 0}$ does not represent the true worth of the asset: if we buy stock for price S_0 at time zero, when we sell it at time t, the value of having held it is not just $S_t - S_0$ but also the total accumulated dividends. In this model these depend on all the values that are taken by the asset price in the time interval $[0, t]$. In this sense, $\{S_t\}_{t\geq 0}$ is not *tradable*.

Our solution, just as for foreign exchange, is to translate the process into something that *is* tradable. The simplest solution is as follows. Suppose that whenever a cash dividend is paid, we immediately reinvest it in stock. The infinitesimal payout $\delta S_t dt$ will buy δdt units of stock. At time t, rather than holding one unit of stock, we hold $e^{\delta t}$ units with total worth

$$Z_t = S_0 \exp((\nu + \delta)t + \sigma W_t).$$

We regard the simple portfolio obtained by holding stock and continuously reinvesting the dividends in this way as a single asset with value Z_t at time t. There is no cost in holding this asset and it makes no dividend payments. We are back in familiar Black–Scholes country.

Remark: Because the dividend payments were a constant proportion of the stock price, it was natural to reinvest them in stock. Had payments been fixed amounts of cash, it would have been more natural to construct our 'tradable' asset as a portfolio in which dividends were immediately reinvested in bonds. This will be the situation of §5.5. □

Any portfolio consisting of $\phi_t e^{\delta t}$ units of our original dividend-paying stock and ψ_t cash bonds at time t can be thought of as a portfolio of ϕ_t units of our new tradable asset and ψ_t units of cash bond.

We can now follow our familiar procedure.

Three steps to replication (continuous dividends): Let $\tilde{Z}_t = B_t^{-1} Z_t = e^{-rt} Z_t$.

1. Find a probability measure $\mathbb{Q}$ under which $\{\tilde{Z}_t\}_{t\geq 0}$ (with its natural filtration) is a martingale.
2. Form the discounted value process,
$$\tilde{V}_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} C_T \middle| \mathcal{F}_t\right].$$
3. Find a predictable process $\{\phi_t\}_{0\leq t\leq T}$ such that
$$d\tilde{V}_t = \phi_t d\tilde{Z}_t.$$

Notice that a portfolio consisting of ϕ_t units of our tradable asset and ψ_t units of cash bond at each time $t \in [0, t]$ is *self-financing* if the value, $\{V_t\}_{t\geq 0}$, satisfies

$$dV_t = \phi_t dZ_t + \psi_t dB_t = \phi_t dS_t + \phi_t \delta S_t dt + \psi_t dB_t.$$

The change in value over $[t, t + dt)$ is due not only to profits and losses of trading but also to dividend payments.

Example 5.4.1 (Call option) *Suppose that a call option with strike price K and maturity T is written on the dividend-paying stock described above. What is the value of the option at time zero and what is the replicating portfolio?*

Solution: We follow our three steps to replication. First we must find the martingale measure $\mathbb{Q}$. The stochastic differential equation satisfied by $\{\tilde{Z}_t\}_{t\geq 0}$ is

$$d\tilde{Z}_t = \left(\nu + \delta + \frac{1}{2}\sigma^2 - r\right)\tilde{Z}_t dt + \sigma \tilde{Z}_t dW_t.$$

As usual we apply the Girsanov Theorem. Under the measure $\mathbb{Q}$ defined by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right)$$

with $\lambda = \left(\nu + \delta + \frac{1}{2}\sigma^2 - r\right)/\sigma$

$$X_t = W_t + \frac{\left(\nu + \delta + \frac{1}{2}\sigma^2 - r\right)}{\sigma} t$$

is a Brownian motion and hence $\{\tilde{Z}_t\}_{t\geq 0}$ is a martingale. We can now just read off the price and the hedge from the corresponding formulae in Example 5.2.2 and Example 5.2.4. The value of the portfolio at time t is

$$\begin{aligned} V_t &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)_+ \middle| \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(e^{-\delta T} Z_T - K\right)_+ \middle| \mathcal{F}_t\right] \\ &= e^{-r(T-t)} e^{-\delta T}\mathbb{E}^{\mathbb{Q}}\left[\left(Z_T - K e^{\delta T}\right)_+ \middle| \mathcal{F}_t\right]. \end{aligned}$$

That is the value is that of $e^{-\delta T}$ copies of a call option on $\{Z_t\}_{t\geq 0}$ with maturity T and strike $Ke^{\delta t}$. We write $F_t = e^{(r-\delta)(T-t)} S_t$ for the forward price of our underlying stock at time t (see Exercise 14). Substituting from Example 5.2.2 we obtain

$$\begin{aligned} V_t &= e^{-\delta T}\left\{ Z_t \Phi\left(\frac{\log\left(\frac{Z_t}{Ke^{\delta T}}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right. \\ &\qquad \left. - K e^{\delta T} e^{-r(T-t)} \Phi\left(\frac{\log\left(\frac{Z_t}{Ke^{\delta T}}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right\} \\ &= e^{-r(T-t)}\left\{ F_t \Phi\left(\frac{\log\left(\frac{F_t}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right. \\ &\qquad \left. - K\Phi\left(\frac{\log\left(\frac{F_t}{K}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right\}. \end{aligned}$$

Now using Example 5.2.4 we see that the replicating portfolio should consist of $e^{-\delta T}\phi_t$ units of our tradable asset Z_t at time t where

$$\phi_t = \Phi\left(\frac{\log\left(\frac{F_t}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right),$$

corresponding to

$$e^{-\delta(T-t)}\Phi\left(\frac{\log\left(\frac{F_t}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)$$

units of the dividend-paying asset. The bond holding in the portfolio will be

$$\psi_t = -Ke^{-rT}\,\Phi\left(\frac{\log\left(\frac{F_t}{K}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right).$$

□

Example 5.4.2 (Guaranteed equity profits) *Let $\{S_t\}_{t\geq 0}$ denote the value of the UK FTSE stock index. Suppose that we buy a five-year contract that pays out Z defined to be 90% of the ratio of the terminal and initial values of the FTSE if this value is in the interval $[1.3, 1.8]$, 1.3 if $Z < 1.3$ and 1.8 if $Z > 1.8$. What is the value of this contract at time zero?*

Solution: The claim C_T is

$$C_T = \min\left\{\max\left\{1.3, 0.9\frac{S_T}{S_0}\right\}, 1.8\right\},$$

where T is five years. Since the claim is based on a ratio, without loss of generality we set $S_0 = 1$. As FTSE is composed of one hundred different stocks, their separate dividend payments will approximate a continuously paying stream. We assume the following data:

$$\begin{aligned} \text{FTSE drift} \quad & \mu = 7\%, \\ \text{FTSE volatility} \quad & \sigma = 15\%, \\ \text{FTSE dividend yield} \quad & \delta = 4\%, \\ \text{UK interest rate} \quad & r = 6.5\%. \end{aligned}$$

We can rewrite the claim as the sum of some cash plus the difference in the payout of two FTSE calls,

$$C_T = 1.3 + 0.9\left\{(S_T - 1.444)_+ - (S_T - 2)_+\right\}.$$

Now the forward price for S_t is

$$F_t = e^{(r-\delta)(T-t)}S_0 = 1.133,$$

and so using the call price formula for continuous dividend-paying stocks of Example 5.4.1 we can value these calls at 0.0422 and 0.0067 (per unit) at time zero. The value of our contract at time zero is then

$$1.3e^{-rT} + 0.9\,(0.0422 - 0.0067) = 0.9712.$$

□

Periodic dividends

In practice, an individual stock does not pay dividends continuously, but rather at regular intervals. Suppose that the times of the payments are known in advance to be $T_1, T_2, \ldots$ and that at each time T_i the current holder of the equity receives a payment

of δS_{T_i}. As shown in Exercise 7 of Chapter 2, in the absence of arbitrage, the stock price must instantaneously decrease by the same amount. So at any of the times T_i, the dividend payout is exactly equal to the instantaneous decrease in the stock price. Between payouts we assume our usual geometric Brownian motion model.

> **Equity model with periodic dividends:** At deterministic times $T_1, T_2, \ldots$ the equity pays a dividend of a fraction δ of the stock price which was current just before the dividend is paid. The stock price itself is modelled as
>
> $$S_t = S_0 (1-\delta)^{n[t]} \exp(\nu t + \sigma W_t),$$
>
> where $n[t] = \max\{i : T_i \le t\}$ is the number of dividend payments made by time t. There is also a riskless cash bond $B_t = \exp(rt)$.

At first sight it looks as though we have two problems. First, although between the times T_i our stock price follows the usual geometric Brownian motion model, at those times it has discontinuous jumps. This doesn't fit our framework. Secondly, as for continuous dividends, the stock price process $\{S_t\}_{t\ge 0}$ does not reflect the true value of the stock. However, by adapting our strategy for the continuous dividends case and reinvesting all dividend payments in stock, we'll overcome both of these obstacles.

We define $\{Z_t\}_{t\ge 0}$ to be the value of the portfolio that starts with one unit of stock at time zero and every time the stockholding pays a dividend it is reinvested by buying more stock. The first dividend payment is δS_{T_1-}, δ times the stock price immediately prior to the payment. Immediately after the payment of the first dividend, the stock price jumps to $S_{T_1+} = (1-\delta) S_{T_1-}$, so our dividend payment will buy us an additional $\delta/(1-\delta)$ units of stock, thereby increasing our total stock holding by a factor of $1/(1-\delta)$. At time t the portfolio will therefore consist of $1/(1-\delta)^{n[t]}$ units of stock. Thus

$$Z_t = (1-\delta)^{-n[t]} S_t = S_0 \exp(\nu t + \sigma W_t).$$

As before we think of our portfolio $\{Z_t\}_{t\ge 0}$ as a non-dividend-paying asset and so our market consists of two tradable assets, the portfolio $\{Z_t\}_{t\ge 0}$ and a riskless cash bond $\{B_t\}_{t\ge 0}$, and we are back in familiar territory.

We mimic exactly what we did for continuous dividend payments. A portfolio consisting of ϕ_t units of Z_t and ψ_t in cash bonds at time t is equivalent to $(1-\delta)^{-n[t]}\phi_t$ units of the dividend-paying underlying stock, S_t, and ψ_t units of cash bond.

The measure $\mathbb{Q}$ that makes the discounted process $\{\tilde{Z}_t\}_{t\ge 0}$ a martingale satisfies

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right)$$

with $\lambda = \left(\nu + \frac{1}{2}\sigma^2 - r\right)/\sigma$. Rewriting the claim as a function of Z_T we can use the classical Black–Scholes analysis to price and hedge the option.

Example 5.4.3 (Forward price) *Find the fair price to be written into a forward contract on a stock that pays periodic dividends.*

Solution: The value of the contract at time T is $C_T = S_T - K$. We seek K so that the time zero value of the contract is zero. As usual, the value at time t is the discounted expected value of the claim under the martingale measure $\mathbb{Q}$. That is

$$\begin{aligned} V_t &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\left(S_T - K\right)\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\left((1-\delta)^{n[T]}Z_T - K\right)\middle|\mathcal{F}_t\right] \\ &= (1-\delta)^{n[T]}Z_t - Ke^{-r(T-t)} \\ &= (1-\delta)^{n[T]-n[t]}S_t - Ke^{-r(T-t)}. \end{aligned}$$

The K for which this is zero at time zero is

$$K = e^{rT}(1-\delta)^{n[T]}S_0. \tag{5.5}$$

□

5.5 Bonds

A pure discount bond is a security that pays off one unit at some future time T. Most market bonds also pay off a series of smaller amounts, c, at predetermined times $T_1, T_2, \ldots, T_n$. Such *coupon* payments resemble dividend payments except that the amount of the coupon is known in advance.

So far we have considered only a riskless cash bond in which the interest rate too is known in advance. In real markets, uncertainty in interest rates causes the price of bonds to move randomly as well. In order to keep the book to a reasonable length we do not intend to enter into a full account of bond market models. An excellent introduction can be found in Baxter & Rennie (1996). So for the purposes of this section, we are going to take a schizophrenic attitude to interest rates. We'll assume that we have a riskless cash bond following $B_t = e^{rt}$, but also a stochastically varying coupon bond whose price between coupon payments evolves as a geometric Brownian motion. Clearly there are links between the short term interest rate and bond prices, but over short time horizons, many practitioners ignore them. In effect we are thinking of a coupon bond as an asset paying predetermined cash dividends at times $T_1, T_2, \ldots, T_n$ where we assume $T_n < T$. Writing $I(t) = \min\{i : t < T_i\}$, the bond price satisfies

$$S_t = \sum_{i=I(t)}^{n} ce^{-r(T_i - t)} + A\exp\left(\nu t + \sigma W_t\right),$$

for some constants A, ν and σ.

As for dividend-paying stock, the price process $\{S_t\}_{t\geq 0}$ is discontinuous at the coupon payment times. Once again however we can manufacture a continuous non-dividend-paying asset from $\{S_t\}_{t\geq 0}$. Whereas when our dividend payment was a fraction of the stock price it was natural to reinvest it in stock, now, since our coupons are fixed cash payments, we invest them in the riskless cash bond. With this investment strategy the coupon paid at time T_i then has value $ce^{-r(T_i-t)}$ for *all* $t \in [0, T]$ and so the portfolio constructed in this way has value

$$Z_t = \sum_{i=1}^{n} ce^{-r(T_i-t)} + A\exp(\nu t + \sigma W_t).$$

Exactly as before we think of our market as consisting of the riskless cash bond $\{B_t\}_{t\geq 0}$ and the tradable asset $\{Z_t\}_{t\geq 0}$.

As usual we want to find $\mathbb{Q}$ under which the discounted asset price $\{\tilde{Z}_t\}_{t\geq 0}$ is a martingale. But $\tilde{Z}_t$ is just the constant cash sum $\sum_{i=1}^{n} ce^{-rT_i}$ plus the geometric Brownian motion $A\exp((\nu - r)t + \sigma W_t)$. This will be a $\mathbb{Q}$-martingale if

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\lambda W_t - \frac{1}{2}\lambda^2 t\right)$$

where $\lambda = \left(\nu + \frac{1}{2}\sigma^2 - r\right)/\sigma$. Under $\mathbb{Q}$,

$$X_t = W_t + \frac{\left(\nu + \frac{1}{2}\sigma^2 - r\right)}{\sigma}t$$

is a Brownian motion. The value at time t of an option with payoff C_T is now

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}C_T\middle|\mathcal{F}_t\right].$$

Under $\mathbb{Q}$, the price of the bond at time T is just

$$S_T = A\exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma X_T\right).$$

The forward price for S_T at time zero is $F = Ae^{rT}$ and the value of a call on S_T with strike price K at maturity T is

$$e^{-rT}\left\{F\Phi\left(\frac{\log\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{\log\left(\frac{F}{K}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right\};$$

see Exercise 20.

5.6 Market price of risk

A definite pattern has emerged. Given a non-tradable stock, we have tied it to a portfolio that can be thought of as a tradable, found the martingale measure corresponding to that tradable process and used that measure to price the option. In deciding what is tradable and what is not we have used only common sense. Indeed it is not something that can be reduced purely to mathematics, but if we decide that an asset with price $\{S_t\}_{t\geq 0}$ is truly tradable and we have a riskless cash bond, $\{B_t\}_{t\geq 0}$, we should like to determine the class of tradables within the market that they create.

Martingales and tradables

Suppose that $\tilde{S}_t = B_t^{-1} S_t$ is the discounted price of our tradable asset at time t. Let $\mathbb{Q}$ be a measure under which $\{\tilde{S}_t\}_{t\geq 0}$ is a martingale. If we take another process, $\{V_t\}_{t\geq 0}$ for which the discounted process $\{\tilde{V}_t\}_{t\geq 0}$ given by $\tilde{V}_t = B_t^{-1} V_t$ is a $\left(\mathbb{Q}, \{\mathcal{F}_t^{\tilde{S}}\}_{t\geq 0}\right)$-martingale, then is $\{V_t\}_{t\geq 0}$ tradable?

Our strategy is to construct a self-financing portfolio consisting of our tradable asset and the tradable discounting process whose value at time t is always exactly V_t. As usual, the first step is an application of the Martingale Representation Theorem. Provided that $B_t^{-1} S_t$ has non-zero volatility, we can find an $\{\mathcal{F}_t^{\tilde{S}}\}_{t\geq 0}$-previsible process $\{\phi_t\}_{t\geq 0}$ such that

$$d\tilde{V}_t = \phi_t d\tilde{S}_t. \tag{5.6}$$

Taking our cue from the construction of the portfolio replicating an option in §5.1, we create a portfolio that at time t consists of ϕ_t units of the tradable S_t and $\psi_t = \tilde{V}_t - \phi_t \tilde{S}_t$ units of (the tradable) B_t. The value of this portfolio at time t is exactly V_t. We must check that it is self-financing. Now

$$\begin{aligned}
dV_t &= B_t d\tilde{V}_t + \tilde{V}_t dB_t \qquad \text{(integration by parts)}\\
&= B_t \phi_t d\tilde{S}_t + \tilde{V}_t dB_t \qquad \text{(equation (5.6))}\\
&= B_t \phi_t d\tilde{S}_t + \left(\psi_t + \phi_t \tilde{S}_t\right) dB_t\\
&= \phi_t \left(B_t d\tilde{S}_t + \tilde{S}_t dB_t\right) + \psi_t dB_t\\
&= \phi_t dS_t + \psi_t dB_t \qquad \text{(integration by parts)}
\end{aligned}$$

and so the change in value of the portfolio over any infinitesimal time interval is due to changes in asset prices. That is we have the self-financing property and $\{V_t\}_{t\geq 0}$ is indeed tradable.

What about the other way round? Suppose that $\{B_t^{-1} V_t\}_{t\geq 0}$ were *not* a $\left(\mathbb{Q}, \{\mathcal{F}_t^{\tilde{S}}\}_{t\geq 0}\right)$-martingale. Then there would have to be times $s < T$ such that with positive probability

$$B_s^{-1} V_s \neq \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} V_T \middle| \mathcal{F}_s^{\tilde{S}} \right].$$

Suppose that $\{V_t\}_{t\geq 0}$ were tradable. We can construct a process $\{U_t\}_{t\geq 0}$ by setting

$$U_t = B_t \mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} V_T \middle| \mathcal{F}_t^{\tilde{S}} \right].$$

Since $\{B_t^{-1} U_t\}_{t\geq 0}$ is a $\left(\mathbb{Q}, \{\mathcal{F}_t^{S}\}_{t\geq 0}\right)$-martingale, we know that $\{U_t\}_{t\geq 0}$ is tradable. That is we have two tradables that take the same value at time T, but with positive probability take *different* values at an earlier time s. Exercise 21 shows that in the absence of arbitrage this is a contradiction. So if $\{B_t^{-1} V_t\}_{t\geq 0}$ is not a martingale, then $\{V_t\}_{t\geq 0}$ is *not* a tradable.

Of course, since interest rates are deterministic, $\mathcal{F}_t^{\tilde{S}} = \mathcal{F}_t^{S}$ and so we have proved the following lemma.

Lemma 5.6.1 *Given a riskless cash bond $\{B_t\}_{t\geq 0}$ and a tradable asset $\{S_t\}_{t\geq 0}$, a process $\{V_t\}_{t\geq 0}$ represents a tradable asset if and only if the discounted value*

$\{B_t^{-1}V_t\}_{t\geq 0}$ is a $\left(\mathbb{Q}, \{\mathcal{F}_t^S\}_{t\geq 0}\right)$-martingale where $\mathbb{Q}$ is the measure under which the discounted asset price $\{B_t^{-1}S_t\}_{t\geq 0}$ is a martingale.

Tradables and the market price of risk

Suppose that we have two tradable risky securities $\{S_t^1\}_{t\geq 0}$ and $\{S_t^2\}_{t\geq 0}$ in a single Black–Scholes market – that is they are both functions of the same Brownian motion. We define them both via their stochastic differential equations,

$$dS_t^i = \mu_i S_t^i dt + \sigma_i S_t^i dW_t.$$

In order for both to be tradable, Lemma 5.6.1 tells us that they must both be martingales with respect to the *same* measure $\mathbb{Q}$. Assuming that $B_t = e^{rt}$ we must have that

$$X_t = W_t + \left(\frac{\mu_i - r}{\sigma_i}\right)t$$

is a $\mathbb{Q}$-Brownian motion for $i = 1, 2$. This can only be the case if

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}.$$

Economists attach a meaning to this quantity. If we think of μ as the rate of growth of the tradable, r as the rate of growth of the riskless bond and σ as a measure of the risk of the asset, then

$$\gamma = \frac{\mu - r}{\sigma}$$

is the excess rate of return (above the risk-free rate) per unit of risk. As such, it is often called the *market price of risk*. It is also known as the *Sharpe ratio*. In such a simple market, every tradable asset should have the same market price of risk, otherwise there would be arbitrage opportunities.

Of course, γ is precisely the change of drift in the underlying Brownian motion when we change measure from $\mathbb{P}$ (the market measure) to $\mathbb{Q}$ (the martingale measure). However, this appealing economic interpretation of γ does *not* provide a new argument for using $\mathbb{Q}$. It is *replication* that makes the Black–Scholes analysis work. Without a replicating portfolio our arbitrage arguments collapse.

Exercises

1 Suppose that an asset price S_t is such that $dS_t = \mu S_t dt + \sigma S_t dW_t$, where $\{W_t\}_{t\geq 0}$ is, as usual, standard $\mathbb{P}$-Brownian motion. Let r denote the risk-free interest rate. The price of a riskless asset then follows $dB_t = rB_t dt$. We write $\{\psi_t, \phi_t\}$ for the portfolio consisting of ψ_t units of the riskless asset B_t and ϕ_t units of S_t at time t. For each of the following choices of ϕ_t, find ψ_t so that the portfolio $\{\psi_t, \phi_t\}$ is self-financing. (Recall that the value of the portfolio at time t is $V_t = \psi_t B_t + \phi_t S_t$ and that the portfolio is self-financing if $dV_t = \psi_t dB_t + \phi_t dS_t$.)

(a) $\phi_t = 1$,
(b) $\phi_t = \int_0^t S_u du$,
(c) $\phi_t = S_t$.

2 Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the natural filtration associated with a $\mathbb{P}$-Brownian motion $\{W_t\}_{t\geq 0}$. Show that if $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ and H_T is an $\mathcal{F}_T$-measurable random variable with $\mathbb{E}^{\mathbb{Q}}\left[H_T^2\right] < \infty$ then

$$M_t \triangleq \mathbb{E}^{\mathbb{Q}}\left[H_T | \mathcal{F}_t\right]$$

defines a square-integrable $\mathbb{Q}$-martingale.

3 Show that, in the notation of Example 5.2.2, the Black–Scholes price at time t of a European put option with strike K and maturity T is $F(t, S_t)$ where

$$F(t,x) = Ke^{-r\theta}\Phi(-d_2) - x\Phi(-d_1).$$

4 Suppose that the value of a European call option can be expressed as $V_t = F(t, S_t)$ (as we prove in Proposition 5.2.3). Then $\tilde{V}_t = e^{-rt}V_t$, and we may define $\tilde{F}$ by

$$\tilde{V}_t = \tilde{F}(t, \tilde{S}_t).$$

Under the risk-neutral measure, the discounted asset price follows $d\tilde{S}_t = \sigma\tilde{S}_t dX_t$, where (under this probability measure) $\{X_t\}_{t\geq 0}$ is a standard Brownian motion.

(a) Find the stochastic differential equation satisfied by $\tilde{F}(t, \tilde{S}_t)$.
(b) Using the fact that $\tilde{V}_t$ is a martingale under the risk-neutral measure, find the partial differential equation satisfied by $\tilde{F}(t, x)$, and hence show that

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} + rx\frac{\partial F}{\partial x} - rF = 0.$$

This is the *Black–Scholes equation.*

5 *Delta hedging* The following derivation of the Black–Scholes equation is very popular in the finance literature. We will suppose, as usual, that an asset price follows a geometric Brownian motion. That is, there are parameters μ, σ, such that

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Suppose that we are trying to value a European option based on this asset. Let us denote the value of the option at time t by $V(t, S_t)$. We know that at time T, $V(T, S_T) = f(S_T)$, for some function f.

(a) Using Itô's formula express V as the solution to a stochastic differential equation.
(b) Suppose that a portfolio, whose value we denote by π, consists of one option and a (negative) quantity $-\delta$ of the asset. Assuming that the portfolio is self-financing, find the stochastic differential equation satisfied by π.
(c) Find the value of δ for which the portfolio you have constructed is 'instantaneously riskless', that is for which the stochastic term vanishes.
(d) An instantaneously riskless portfolio must have the same rate of return as the risk-free interest rate. Use this observation to find a (deterministic) partial differential equation for the $V(t, x)$. Notice that this is the Black–Scholes equation obtained in Exercise 4.

Of course δ is precisely the stock holding in our replicating portfolio. In fact this derivation is not entirely satisfactory as it can be checked that the portfolio that we have constructed is *not* self-financing, violating our assumption in (b). A rigorous approach requires a portfolio consisting of one option, $-\delta$ assets and $e^{-rt}(V(t, S_t) - \delta S_t)$ cash bonds at time t.

6 Calculate the values of the Greeks for a European call option with strike price K at maturity time T.

7 An alternative approach to solving the Black–Scholes equation is to transform it via a change of variables into the heat equation. Suppose that $F(t, x) : [0, T] \times [0, \infty) \to \mathbb{R}$ satisfies

$$\frac{\partial F}{\partial t}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t, x) + rx\frac{\partial F}{\partial x}(t, x) - rF(t, x) = 0,$$

subject to the boundary conditions

$$F(t, 0) = 0, \qquad \frac{F(t, x)}{x} \to 1 \text{ as } x \to \infty, \qquad F(T, x) = (x - K)_+.$$

That is F solves the Black–Scholes equation with the boundary conditions appropriate for pricing a European call option with strike price K at time T.

(a) Show that the change of variables

$$x = Ke^y, \qquad t = T - \frac{2\tau}{\sigma^2}, \qquad F = Kv(\tau, y)$$

results in the equation

$$\frac{\partial v}{\partial \tau}(\tau, y) = \frac{\partial^2 v}{\partial y^2}(\tau, y) + (k-1)\frac{\partial v}{\partial y}(\tau, y) - kv(\tau, y), \qquad y \in \mathbb{R}, \tau \in \left[0, \frac{1}{2}\sigma^2 T\right],$$

where $k = 2r/\sigma^2$ and $v(0, y) = (e^y - 1)_+$.

(b) Now set $v(\tau, y) = e^{\alpha y + \beta\tau}u(\tau, y)$. Find α and β such that

$$\frac{\partial u}{\partial \tau}(\tau, y) = \frac{\partial^2 u}{\partial y^2}(\tau, y), \qquad y \in \mathbb{R},$$

and find the corresponding initial condition for u.

(c) Solve for u and retrace your steps to obtain the Black–Scholes pricing formula for a European call option.

8 Show that, for each constant A, $V(t, x) = Ax$ and $V(t, x) = Ae^{rt}$ are both exact solutions of the Black–Scholes differential equation. What do they represent and what is the hedging portfolio in each case?

9 Find the most general solution of the Black–Scholes equation that has the special form

(a) $V(t, x) = V(x)$,

(b) $V(t, x) = f(t)g(x)$.

These are examples of similarity solutions. The solutions in (a) give prices of *perpetual* options.

10 Let $C(t, S_t)$ and $P(t, S_t)$ denote the values of a European call and put option with the same exercise price, K, and expiry time, T. Show that $C(t, x) - P(t, x)$ also satisfies the Black–Scholes equation with the final data $C(T, x) - P(T, x) = x - K$. Deduce that $x - Ke^{-r(T-t)}$ is also a solution of the Black–Scholes equation. Interpret these results financially.

11 Assuming the model of §5.3, find the Black–Scholes price for a Sterling call option that gives us the right to buy a pound Sterling at time T for K dollars. What is the corresponding hedging portfolio?

12 Check that the Sterling and dollar investors of §5.3 use exactly the same replicating strategy.

13 Suppose that the US dollar/Japanese Yen exchange rate follows the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

for some constants μ and σ. You are told that the expected \$/¥and ¥/\$ exchange rates in one years time are both $2S_0$. Is this possible?

14 In our usual notation suppose that an asset price follows geometric Brownian motion with $S_t = S_0 \exp(\nu t + \sigma W_t)$ at time t. If in each infinitesimal time interval the asset pays to its holder a dividend of $\delta S_t dt$, find an expression for the fair price in a forward contract based on the stock with maturity time T. What is the corresponding hedging portfolio?

15 What is the 'put–call parity' relation for the market in Exercise 14?

16 Suppose that in valuing the contract in Example 5.4.2 we had failed to take account of the dividend stream from the constituent stocks of the FTSE. Find a financial argument to indicate whether the price obtained for the contract will be too high or too low. Find the exact value that we would have obtained for the contract.

17 Suppose that V_t is the value of a self-financing portfolio consisting of ϕ_t units of stock that pays periodic dividends, as in §5.4, and ψ_t units of cash bond. Find the differential equation that characterises the self-financing property of V_t in this setting.

18 Find a portfolio that replicates the forward of Example 5.4.3.

19 Value and hedge a European call option with maturity T and strike K based on the periodic dividend-paying stock of §5.4. Express your answer in terms of the usual Black–Scholes formula evaluated on the forward price of equation (5.5).

20 Check the forward price and the value of a call option claimed in §5.5 and find the corresponding self-financing replicating portfolios.

21 Show that if two tradable assets have the same value at time T, but with positive probability take different values at time $s < T$, then there are arbitrage opportunities in the market.