he previous part to show that

$$\infty]\,\phi(\sqrt{t}x)\,dx + 1 - \Phi\left(\frac{a+bt}{\sqrt{t}}\right).$$

l and deduce that

$$\left.\frac{a}{}\right) + 1 - \Phi\left(\frac{a+bt}{\sqrt{t}}\right).$$

nder the measure $\mathbb{P}$ and let $\{\mathcal{F}_t\}_{t\geq 0}$
wing are $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingales?

associated to a standard $\mathbb{P}$-Brownian
.6.3 can be rewritten as

$$V_s - \left.\frac{1}{2}\sigma^2 s\right)\mathbf{1}_A, \quad \text{for all } A \in \mathcal{F}_s.$$

provide another proof that $\{W_t^2 - t\}_{t\geq 0}$
the following are also $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-

se that the real random variable $T : \Omega \to$
e measure $\mathbb{P}$. Define $\{X_t\}_{t\geq 0}$ by

$$T(\omega) = t,$$
$$T(\omega) \neq t.$$

ith respect to its own filtration. [Hint:
within a random variable that is almost

ne Optional Stopping Theorem fails.

g times of levels a and b respectively of
$_0$ is not necessarily zero (see the remarks
$< b$ then

$$= x] = \frac{(b-x)}{(b-a)}.$$

ding result for random walk, cf. Proposi-

$= T_a \wedge T_b$. Prove that if $a < 0 < b$ then

$$= 0] = -ab.$$

4 Stochastic calculus

Summary

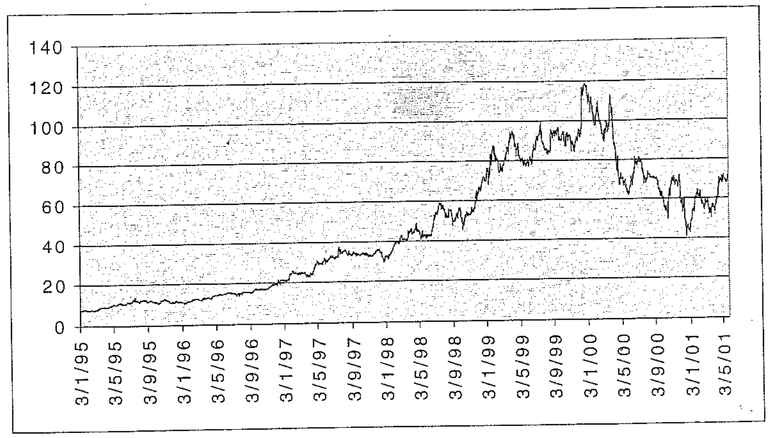
Brownian motion is clearly inadequate as a market model, not least because it would predict negative stock prices. However, by considering *functions* of Brownian motion we can produce a wide class of potential models. The basic model underlying the Black–Scholes pricing theory, geometric Brownian motion, arises precisely in this way. It will inherit from the Brownian motion very irregular paths. In §4.1 we shall see why a stock price model with rough paths is forced upon us by arbitrage arguments. This is not in itself sufficient to justify the geometric Brownian motion model. However in §4.7 we provide a further argument that suggests that it is at least a sensible starting point. A more detailed discussion of the shortcomings of the geometric Brownian motion model is deferred until Chapter 7.

In order to study models built in this way, we need to develop a calculus based on Brownian motion. The Itô stochastic calculus is the main topic of this chapter. In §4.2 we define the Itô stochastic integral and then in §4.3 we derive the corresponding chain rule of stochastic calculus and learn how to integrate by parts.

Just as in the discrete world, there will be two key ingredients to pricing and hedging in the Black–Scholes framework. First we need to be able to *change the probability measure* so that discounted asset prices are martingales. The tool for doing this is the Girsanov Theorem of §4.5. The construction of the hedging portfolio depends on the continuous analogue of the Binomial Representation Theorem, the Martingale Representation Theorem of §4.6.

Again as in the discrete world, the pricing formula will be in the form of the discounted expected value of a claim. Black and Scholes obtained this result via a completely different argument (see Exercise 5 of Chapter 5) in which the price is obtained as the solution of a partial differential equation. The connection with the probabilistic approach is via the Feynman–Kac stochastic representation formula of §4.8 which exposes the intricate connection between stochastic differential equations and certain second order parabolic (deterministic) partial differential equations.

Once again our coverage of this material is necessarily rather sketchy. Even so readers eager to get back to some finance may wish to skip the proofs in this



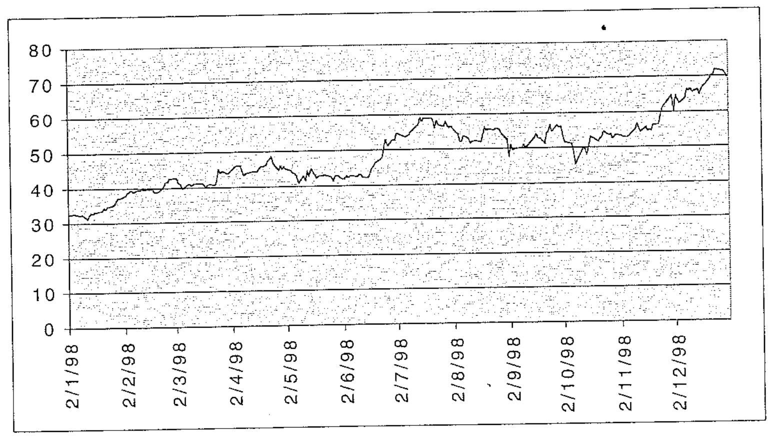
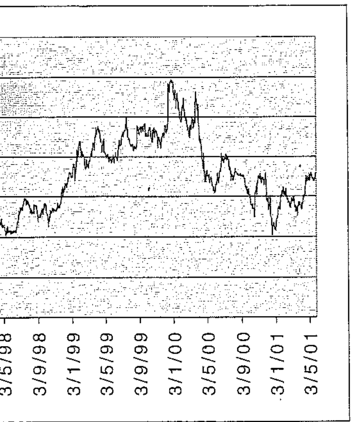


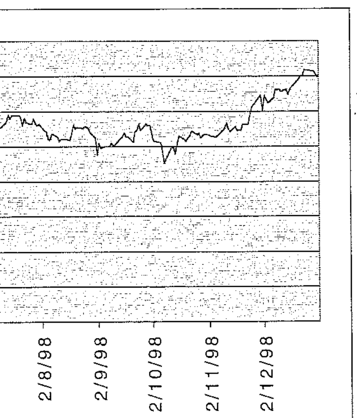
Figure 4.1 Two graphs of non-dividend-paying stock over long period ($6\frac{1}{3}$ years) and short perioc (1 year).

chapter. There is no shortage of excellent stochastic calculus texts to refer to. suggestions are included in the bibliography.

4.1 Stock prices are not differentiable

Figure 4.1 shows the Microsoft share price over $6\frac{1}{3}$ year and 1 year peri certainly doesn't look like a particularly nice function of time. Even ovel time scales, the path followed by the price looks rough. There are many sta studies that investigate the irregularity of paths of stock prices. In this secti explore through a purely mathematical argument of Lyons (1995) just how paths of our stock price model should be, at least under the assumption that trade continuously without incurring transaction costs and, as usual, that there arbitrage opportunities. We continue to suppose that our market contains a r cash bond.





ng period ($6\frac{1}{3}$ years) and short period

ochastic calculus texts to refer to. Some
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e over $6\frac{1}{3}$ year and 1 year periods. It
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least under the assumption that we can
tion costs and, as usual, that there are no
pose that our market contains a riskless

uantifying ughness

First we need a means of quantifying 'roughness'. For a function $f : [0, T] \to \mathbb{R}$, its variation is defined in terms of partitions.

Definition 4.1.1 *Let π be a partition of $[0, T]$, $N(\pi)$ the number of intervals that make up π and $\delta(\pi)$ be the* mesh *of π (that is the length of the largest interval in the partition). Write $0 = t_0 < t_1 < \cdots < t_{N(\pi)} = T$ for the endpoints of the intervals of the partition. Then the* variation *of f is*

$$\lim_{\delta\to 0}\left\{\sup_{\pi:\delta(\pi)=\delta}\sum_{j=1}^{N(\pi)}\left|f(t_j)-f(t_{j-1})\right|\right\}.$$

If the function is 'nice', for example differentiable, then it has bounded variation. Our 'rough' paths will have *unbounded* variation. To quantify roughness we can extend the idea of variation to that of p-variation.

Definition 4.1.2 *In the notation of Definition 4.1.1, the p-*variation *of a function $f : [0, T] \to \mathbb{R}$ is defined as*

$$\lim_{\delta\to 0}\left\{\sup_{\pi:\delta(\pi)=\delta}\sum_{j=1}^{N(\pi)}\left|f(t_j)-f(t_{j-1})\right|^p\right\}.$$

Notice that if $p > 1$ the p-variation will be finite for functions that are much rougher than those for which the variation is bounded. For example, roughly speaking, finite 2-variation will follow if the fluctuation of the function over an interval of order δ is order $\sqrt{\delta}$.

unded riation and bitrage

We now argue that if stock prices had bounded variation, then either they would be constant multiples of the riskless cash bond or (provided we can trade continuously and there are no transaction costs) there would be unbounded arbitrage opportunities.

In the discrete time world of Chapter 2 we showed (equation (2.5)) that if a portfolio consisting of ϕ_{i+1} units of stock and ψ_{i+1} cash bonds over the time interval $[i\delta t, (i+1)\delta t)$ is self-financing, then its discounted value at time $N\delta t = T$ is

$$\tilde{V}_N = V_0 + \sum_{j=0}^{N-1}\phi_{j+1}\left(\tilde{S}_{j+1}-\tilde{S}_j\right). \tag{4.1}$$

Here ϕ_{j+1} is known at time $j\delta t$, but is typically a function of $\tilde{S}_j$. In our continuous world, we can let the trading interval δt tend to zero and, if the discounted stock price process has bounded variation, as $\delta t \downarrow 0$ the Riemann sum in (4.1) will converge to the Riemann integral

$$\int_0^T \phi_t(\tilde{S}_t)\,d\tilde{S}_t$$

where ϕ_t denotes our stock holding at time t. This says that for any choice of $\{\phi_t(\cdot)\}_{0\le t\le T}$, we can construct a self-financing portfolio whose discounted value at time T is

$$V_0 + \int_0^T \phi_t(\tilde{S}_t)d\tilde{S}_t.$$

Now choose a differentiable function $F(x)$ that is small near $x = S_0$ and ve everywhere else. Then by investing $F(S_0)$ at time zero and holding a self-fi portfolio with $\phi(\tilde{S}_t)$ units of stock at time t, where $\phi(x) = F'(x)$, we ge portfolio at time T whose discounted value is

$$F(S_0) + \int_0^T F'(\tilde{S}_t) d\tilde{S}_t,$$

which, by the Fundamental Theorem of Calculus, is $F(\tilde{S}_T)$.

We only have to wait for the discounted stock price to move away fro generate a lot of wealth. For example, the strategy that holds $\left(\tilde{S}_t - \tilde{S}_0\right)$ stock at time t generates

$$e^{rT} \int_0^T (\tilde{S}_t - \tilde{S}_0) d\tilde{S}_t = e^{rT} \left(\tilde{S}_T - \tilde{S}_0\right)^2$$

units of wealth at time T (where we have multiplied by e^{rT} to 'undo' the di ing).

In the absence of arbitrage then we do not expect the paths of our stock have bounded variation. In fact, as Lyons points out, arguments of L C Young this. Again assuming continuous trading and no transaction costs, if the path stock price have finite p-variation for some $p < 2$, then there are arbitraril profits to be made.

4.2 Stochastic integration

The work of §4.1 suggests that we should be looking for models in wh stock price has infinite p-variation for $p < 2$. A large class of such models constructed using Brownian motion as a building block, but this will requir calculus. The paths of Brownian motion are too rough for the familiar Ne calculus to help us and, indeed, if it did the Fundamental Theorem of Calculus once again lead us to discard Brownian motion as a basis for our models.

A differential equation for the stock price

The processes used to model stock prices are usually functions of one o Brownian motions. Here, for simplicity, we restrict ourselves to functions one Brownian motion. The first thing that we should like to do is to write differential equation for the way in which the stock price evolves.

Suppose that the stock price is of the form $S_t = f(t, W_t)$. Using Taylor's T (and assuming that f at least is 'nice'),

$$\begin{aligned} f(t+\delta t, W_{t+\delta t}) - f(t, W_t) &= \delta t \dot{f}(t, W_t) + O(\delta t^2) + (W_{t+\delta t} - W_t) f' \\ &+ \frac{1}{2!} (W_{t+\delta t} - W_t)^2 f''(t, W_t) + \cdots \end{aligned}$$

where we have used the notation

$$\dot{f}(t,x) = \frac{\partial f}{\partial t}(t,x), \quad f'(t,x) = \frac{\partial f}{\partial x}(t,x) \text{ and } f''(t,x) = \frac{\partial^2 f}{\partial x^2}(t,x).$$

Now in our usual derivation of the chain rule, when $\{W_t\}_{t\geq 0}$ is replaced by a bounded variation function, the last term on the right hand side is order $O(\delta t^2)$. However, for Brownian motion, we know that $\mathbb{E}[(W_{t+\delta t} - W_t)^2]$ is δt. Consequently we cannot ignore the term involving the second derivative. Of course, now we have a problem, because we must interpret the term involving the *first* derivative. If $(W_{t+\delta t} - W_t)^2$ is $O(\delta t)$, then $(W_{t+\delta t} - W_t)$ should be $O(\sqrt{\delta t})$, which could lead to unbounded changes in $\{S_t\}_{t\geq 0}$ over a bounded time interval. However, things are not hopeless. The expected value of $W_{t+\delta t} - W_t$ is zero, and the fluctuations around zero are on the order of $\sqrt{\delta t}$. By comparison with the Central Limit Theorem, it is plausible that $S_t - S_0$ is a well-defined random variable. Assuming that we can make this rigorous, the differential equation governing $S_t = f(t, W_t)$ will take the form

$$dS_t = \dot{f}(t, W_t)dt + f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt.$$

It is convenient to write this in integrated form,

$$S_t = S_0 + \int_0^t \dot{f}(s, W_s)ds + \int_0^t f'(W_s)dW_s + \int_0^t \frac{1}{2}f''(W_s)ds. \tag{4.2}$$

In order to make sense of a calculus based on Brownian motion, we must find a rigorous mathematical interpretation of the *stochastic integral* (that is, the first integral) on the right hand side of equation (4.2). The key is to study the *quadratic variation* of Brownian motion.

For a typical Brownian path, the 2-variation will be infinite. However, a slightly weaker analogue of 2-variation *does* exist.

Theorem 4.2.1 *Let W_t denote Brownian motion under $\mathbb{P}$ and for a partition π of $[0, T]$ define*

$$S(\pi) \doteq \sum_{j=1}^{N(\pi)} \left|W_{t_j} - W_{t_{j-1}}\right|^2.$$

Let π_n be a sequence of partitions with $\delta(\pi_n) \to 0$. Then

$$\mathbb{E}\left[\left|S(\pi_n) - T\right|^2\right] \to 0 \qquad \text{as } n \to \infty. \tag{4.3}$$

We say that the quadratic variation process of Brownian motion, denoted by $\{[W]_t\}_{t\geq 0}$, is $[W]_t = t$. More generally, we can define the quadratic variation process associated with any bounded continuous martingale.

Definition 4.2.2 *Suppose that $\{M_t\}_{t\geq 0}$ is a bounded continuous $\mathbb{P}$-martingale. The* quadratic variation *process associated with $\{M\}_{t\geq 0}$ is the process $\{[M]_t\}_{t\geq 0}$ such that for any sequence of partitions π_n of $[0, T]$ with $\delta(\pi_n) \to 0$,*

$$\mathbb{E}\left[\left|\sum_{j=1}^{N(\pi)} \left|M_{t_j} - M_{t_{j-1}}\right|^2 - [M]_T\right|^2\right] \to 0 \qquad \text{as } n \to \infty. \tag{4.4}$$

Remark: We don't prove it here, but the limit in (4.4) will be independent of the sequence of partitions. $\square$

Proof of Theorem 4.2.1: We expand the expression inside the expectation in (4.3) make use of our knowledge of the normal distribution. Let $\{t_{n,j}\}_{j=0}^{N(\pi_n)}$ denote endpoints of the intervals that make up the partition π_n. First observe that

$$|S(\pi_n) - T|^2 = \left|\sum_{j=1}^{N(\pi_n)} \left\{\left|W_{t_{n,j}} - W_{t_{n,j-1}}\right|^2 - \left(t_{n,j} - t_{n,j-1}\right)\right\}\right|^2.$$

It is convenient to write $\delta_{n,j}$ for $\left|W_{t_{n,j}} - W_{t_{n,j-1}}\right|^2 - (t_{n,j} - t_{n,j-1})$. Then

$$|S(\pi_n) - T|^2 = \sum_{j=1}^{N(\pi_n)} \delta_{n,j}^2 + 2\sum_{j<k} \delta_{n,j}\delta_{n,k}.$$

Note that since Brownian motion has independent increments,

$$\mathbb{E}\left[\delta_{n,j}\delta_{n,k}\right] = \mathbb{E}\left[\delta_{n,j}\right]\mathbb{E}\left[\delta_{n,k}\right] = 0 \qquad \text{if } j \neq k.$$

Also

$$\begin{aligned}\mathbb{E}[\delta_{n,j}^2] &= \mathbb{E}\big[|W_{t_{n,j}} - W_{t_{n,j-1}}|^4 \\ &\quad - 2|W_{t_{n,j}} - W_{t_{n,j-1}}|^2(t_{n,j} - t_{n,j-1}) + (t_{n,j} - t_{n,j-1})^2\big].\end{aligned}$$

For a normally distributed random variable, X, with mean zero and variance λ, : Exercise 5 of Chapter 3, $\mathbb{E}[|X|^4] = 3\lambda^2$, so we have

$$\begin{aligned}\mathbb{E}[\delta_{n,j}^2] &= 3\left(t_{n,j} - t_{n,j-1}\right)^2 - 2\left(t_{n,j} - t_{n,j-1}\right)^2 + \left(t_{n,j} - t_{n,j-1}\right)^2 \\ &= 2\left(t_{n,j} - t_{n,j-1}\right)^2 \\ &\leq 2\delta(\pi_n)\left(t_{n,j} - t_{n,j-1}\right).\end{aligned}$$

Summing over j

$$\begin{aligned}\mathbb{E}\left[\left|S(\pi_n) - T\right|^2\right] &\leq 2\sum_{j=1}^{N(\pi_n)} \delta(\pi_n)\left(t_{n,j} - t_{n,j-1}\right) \\ &= 2\delta(\pi_n)T \\ &\to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

Integrating Brownian motion against itself

This result is not enough to define the integral $\int_0^T f(s, W_s)dW_s$ in the classical but it is enough to allow us to essentially mimic the construction of the (Lebes integral, as limits of integrals of simple functions, at least for functions for w $\int_0^T \mathbb{E}[f^2(s, W_s)]ds < \infty$, provided we only require that the limit exist in *a sense*. That is, if $\{f^{(n)}\}_{n\geq 1}$ is a sequence of step functions converging to f, $\int_0^t f(s, W_s)dW_s$ will be a random variable for which

$$\mathbb{E}\left[\left|\int_0^t f(s, W_s)dW_s - \int_0^t f^{(n)}(s, W_s)dW_s\right|^2\right] \to 0 \quad \text{as } n \to \infty.$$

This corresponds to replacing the notion of 2-variance by that of quadratic variation in Definition 4.2.2.

Although the construction of the integral may look familiar, its behaviour is far from familiar. We first illustrate this by defining $\int_0^T W_s dW_s$.

From classical integration theory we are used to the idea that

$$\int_0^T f(s, x_s) dx_s = \lim_{\delta(\pi)\to 0} \sum_{j=0}^{N(\pi)-1} f(t_j, x_{t_j}) \left(x_{t_{j+1}} - x_{t_j}\right). \tag{4.5}$$

Let us define the stochastic integral in the same way, that is

$$\int_0^T W_s dW_s \triangleq \lim_{\delta(\pi)\to 0} \sum_{j=0}^{N(\pi)-1} W_{t_j} \left(W_{t_{j+1}} - W_{t_j}\right),$$

but now with the caveat that the limit may only exist in the L^2 sense.

Consider again the quantity $S(\pi)$ of Theorem 4.2.1.

$$\begin{aligned} S(\pi) &= \sum_{j=1}^{N(\pi)} \left(W_{t_j} - W_{t_{j-1}}\right)^2 \\ &= \sum_{j=1}^{N(\pi)} \left\{\left(W_{t_j}^2 - W_{t_{j-1}}^2\right) - 2W_{t_{j-1}}\left(W_{t_j} - W_{t_{j-1}}\right)\right\} \\ &= W_T^2 - W_0^2 - 2\sum_{j=0}^{N(\pi)-1} W_{t_j}\left(W_{t_{j+1}} - W_{t_j}\right). \end{aligned}$$

The left hand side converges to T as $\delta(\pi) \to 0$ (by Theorem 4.2.1) and so letting $\delta(\pi) \to 0$ and rearranging we obtain

$$\int_0^T W_s dW_s = \frac{1}{2}\left(W_T^2 - W_0^2 - T\right).$$

Remark: Notice that this is *not* what one would have predicted from classical integration theory. The extra term in the stochastic integral arises from $\lim_{\delta(\pi)\to 0} S(\pi)$. □

In equation (4.5), we use $f(t_j, x_{t_j})$ to approximate the value of f on the interval $(t_j, t_{j+1}]$, but in the classical theory we could equally have taken any point inside the interval in place of t_j and, in the limit, the result ~~would~~ have been the same. In the stochastic theory this is no longer the case. In Exercise 3 you are asked to calculate two further limits:

(a)

$$\lim_{\delta(\pi)\to 0} \sum_{j=0}^{N(\pi)-1} W_{t_{j+1}} \left(W_{t_{j+1}} - W_{t_j}\right),$$

(b)

$$\lim_{\delta(\pi)\to 0} \sum_{j=0}^{N(\pi)-1} \left(\frac{W_{t_j} + W_{t_{j+1}}}{2}\right)\left(W_{t_{j+1}} - W_{t_j}\right).$$

By choosing different points within each subinterval of the partition with wh approximate f over the subinterval we obtain *different* integrals. The *Itô integ* a function $f(s, W_s)$ with respect to W_s is defined (up to a set of $\mathbb{P}$-probability as

$$\int_0^T f(s, W_s)dW_s = \lim_{\delta(\pi)\to 0} \sum_{j=0}^{N(\pi)-1} f(t_j, W_{t_j})\left(W_{t_{j+1}} - W_{t_j}\right).$$

The *Stratonovich integral* is defined as

$$\int_0^T f(s, W_s) \circ dW_s = \lim_{\delta(\pi)\to 0} \sum_{j=1}^{N(\pi)} \left(\frac{f(t_j, W_{t_j}) + f(t_{j+1}, W_{t_{j+1}})}{2}\right)\left(W_{t_{j+1}} -\right.$$

Both limits are to be understood in the L^2 sense. The Stratonovich integral h advantage from the calculational point of view that the rules of Newtonian cal hold good; cf. Exercise 8. From a modelling point of view, at least for our pur it is the *wrong* choice. To see why, think of what is happening over an infinite time interval. We might be modelling, for example, the value of a portfoli readjust our portfolio at the *beginning* of the time interval and its change in over the infinitesimal tick of the clock is beyond our control. A Stratonovich i would allow us to change our portfolio *now* on the basis of the average of two v depending respectively on the current stock price and prices after the next tick clock. We don't have that information when we make our investment decisions

We are simply reiterating what was said in the discrete world. The composit our portfolio was *previsible*. We make an analogous definition in continuous ti

Definition 4.2.3 *Given a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, the stochastic process $\{X_t\}_t$* $\{\mathcal{F}_t\}_{t\geq 0}$-previsible *or* $\{\mathcal{F}_t\}_{t\geq 0}$-predictable *if X_t is $\mathcal{F}_{t-}$-measurable for all t wh*

$$\mathcal{F}_{t-} = \bigcup_{s<t} \mathcal{F}_s.$$

Remark: If $\{X_t\}_{t\geq 0}$ is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted and left continuous (so, in particular, i continuous) then it is automatically predictable.

In our Itô stochastic integrals the integrand will always be predictable.

Integrating simple functions

We have evaluated the Itô integral in just one special case, when the integrand is Brownian motion. We now extend our repertoire in the same way as in the cla setting by first considering the integral of simple functions. Throughout we as that $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion generating the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Definition 4.2.4 *A simple function is one of the form*

$$f(s, \omega) = \sum_{i=1}^{n} a_i(\omega)\mathbf{1}_{I_i}(s),$$

where

$$I_i = (s_i, s_{i+1}], \quad \bigcup_{i=1}^{n} I_i = (0, T], \quad I_i \cap I_j = \{\emptyset\} \text{ if } i \neq j$$

and, for each $i = 1, \ldots, n$, $a_i : \Omega \to \mathbb{R}$ *is an* $\mathcal{F}_{s_i}$*-measurable random variable with* $\mathbb{E}[a_i(\omega)^2] < \infty$.

Remark: We have temporarily abandoned our convention of not mentioning Ω. However, this notation has the advantage of capturing both of the key examples that we have in mind, namely a_i a function of W_{s_i} and a_i a function of $\{W_r\}_{0 \leq r \leq s_i}$. We continue to suppress dependence of $\{W_s\}_{0 \leq s \leq T}$ on ω in our notation. □

> **Warning:** We have defined simple functions to be $\{\mathcal{F}_t\}_{t \geq 0}$-predictable. Some texts would call such functions *simple predictable functions.*

If f is a simple function, then so is $f(s, \omega)\mathbf{1}_{[0,t]}(s)$. We define

$$\int_0^t f(s, \omega) dW_s = \int f(s, \omega)\mathbf{1}_{[0,t]}(s) dW_s.$$

Following (4.6),

$$\int_0^t f(s, \omega) dW_s \triangleq \sum_{i=1}^{n} a_i(\omega)\mathbf{1}_{[0,t]}(s_i)\left(W_{s_{i+1} \wedge t} - W_{s_i}\right).$$

Now, just as for classical integration theory, for a more general (predictable) function, f, we find a sequence of simple functions $\{f^{(n)}\}_{n \geq 1}$ such that $f^{(n)} \to f$ as $n \to \infty$ and define the integral of f with respect to $\{W_s\}_{0 \leq s \leq t}$ to be $\lim_{n \to \infty} \int f^{(n)}(s, \omega) dW_s$ if this limit exists. This won't work for arbitrary f. The next lemma helps identify a space of functions for which we can reasonably expect a nice limit.

Lemma 4.2.5 *Suppose that* f *is a simple function; then*

1 *the process*

$$\int_0^t f(s, \omega) dW_s$$

is a continuous $\left(\mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}\right)$*-martingale,*

2

$$\mathbb{E}\left[\left(\int_0^t f(s, \omega) dW_s\right)^2\right] = \int_0^t \mathbb{E}[f(s, \omega)^2] ds,$$

3

$$\mathbb{E}\left[\sup_{t\leq T}\left(\int_0^t f(s,\omega)dW_s\right)^2\right] \leq 4\int_0^T \mathbb{E}\left[f(s,\omega)^2\right]ds.$$

Remark: The second assertion is the famous *Itô isometry*. It suggests that we be able to extend our definition of the integral over $[0, t]$ to predictable functioi that $\int_0^t \mathbb{E}[f(s,\omega)^2]ds < \infty$. Moreover, for such functions, all three assertions remain true. In fact, one can extend the definition a little further, but the integr then fail to be a martingale and this property will be important to us.

Before proving Lemma 4.2.5 we quote a famous result of Doob.

Theorem 4.2.6 (Doob's inequality) *If $\{M_t\}_{0\leq t\leq T}$ is a continuous martinga*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} M_t^2\right] \leq 4\mathbb{E}\left[M_T^2\right].$$

The proof of this remarkable theorem can be found, for example, in Ch Williams (1990).

Proof of Proof of Lemma 4.2.5.: The proof of the first assertion is Exercise 5 a third follows from the second by an application of Doob's inequality, so we c ourselves to proving the second statement.

We simplify notation by supposing that, in the notation of Definition 4.2.4

$$f(s,\omega)\mathbf{1}_{[0,t]}(s) = \sum_{i=1}^n a_i(\omega)\mathbf{1}_{I_i}(s)$$

where the intervals I_i are disjoint and $\bigcup_{i=1}^n I_i = (0, t]$. By our definition we

$$\int_0^t f(s,\omega)dW_s = \sum_{i=1}^n a_i(\omega)\left(W_{s_{i+1}} - W_s\right)$$

and so

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t f(s,\omega)dW_s\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^n a_i(\omega)\left(W_{s_{i+1}} - W_{s_i}\right)\right)^2\right]\\ &= \mathbb{E}\left[\sum_{i=1}^n a_i^2(\omega)\left(W_{s_{i+1}} - W_{s_i}\right)^2\right]\\ &\quad + 2\mathbb{E}\left[\sum_{i<j} a_i(\omega)a_j(\omega)\left(W_{s_{i+1}} - W_{s_i}\right)\left(W_{s_{j+1}} -\right.\right.\end{aligned}$$

Suppose that $j > i$; then by the tower property of conditional expectations

$$\begin{aligned}\mathbb{E}&\left[a_i(\omega)a_j(\omega)\left(W_{s_{i+1}}-W_{s_i}\right)\left(W_{s_{j+1}}-W_{s_j}\right)\right]\\ &=\mathbb{E}\left[a_i(\omega)a_j(\omega)\left(W_{s_{i+1}}-W_{s_i}\right)\mathbb{E}\left[\left(W_{s_{j+1}}-W_{s_j}\right)\middle|\,\mathcal{F}_{s_j}\right]\right]=0.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbb{E}\left[a_i^2(\omega)\left(W_{s_{i+1}}-W_{s_i}\right)^2\right] &= \mathbb{E}\left[a_i^2(\omega)\mathbb{E}\left[\left(W_{s_{i+1}}-W_{s_i}\right)^2\middle|\,\mathcal{F}_{s_i}\right]\right]\\ &= \mathbb{E}\left[a_i^2(\omega)\right](s_{i+1}-s_i).\end{aligned}$$

Substituting we obtain

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t f(s,\omega)dW_s\right)^2\right] &= \sum_{i=1}^{n}\mathbb{E}\left[a_i^2(\omega)\right](s_{i+1}-s_i)\\ &= \int_0^t \mathbb{E}\left[f(s,\omega)^2\right]ds\end{aligned}$$

as required. □

> **Notation:** We write $\mathcal{H}_T$ for the set of functions $f:\mathbb{R}_+\times\Omega\to\mathbb{R}$ for which $f(t,\omega)$ is $\{\mathcal{F}_t\}_{t\geq 0}$-predictable for $0\leq t\leq T$ and
>
> $$\int_0^T \mathbb{E}\left[f(s,\omega)^2\right]ds<\infty.$$

Construction of the Itô integral

This will be our class of integrable functions. We proceed as advertised: approximate a general $f\in\mathcal{H}_T$ by a sequence of simple functions, $\{f^{(n)}\}_{n\geq 1}$, and define

$$\int_0^t f(s,\omega)ds \triangleq \lim_{n\to\infty}\int_0^t f^{(n)}(s,\omega)dW_s.$$

The following theorem confirms that this really works.

Theorem 4.2.7 *Suppose that $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion and let $\{\mathcal{F}_t\}_{t\geq 0}$ denote its natural filtration. There exists a linear mapping, J, from $\mathcal{H}_T$ to the space of continuous $\left(\mathbb{P},\{\mathcal{F}_t\}_{t\geq 0}\right)$-martingales defined on $[0,T]$ such that*

1. *if f is simple and $t\leq T$,*

$$J(f)_t=\int_0^t f(s,\omega)dW_s,$$

2. *if $t\leq T$,*

$$\mathbb{E}\left[J(f)_t^2\right]=\int_0^t \mathbb{E}\left[f(s,\omega)^2\right]ds,$$

3

$$\mathbb{E}\left[\sup_{0\leq t\leq T} J(f)_t^2\right] \leq 4\int_0^T \mathbb{E}\left[f(s,\omega)^2\right] ds.$$

Proof: By Doob's inequality, the last part follows from the second once
that $\{J(f)_t\}_{0\leq t\leq T}$ is a $\mathbb{P}$-martingale. To define J and prove the first two as
we follow the approximation procedure outlined above.

Let $\{f^{(n)}\}_{n\geq 1}$ be a sequence of simple functions such that

$$\mathbb{E}\left[\int_0^t \left|f(s,\omega) - f^{(n)}(s,\omega)\right|^2 ds\right] \to 0 \quad \text{as } n\to\infty.$$

Since the difference of two simple functions is simple, by Lemma 4.2.5

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T}\left(J(f^{(n)})_t - J(f^{(m)})_t\right)^2\right] &\leq 4\int_0^T \mathbb{E}\left[\left|f^{(n)}(s,\omega) - f^{(m)}(s,\right.\right. \\ &\to 0 \quad \text{as } n, m\to\infty.\end{aligned}$$

We now define $J(f)_t$ to be $\lim_{n\to\infty} J(f^{(n)})_t$. From (4.7) the limit exists u
for $0\leq t\leq T$, except possibly on a set of $\mathbb{P}$-probability zero, where we set
be identically equal to zero. Moreover,

$$\mathbb{E}\left[J(f)_t^2\right] = \lim_{n\to\infty}\mathbb{E}\left[J(f^{(n)})_t^2\right] = \lim_{n\to\infty}\int_0^t \mathbb{E}\left[f^{(n)}(s,\omega)^2\right]ds = \int_0^t \mathbb{E}\left[f(s,\right.$$

It remains to check the martingale property.

Now by Jensen's inequality (stated in Exercise 16 of Chapter 2), which
equally well for conditional expectations,

$$\begin{aligned}\mathbb{E}\left[\left|\mathbb{E}\left[J(f^{(n)})_t\middle|\mathcal{F}_s\right] - \mathbb{E}\left[J(f)_t\middle|\mathcal{F}_s\right]\right|^2\right] &= \mathbb{E}\left[\mathbb{E}\left[J(f^{(n)})_t - J(f)_t\middle|\mathcal{F}\right.\right. \\ &\leq \mathbb{E}\left[\mathbb{E}\left[\left|J(f^{(n)})_t - J(f)_t\right|^2\right.\right. \\ &= \mathbb{E}\left[\left|J(f^{(n)})_t - J(f)_t\right|^2\right] \\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

So using

$$J(f^{(n)})_s = \mathbb{E}\left[J(f^{(n)})_t\middle|\mathcal{F}_s\right]$$

and taking limits

$$\mathbb{E}\left[\left(J(f)_s - \mathbb{E}\left[J(f)_t|\mathcal{F}_s\right]\right)^2\right] = 0.$$

This implies

$$J(f)_s = \mathbb{E}\left[J(f)_t|\mathcal{F}_s\right] \quad \text{with } \mathbb{P}\text{-probability one.}$$

This is almost the martingale property but we want to remove the 'almost s
qualification. To do this choose a version of $J(f)$ such that the martingale pr

holds for all $s, t \in \mathbb{Q}$ (we can do this by redefining $J(f)$ on a set of $\mathbb{P}$-measure zero). Since $J(f)$ is a uniform limit of continuous functions (or identically zero) it is continuous and so with this definition the martingale property holds for all s, t as required. □

Definition 4.2.8 *For $f \in \mathcal{H}_T$, we write*

$$J(f)_t = \int_0^t f(s,\omega)dW_s$$

and call this quantity the Itô stochastic integral *of f with respect to $\{W_t\}_{t\geq 0}$.*

Notice that $J(f)$ really does agree with the prescription (4.6) except possibly on a set of $\mathbb{P}$-probability zero.

Other integrators

We have defined the stochastic integral with respect to Brownian motion. An easy extension is to any process $\{X_t\}_{t\geq 0}$ that can be written as $X_t = W_t + A_t$ where $\{W_t\}_{t\geq 0}$ is Brownian motion and $\{A_t\}_{t\geq 0}$ is a continuous process of bounded variation. In that case we can define the integral with respect to $\{X_t\}_{t\geq 0}$ as the sum of two parts: the integral with respect to the Brownian motion plus that with respect to $\{A_t\}_{t\geq 0}$. The latter exists in the classical sense. We can also replace Brownian motion by other martingales and that is our next goal.

Suppose that $\{M_t\}_{t\geq 0}$ is a continuous $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale with $\mathbb{E}\left[M_t^2\right] < \infty$ for each $t > 0$. By analogy with the Itô integral with respect to Brownian motion, for a simple function

$$f(s,\omega) = \sum_{i=1}^n a_i(\omega) 1_{I_i}(s),$$

we define

$$\int f(s,\omega)dM_s \triangleq \sum_{i=1}^n a_i(\omega)\left(M_{s_{i+1}} - M_{s_i}\right).$$

Passing to limits we can then define

$$J^M(f)_t = \int_0^t f(s,\omega)dM_s$$

for all $f \in \mathcal{H}_T^M$ where $\mathcal{H}_T^M$ is the set of predictable functions $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^T \mathbb{E}\left[f(s,\omega)^2\right] d[M]_s < \infty.$$

By redefining $J^M(f)$ to be zero on a set of $\mathbb{P}$-measure zero if necessary, we obtain the following analogue of Theorem 4.2.7.

Theorem 4.2.9 *Assume that $\{M_t\}_{t\geq 0}$ is a bounded continuous $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale with $\mathbb{E}\left[M_t^2\right] < \infty$ for each $t > 0$. Then there exists a linear mapping J^M from $\mathcal{H}_T^M$ to the space of continuous $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingales defined on $[0, T]$ such that*

1 *if f is simple, then*

$$J^M(f)_t = \int_0^t f(s,\omega)dM_s = \int f(s,\omega)1_{[0,t]}(s)dM_s,$$

as defined above,

2 *if $t \le T$*

$$\mathbb{E}\left[J^M(f)_t^2\right] = \mathbb{E}\left[\int_0^t f(s,\omega)^2 d[M]_s\right],$$

where $\{[M]_t\}_{t\ge 0}$ is the quadratic variation process associated with $\{M_t\}_{t\ge 0}$, an

3

$$\mathbb{E}\left[\sup_{0\le t\le T} J^M(f)_t^2\right] \le 4\mathbb{E}\left[\int_0^T f(s,\omega)^2 d[M]_s\right].$$

Except possibly on a set of $\mathbb{P}$-probability zero,

$$\int_0^t f(s,\omega)dM_s = \lim_{\delta(\pi)\to 0} \sum_{i=0}^{N(\pi)-1} f(s_i,\omega)\left(M_{s_{i+1}} - M_{s_i}\right).$$

We can now extend the definition still further to define the integral with respe any process $\{X_t\}_{t\ge 0}$ that can be written as $X_t = M_t + A_t$ for a continuous martin $\{M_t\}_{t\ge 0}$ (with $\mathbb{E}\left[M_t^2\right] < \infty$) and a process $\{A_t\}_{t\ge 0}$ of bounded variation.

We'll exploit this greater generality in proving Lemma 4.2.11, a useful result tells us that martingales of bounded variation are constant.

Definition 4.2.10 *Suppose that $\{M_t\}_{t\ge 0}$ is a continuous martingale and $\{A_t$ is a process of bounded variation; then the process $\{X_t\}_{t\ge 0}$ defined by $X_t = M_t$ - is said to be a* semimartingale.

A continuous semimartingale is any process that can be decomposed in this wa we insist that $A_0 = 0$ then the decomposition is unique.

Warning: Strictly we should replace 'martingale' by 'local martingale' i Definition 4.2.10. See, for example, Ikeda & Watanabe (1989) or Chung & Williams (1990) for a more general treatment.

Lemma 4.2.11 *Let $\{A_t\}_{0\le t\le T}$ be a continuous $(\mathbb{P}, \{\mathcal{F}_t\}_{0\le t\le T})$-martingale $\mathbb{E}\left[A_t^2\right] < \infty$ for each $0 \le t \le T$. If $\{A_t\}_{0\le t\le T}$ has bounded variation on* $[0,T]$

$$\mathbb{P}\left[A_t = A_0, \forall t \in [0,T]\right] = 1.$$

Proof: Let $\hat{A}_t = A_t - A_0$. Since $\{\hat{A}_t\}_{0\le t\le T}$ is a continuous process of bour variation we can define the integral $\int_0^t \hat{A}_s d\hat{A}_s$ in the classical way, and by

Fundamental Theorem of Calculus,

$$\hat{A}_t^2 - \hat{A}_0^2 = \hat{A}_t^2 = 2\int_0^t \hat{A}_s d\hat{A}_s.$$

The integral will be the same whether viewed as a classical or as a stochastic integral and so Theorem 4.2.9 tells us that it is a martingale and hence

$$\mathbb{E}\left[2\int_0^t \hat{A}_s d\hat{A}_s\right] = 0.$$

Thus $\mathbb{E}\big[\hat{A}_t^2\big] = 0$ for all t and so by continuity of $\{\hat{A}_t\}_{0\le t\le T}$, $\mathbb{P}\big[\hat{A}_t = 0, \forall t \in [0,T]\big] = 1$ as required. □

4.3 Itô's formula

Having made some sense of the stochastic integral, we are now in a position to establish some of the rules of Itô stochastic calculus. We begin with the chain rule and some of its ramifications.

The stochastic chain rule

Theorem 4.3.1 (Itô's formula) *For f such that the partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ exist and are continuous and $\frac{\partial f}{\partial x} \in \mathcal{H}$, almost surely for each t we have*

$$\begin{aligned} f(t, W_t) - f(0, W_0) &= \int_0^t \frac{\partial f}{\partial x}(s, W_s)dW_s + \int_0^t \frac{\partial f}{\partial s}(s, W_s)ds \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, W_s)ds. \end{aligned}$$

Notation: Often one writes Itô's formula in differential notation as

$$df(t, W_t) = f'(t, W_t)dW_t + \dot{f}(t, W_t)dt + \frac{1}{2}f''(t, W_t)dt.$$

Outline of proof: To avoid too many cumbersome formulae, suppose that $\dot{f}_t \equiv 0$. (The proof extends without difficulty to the general case.) The formula then becomes

$$f(W_t) - f(W_0) = \int_0^t \frac{\partial f}{\partial x}(W_s)dW_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(W_s)ds. \tag{4.8}$$

Let π be a partition of $[0,t]$ and as usual write t_i, t_{i+1} for the endpoints of a generic interval. Then

$$f(W_t) - f(W_0) = \sum_{j=0}^{N(\pi)-1}\left(f(W_{t_{j+1}}) - f(W_{t_j})\right).$$

We apply Taylor's Theorem on each interval of the partition.

$$f(W_t) - f(W_0) = \sum_{j=0}^{N(\pi)-1} f'(W_{t_j})\left(W_{t_{j+1}} - W_{t_j}\right) + \frac{1}{2}\sum_{j=0}^{N(\pi)-1} f''(\xi_j)\left(W_{t_{j+1}} - W\right.$$

for some points $\xi_j \in \left[W_{t_j} \wedge W_{t_{j+1}}, W_{t_j} \vee W_{t_{j+1}}\right]$. By continuity of the Brow path, we can write this as

$$f(W_t) - f(W_0) = \sum_{j=0}^{N(\pi)-1} f'(W_{t_j})(W_{t_{j+1}} - W_{t_j}) + \frac{1}{2}\sum_{j=0}^{N(\pi)-1} f''(W_{\eta_j})(W_{t_{j+1}} - W$$

where $\eta_j \in [t_j, t_{j+1}]$. We rewrite the second term as

$$\frac{1}{2}\sum_{j=0}^{N(\pi)-1}\left(f''(W_{t_j}) + \epsilon_j\right)\left(W_{t_{j+1}} - W_{t_j}\right)^2,$$

where $\epsilon_j = f''(W_{\eta_j}) - f''(W_{t_j})$.

A special case: Suppose that $\frac{\partial^2 f}{\partial x^2}$ is bounded. Then, for each fixed T, since r $\frac{\partial^2 f}{\partial x^2}(W_r)$ is uniformly continuous on $[0, T]$, $\sup_j \epsilon_j \to 0$ as the mesh of the part tends to zero. Now we mimic our proof of Theorem 4.2.1.

$$\begin{aligned}
&\mathbb{E}\left[\left|\sum_{j=0}^{N(\pi)-1} f''(W_{t_j})\left(\left(W_{t_{j+1}} - W_{t_j}\right)^2 - \left(t_{j+1} - t_j\right)\right)\right|^2\right] \\
&\quad = \mathbb{E}\left[\sum_{j=0}^{N(\pi)-1}(f''(W_{t_j}))^2\left(\left(W_{t_{j+1}} - W_{t_j}\right)^2 - \left(t_{j+1} - t_j\right)\right)^2\right] \\
&\quad\quad + 2\mathbb{E}\left[\sum_{0\le i<j\le N(\pi)-1} f''(W_{t_i})f''(W_{t_j})\left(\left(W_{t_{i+1}} - W_{t_i}\right)^2 - (t_{i+1} - t_i)\right)\right. \\
&\quad\quad\quad \left.\times\left(\left(W_{t_{j+1}} - W_{t_j}\right)^2 - \left(t_{j+1} - t_j\right)\right)\right].
\end{aligned}$$

Exactly as before, conditioning on $\mathcal{F}_{t_j}$ and using the tower property of conditi expectations, coupled now with boundedness of $\frac{\partial^2 f}{\partial x^2}$, shows that the right hand of (4.9) tends to zero as the mesh of the partition tends to zero.

Finally, using that

$$\sum_{j=0}^{N(\pi)-1}\frac{\partial f}{\partial x}(W_{t_j})\left(W_{t_{j+1}} - W_{t_j}\right) \to \int_0^t \frac{\partial f}{\partial x}(W_s)dW_s,$$

and exploiting continuity, we see that if $\frac{\partial^2 f}{\partial x^2}$ is bounded, the formula (4.8) h except possibly on a set of $\mathbb{P}$-measure zero.

The general case: In order to drop the assumption that $\frac{\partial^2 f}{\partial x^2}$ is bounded, we can use what is called a 'localising sequence'. Let

$$\tau_n = \inf\{t \geq 0 : |W_t| > n\}.$$

Then replacing $\{W_s\}_{s\geq 0}$ by $\{W_{s\wedge\tau_n}\}_{s\geq 0}$, since $\frac{\partial^2 f}{\partial x^2}$ is continuous, $\left\{\frac{\partial^2 f}{\partial x^2}(W_{s\wedge\tau_n})\right\}_{s\geq 0}$ is uniformly bounded. Our proof goes through with $\{W_s\}_{s\geq 0}$ replaced by $\{W_{s\wedge\tau_n}\}_{s\geq 0}$ throughout. (Note that we need the fact that τ_n is a stopping time to make this work.) The full result then follows by letting $n \to \infty$. □

For full details of the proof of Itô's formula, see, for example, Ikeda & Watanabe (1989) or Chung & Williams (1990).

Example 4.3.2 *Use Itô's formula to compute* $\mathbb{E}[W_t^6]$.

Solution: Let us define $\{Z_t\}_{t\geq 0}$ by $Z_t = W_t^6$. Then by Itô's formula

$$dZ_t = 6W_t^5 dW_t + 15W_t^4 dt,$$

and, of course, $Z_0 = 0$. In integrated form,

$$Z_t - Z_0 = \int_0^t 6W_s^5 dW_s + \int_0^t 15W_s^4 ds.$$

The expectation of the stochastic integral vanishes (by the martingale property) and so

$$\mathbb{E}[Z_t] = \int_0^t 15\mathbb{E}\left[W_s^4\right] ds.$$

Now from Exercise 5 of Chapter 3 (or Exercise 9 of this chapter), $\mathbb{E}\left[W_s^4\right] = 3s^2$, and so substituting

$$\mathbb{E}\left[W_t^6\right] = \mathbb{E}\left[Z_t\right] = 15\int_0^t 3s^2 ds = 15t^3.$$

□

Geometric Brownian motion

The basic reference model for stock prices in continuous time is *geometric Brownian motion*, defined by

$$S_t = S_0 \exp\left(\nu t + \sigma W_t\right), \tag{4.10}$$

where, as usual, $\{W_t\}_{t\geq 0}$ is a standard $\mathbb{P}$-Brownian motion. Applying Itô's formula,

$$dS_t = \sigma S_t dW_t + \left(\nu + \frac{1}{2}\sigma^2\right) S_t dt.$$

This expression is called the *stochastic differential equation* for S_t. It is common to write such symbolic equations even though it is the *integral* equation that makes sense.

Lemma 4.3.3 *Writing $\mu = \nu + \sigma^2/2$, the geometric Brownian motion process defined above is a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale if and only if $\mu = 0$. Moreover*

$$\mathbb{E}[S_t] = S_0 \exp(\mu t).$$

Proof: Writing the stochastic differential equation for geometric Brownian motion in integrated form,

$$S_t = S_0 + \int_0^t \left(\nu + \frac{1}{2}\sigma^2\right) S_s ds + \int_0^t \sigma S_s dW_s. \tag{4.11}$$

Notice that $\{S_t\}_{t\geq 0}$ is a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-semimartingale. Proving that it is a martingale if and only if $\mu = 0$ amounts to proving uniqueness of the decomposition of a semimartingale into a martingale and a bounded variation process in this special case.

Suppose $\mu = 0$: The classical integral in (4.11) then vanishes. Since $\{S_t\}_{t\geq 0}$ inherits continuity from the Brownian motion and by (4.10) it is adapted, by the remark after Definition 4.2.3 it is predictable and so by Theorem 4.2.7 the stochastic integral in (4.11) is a martingale.

Suppose that $\{S_t\}_{t\geq 0}$ is a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale: Since the difference of two martingales (with respect to the same filtration) is again a martingale, we see that $\{A_t\}_{t\geq 0}$ defined by

$$A_t = S_t - S_0 - \int_0^t \sigma S_s dW_s = \int_0^t \mu S_s ds$$

is a martingale. This classical integral has bounded variation and so by Lemma 4.2.11 with probability one it is equal to $A_0 = 0$. Since $S_s > 0$ for all s, it follows that $\mu = 0$.

The expectation: To verify the second claim, we take expectations in (4.11) and use once again that the stochastic integral term is a mean zero martingale to obtain

$$\mathbb{E}[S_t] - S_0 = \mathbb{E}\left[\int_0^t \mu S_s ds\right] = \int_0^t \mu \mathbb{E}[S_s]\, ds.$$

(The interchange of time integral and expectation is justified by classical integration theory.) Solving this integral equation gives

$$\mathbb{E}[S_t] = S_0 \exp(\mu t)$$

as required. □

Itô's formula for geometric Brownian motion

It is convenient to have a version of Itô's formula that allows us to work directly with $\{S_t\}_{t\geq 0}$ (so that we can write down a stochastic differential equation for $f(t, S_t)$). We now know how to make our original heuristic calculations rigorous, so with a

clear conscience we proceed as follows:

$$
\begin{aligned}
f(t+\delta t, S_{t+\delta t}) - f(t, S_t) &\approx \dot{f}(t, S_t)\delta t + f'(t, S_t)\left(S_{t+\delta t} - S_t\right) \\
&\quad + \frac{1}{2} f''(t, S_t)\left(S_{t+\delta t} - S_t\right)^2 \\
&\approx \dot{f}(t, S_t)dt + f'(t, S_t)dS_t \\
&\quad + \frac{1}{2} f''(t, S_t)\left\{\sigma^2 S_t^2 dW_t^2 + \mu^2 S_t^2 dt^2 + 2\sigma\mu S_t^2 dW_t dt\right\}.
\end{aligned}
$$

That is

$$
df(t, S_t) = \dot{f}(t, S_t)dt + f'(t, S_t)dS_t + \frac{1}{2} f''(t, S_t)\sigma^2 S_t^2 dt,
$$

where we have used the multiplication table

$\times$	dW_t	dt
dW_t	dt	0
dt	0	0

Writing this version of Itô's formula in integrated form gives then

$$
\begin{aligned}
f(t, S_t) - f(0, S_0) &= \int_0^t \frac{\partial f}{\partial u}(u, S_u)du + \int_0^t \frac{\partial f}{\partial x}(u, S_u)dS_u \\
&\quad + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(u, S_u)\sigma^2 S_u^2 du \\
&= \int_0^t \frac{\partial f}{\partial u}(u, S_u)du + \int_0^t \frac{\partial f}{\partial x}(u, S_u)\sigma S_u dW_u \\
&\quad + \int_0^t \frac{\partial f}{\partial x}(u, S_u)\mu S_u du + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(u, S_u)\sigma^2 S_u^2 du.
\end{aligned}
$$

> **Warning:** Be aware that the stochastic integral with respect to $\{S_t\}_{t\geq 0}$ will not be a martingale with respect to the probability $\mathbb{P}$ under which $\{W_t\}_{t\geq 0}$ is a martingale except in the special case when $\{S_t\}_{t\geq 0}$ is a $\mathbb{P}$-martingale, that is when $\mu = 0$. To actually *calculate* it is often wise to separate the martingale part by expanding the 'stochastic' integral as in the last line.

Example 4.3.4 *Suppose that $\{S_t\}_{t\geq 0}$ is a geometric Brownian motion satisfying*

$$
dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{4.12}
$$

where $\{W_t\}_{t\geq 0}$ is standard Brownian motion under $\mathbb{P}$. Calculate $\mathbb{E}\left[S_t^n\right]$ for $n \in \mathbb{N}$.

Solution: From our calculations above,

$$
S_t = \exp\left(\nu t + \sigma W_t\right)
$$

where $\nu = \mu - \sigma^2/2$. Thus $\{S_t^n\}_{t\geq 0}$ is also a geometric Brownian motion with parameters $\nu^{(n)} = n\nu$ and $\sigma^{(n)} = n\sigma$. By Lemma 4.3.3 we then have

$$\mathbb{E}\left[S_t^n\right] = S_0^n \exp\left(\left(n\nu + \frac{1}{2}n^2\sigma^2\right)t\right) = S_0^n \exp\left(\left(n\mu + \frac{1}{2}n(n-1)\sigma^2\right)t\right). \tag{4.13}$$

To gain some more practice with stochastic calculus, suppose that we did not know how to express the solution to the stochastic differential equation (4.12) explicitly as a function of $\{W_t\}_{t\geq 0}$. An alternative approach to calculating $\mathbb{E}\left[S_t^n\right]$, which we now sketch, is to apply Itô's lemma.

$$\begin{aligned} d\left(S_t^n\right) &= nS_t^{n-1}dS_t + \frac{1}{2}n(n-1)S_t^{n-2}\sigma^2 S_t^2 dt \\ &= S_t^n\left(n\mu + \frac{1}{2}n(n-1)\sigma^2\right)dt + n\sigma S_t^n dW_t. \end{aligned}$$

Writing this equation in integrated form and taking expectations yields

$$\mathbb{E}\left[S_t^n\right] - \mathbb{E}\left[S_0^n\right] = \int_0^t \left(n\mu + \frac{1}{2}n(n-1)\sigma^2\right)\mathbb{E}\left[S_s^n\right]ds.$$

This leads us once again to the expression (4.13). □

Lévy's characterisation of Brownian motion

The Itô formula provides a quick route to a useful characterisation of Brownian motion due to Lévy. We have seen that Brownian motion is a martingale. It is useful to be able to identify when a martingale is in fact Brownian motion.

Theorem 4.3.5 *Let $\{W_t\}_{0\leq t\leq T}$ be a continuous $(\mathbb{P}, \{\mathcal{F}_t\}_{0\leq t\leq T})$-martingale such that $W_0 = 0$ and $[W]_t = t$ for $0 \leq t \leq T$. Then $\{W_t\}_{0\leq t\leq T}$ is a $(\mathbb{P}, \{\mathcal{F}_t\}_{0\leq t\leq T})$-Brownian motion.*

Proof: We must check that for any $0 \leq s < t \leq T$, $W_t - W_s$ is normally distributed with mean zero and variance $t - s$ and is independent of $\mathcal{F}_s$.

Let

$$M_t^\theta \triangleq \exp\left(\theta W_t - \frac{\theta^2}{2}t\right).$$

Applying Itô's formula we see that $\{M_t\}_{0\leq t\leq T}$ is a $\left(\mathbb{P}, \{\mathcal{F}_t\}_{0\leq t\leq T}\right)$-martingale and so for $0 \leq s \leq t \leq T$,

$$\mathbb{E}\left[\left.\frac{M_t}{M_s}\right|\mathcal{F}_s\right] = 1.$$

Substituting and rearranging gives

$$\mathbb{E}\left[\exp\left(\theta\left(W_t - W_s\right)\right)\middle|\mathcal{F}_s\right] = \exp\left(\frac{1}{2}(t-s)\theta^2\right).$$

Since the normal distribution is characterised by its moment generating function, the result follows. □

Stochastic differential equations

Let us return to processes of the form $Z_t = f(t, W_t)$. Using Itô's formula

$$dZ_t = \left(\dot{f}(t, W_t) + \frac{1}{2}f''(t, W_t)\right)dt + f'(t, W_t)dW_t.$$

Suppose that f is invertible; then we may write this as

$$dZ_t = \mu(t, Z_t)dt + \sigma(t, Z_t)dW_t. \tag{4.14}$$

Definition 4.3.6 *An equation of the form (4.14) for some deterministic functions $\mu(t, x)$ and $\sigma(t, x)$ on $\mathbb{R}_+ \times \mathbb{R}$ is called a* stochastic differential equation *for* $\{Z_t\}_{t \geq 0}$.

It is often easier to write down a stochastic differential equation for $\{Z_t\}_{t \geq 0}$ than to produce the function $f(t, x)$ explicitly.

> **Warning:** Just as for (Newtonian) ordinary differential equations, in general a stochastic differential equation may not have a solution and, even if it does, the solution may not be unique.

If the functions $\mu(t, x)$ and $\sigma(t, x)$ are, for example, bounded and uniformly Lipschitz-continuous in x then a unique solution does exist, but these conditions are certainly not necessary (see, for example, Chung & Williams (1990) and Ikeda & Watanabe (1989) for more details).

Of course, we should really understand equation (4.14) in integrated form:

$$Z_t - Z_0 = \int_0^t \mu(s, Z_s)ds + \int_0^t \sigma(s, Z_s)dW_s.$$

It is left to the reader to justify the following version of the Itô formula.

Theorem 4.3.7 *If $Z_t = f(t, W_t)$ satisfies*

$$dZ_t = \sigma(t, Z_t)dW_t + \mu(t, Z_t)dt,$$

and

$$Y_t = g(t, Z_t),$$

for some twice differentiable functions f and g, then

$$dY_t = \dot{g}(t, Z_t)dt + g'(t, Z_t)dZ_t + \frac{1}{2}g''(t, Z_t)\sigma^2(t, Z_t)dt. \tag{4.15}$$

Remark: Notice that

$$M_t = Z_t - Z_0 - \int_0^t \mu(s, Z_s)ds$$

is a martingale with mean zero. From the Itô isometry, we know that its variance is

$$\mathbb{E}[M_t^2] = \mathbb{E}\left[\int_0^t \sigma(s, Z_s)^2 ds\right].$$

The expression $\sigma^2(t, Z_t)dt$ appearing in (4.15) is just $d[M]_t$ where $\{[M]_t\}_{t\geq 0}$ is the quadratic variation associated with $\{M_t\}_{t\geq 0}$.

If $\{Z_t\}_{t\geq 0}$ is defined by $Z_t = M_t + A_t$ where $\{M_t\}_{t\geq 0}$ is a continuous martingale with $\mathbb{E}\left[M_t^2\right] < \infty$ and $\{A_t\}_{t\geq 0}$ has bounded variation, then setting $Y_t = g(t, Z_t)$ we have

$$dY_t = \dot{g}(t, Z_t)dt + g'(t, Z_t)dZ_t + \frac{1}{2}g''(t, Z_t)d[M]_t.$$

In particular, by applying this to $Y_t = M_t^2$, one shows (Exercise 15) that $M_t^2 - [M]_t$ is a martingale. □.

Solving stochastic differential equations

Even when a stochastic differential equation has a unique solution it is rare to be able to express it in closed form as a function of Brownian motion. However, if this can be done, then Itô's formula provides a route to finding the solution.

Example 4.3.8 *Solve*

$$dX_t = X_t^3 dt - X_t^2 dW_t, \qquad X_0 = 1. \tag{4.16}$$

Solution: If $X_t = f(t, W_t)$, then

$$dX_t = \dot{f}(t, W_t)dt + f'(t, W_t)dW_t + \frac{1}{2}f''(t, W_t)dt,$$

and, substituting in (4.16),

$$dX_t = f(t, W_t)^3 dt - f(t, W_t)^2 dW_t.$$

Equating coefficients we obtain

$$-f(t, W_t)^2 = f'(t, W_t)$$

and

$$\dot{f}(t, W_t) + \frac{1}{2}f''(t, W_t) = f(t, W_t)^3,$$

which yields

$$f(t, x) = \frac{1}{x + c}$$

where c is a constant. Using the initial condition, we find

$$X_t = \frac{1}{W_t + 1}.$$

Notice that this solution blows up in finite time. □

4.4 Integration by parts and a stochastic Fubini Theorem

We shall need two more rules for manipulating stochastic integrals: the integration by parts formula and a 'stochastic Fubini Theorem'. The first is the product rule of stochastic differentiation, the second is used to justify interchange of order of stochastic and classical integrals.

Suppose that we have *two* stochastic differential equations:

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t,$$

$$dZ_t = \tilde{\mu}(t, Z_t)dt + \tilde{\sigma}(t, Z_t)dW_t.$$

We assume for now that these equations are driven by the *same* Brownian motion $\{W_t\}_{t\geq 0}$.

Consider the $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingales defined by

$$M_t^Y = \int_0^t \sigma(s, Y_s)dW_s \quad \text{and} \quad M_t^Z = \int_0^t \tilde{\sigma}(s, Z_s)dW_s,$$

with associated quadratic variation processes

$$[M^Y]_t = \int_0^t \sigma^2(s, Y_s)ds \quad \text{and} \quad [M^Z]_t = \int_0^t \tilde{\sigma}^2(s, Z_s)ds.$$

Covariation

Clearly $\{M_t^Y\}_{t\geq 0}$ and $\{M_t^Z\}_{t\geq 0}$ are not independent. One way of quantifying the dependence between them is through their covariance. Evidently $\{(M_t^Y + M_t^Z)\}_{t\geq 0}$ and $\{(M_t^Y - M_t^Z)\}_{t\geq 0}$ are also $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingales with quadratic variation

$$[M^Y \pm M^Z]_t = \int_0^t (\sigma(s, Y_s) \pm \tilde{\sigma}(s, Z_s))^2 \, ds.$$

We're interested in the process $\{M_t^Y M_t^Z\}_{t\geq 0}$. We attack it via polarisation:

$$M_t^Y M_t^Z = \frac{1}{4}\left(\left(M_t^Y + M_t^Z\right)^2 - \left(M_t^Y - M_t^Z\right)^2\right).$$

Taking expectations,

$$\begin{aligned}\mathbb{E}\left[M_t^Y M_t^Z\right] &= \frac{1}{4}\mathbb{E}\left[\int_0^t (\sigma(s, Y_s) + \tilde{\sigma}(s, Z_s))^2 \, ds - \int_0^t (\sigma(s, Y_s) - \tilde{\sigma}(s, Z_s))^2 \, ds\right] \\ &= \mathbb{E}\left[\int_0^t \sigma(s, Y_s)\tilde{\sigma}(s, Z_s)ds\right].\end{aligned}$$

We write

$$\left[M^Y, M^Z\right]_t = \int_0^t \sigma(s, Y_s)\tilde{\sigma}(s, Z_s)ds.$$

Since for a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale $\{M_t^2 - [M]_t\}_{t\geq 0}$ is a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale (Exercise 15), again exploiting polarisation we see that $\{M_t^Y M_t^Z - [M^Y, M^Z]_t\}_{t\geq 0}$ is a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale.

More generally, we could consider stochastic differential equations driven by *different* (but not necessarily independent) Brownian motions. For example, suppose that

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t,$$
$$dZ_t = \tilde{\mu}(t, Z_t)dt + \tilde{\sigma}(t, Z_t)d\tilde{W}_t,$$

where $\mathbb{E}\left[(W_t - W_s)(\tilde{W}_t - \tilde{W}_s)\right] = \rho(t-s)$ for some $0 < \rho < 1$. This means that the Brownian motions driving the two equations are positively correlated, an increase in one tends to be associated with an increase in the other, but they are not identical. We define M_t^Y and M_t^Z exactly as before and again study $M_t^Y M_t^Z$ via polarisation. Writing

$$\left[M_t^Y M_t^Z\right] = \frac{1}{4}\left(\left[M^Y + M^Z\right]_t - \left[M^Y - M^Z\right]_t\right),$$

and using the definition of quadratic variation,

$$\left[M^Y, M^Z\right]_t = \lim_{\delta(\pi)\to 0} \sum_{j=0}^{N(\pi)-1} \left(M_{t_{j+1}}^Y - M_{t_j}^Y\right)\left(M_{t_{j+1}}^Z - M_{t_j}^Z\right).$$

In our example, provided σ and $\tilde{\sigma}$ are continuous say, we have

$$\left[M^Y, M^Z\right]_t = \lim_{\delta(\pi)\to 0} \sum_{j=0}^{N(\pi)-1} \sigma(t_j, Y_{t_j})\tilde{\sigma}(t_j, Z_{t_j})\left(W_{t_{j+1}} - W_{t_j}\right)\left(\tilde{W}_{t_{j+1}} - \tilde{W}_{t_j}\right),$$

and by mimicking the argument of the proof of Theorem 4.2.1 we obtain

$$\left[M^Y, M^Z\right]_t = \int_0^t \sigma(s, Y_s)\tilde{\sigma}(s, Z_s)\rho ds.$$

Definition 4.4.1 *For continuous* $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$*-martingales* $\{M_t\}_{t\geq 0}$ *and* $\{N_t\}_{t\geq 0}$

$$[M, N]_t \triangleq \frac{1}{4}\left([M+N]_t - [M-N]_t\right)$$

is called the mutual variation *or* covariation *process of M and N.*

Of course $\{[M, M]_t\}_{t\geq 0}$ is just the quadratic variation process associated with $\{M_t\}_{t\geq 0}$. In the notation of Definition 4.4.1, if we write $\delta(\pi)$ for a generic partition of $[0, t]$ then

$$[M, N]_t = \lim_{\delta(\pi)\to 0} \sum_{j=0}^{N(\pi)-1} \left(M_{t_{j+1}} - M_{t_j}\right)\left(N_{t_{j+1}} - N_{t_j}\right).$$

We now have the technology required for manipulating products of semimartingales.

Theorem 4.4.2 (Integration by parts) *If $Y_t = M_t^Y + A_t^Y$ and $Z_t = M_t^Z + A_t^Z$, where $\{M_t^Y\}_{t\geq 0}$ and $\{M_t^Z\}_{t\geq 0}$ are continuous $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingales and $\{A_t^Y\}_{t\geq 0}$ and $\{A_t^Z\}_{t\geq 0}$ are continuous processes of bounded variation, then*

$$d(Y_t Z_t) = Y_t dZ_t + Z_t dY_t + d\left[M^Y, M^Z\right]_t.$$

Proof: We apply the Itô formula to $(Y_t + Z_t)^2$, Y_t^2 and Z_t^2, and subtract the second two from the first to obtain the result. □

Example 4.4.3 *Suppose that the Sterling price of an asset follows the stochastic differential equation*

$$dS_t = \mu_1 S_t dt + \sigma_1 S_t dW_t$$

and the dollar cost of £1 at time t is E_t where

$$dE_t = \mu_2 E_t dt + \sigma_2 E_t d\tilde{W}_t.$$

Here $\{W_t\}_{t\geq 0}$ and $\{\tilde{W}_t\}_{t\geq 0}$ are $\mathbb{P}$-Brownian motions with

$$\mathbb{E}\left[\left(W_t - W_s\right)\left(\tilde{W}_t - \tilde{W}_s\right)\right] = \rho(t-s)$$

for some constant $\rho > 0$.

If the riskless interest rate is r in the UK and s in the USA, find the stochastic differential equation for the discounted asset price in the Sterling and dollar markets respectively. For what values of the parameters are the discounted asset prices martingales in each market?

Solution: In the Sterling market, write $\{\tilde{S}\}_{t\geq 0}$ for the discounted stock price. That is $\tilde{S}_t = e^{-rt} S_t$. Since the function e^{-rt} has bounded variation, our integration by parts formula gives

$$\begin{aligned} d\tilde{S}_t &= -re^{-rt} S_t dt + e^{-rt} dS_t \\ &= (\mu_1 - r)\,\tilde{S}_t dt + \sigma_1 \tilde{S}_t dW_t. \end{aligned}$$

In Sterling markets, this is the stochastic differential equation governing the discounted asset price. The solution is a martingale if and only if $\mu_1 = r$.

Let us write $\{X_t\}_{t\geq 0}$ for the dollar price of the asset. Then $X_t = E_t S_t$ and, again by integration by parts,

$$\begin{aligned} dX_t &= E_t dS_t + S_t dE_t + \sigma_1\sigma_2 E_t S_t \rho dt \\ &= (\mu_1 + \mu_2 + \rho\sigma_1\sigma_2)\, X_t dt + \sigma_1 X_t dW_t + \sigma_2 X_t d\tilde{W}_t. \end{aligned}$$

The discounted asset price in the dollar market, denoted by $\{\tilde{X}_t\}_{t\geq 0}$, then follows

$$d\tilde{X}_t = (\mu_1 + \mu_2 + \rho\sigma_1\sigma_2 - s)\,\tilde{X}_t dt + \sigma_1 \tilde{X}_t dW_t + \sigma_2 \tilde{X}_t d\tilde{W}_t.$$

The discounted price in the dollar market is a martingale if and only if

$$\mu_1 + \mu_2 + \rho\sigma_1\sigma_2 - s = 0.$$

Notice that it is perfectly possible for the discounted asset price to be a martingale in one market but not the other. It is important to keep this in mind when valuing options in the foreign exchange market (see §5.3) or when valuing quantos (see §7.2). □

A stochastic Fubini Theorem

We finish this section with one more useful result. This is a 'stochastic Fubini Theorem' that will allow us to interchange the order of a stochastic and a classical integral. We shall need this result in valuation of certain path-dependent exotic options in Exercise 23 of Chapter 6.

We state the theorem in a very special form that will be sufficient for our needs. For a more general version see, for example, Ikeda & Watanabe (1989).

Theorem 4.4.4 *Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space and let $\{M_t\}_{t\geq 0}$ be a continuous $\left(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}\right)$-martingale with $M_0 = 0$. Suppose that $\Phi(t, r, \omega) : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is a bounded $\{\mathcal{F}_t\}_{t\geq 0}$-predictable random variable. Then for each fixed $T > 0$,*

$$\int_0^t \int_{\mathbb{R}} \Phi(s, r, \omega)\, 1_{[0,T]}(r) dr dM_s = \int_{\mathbb{R}} \int_0^t \Phi(s, r, \omega)\, 1_{[0,T]}(r) dM_s dr.$$

Example 4.4.5 *Suppose that $\{W_t\}_{t\geq 0}$ is a $\left(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}\right)$-Brownian motion. Evaluate the mean and variance of*

$$Y_t \triangleq \int_0^t W_r dr.$$

Solution: The classical Fubini Theorem tells us that

$$\mathbb{E}[Y_t] = \mathbb{E}\left[\int_0^t W_r dr\right] = \int_0^t \mathbb{E}[W_r]\, dr = 0.$$

The difficulty is to calculate $\mathbb{E}\left[Y_t^2\right]$ and this is where we exploit our stochastic Fubini Theorem with $\Phi(s, r, \omega) = \mathbf{1}_{[0,r]}(s)$.

$$\begin{aligned} Y_t = \int_0^t W_r dr &= \int_0^t \int_0^r 1 dW_s dr \\ &= \int_0^t \int_s^t 1 dr dW_s \quad \text{(Fubini's Theorem)} \\ &= \int_0^t (t-s) dW_s. \end{aligned}$$

Now using the Itô isometry we can calculate $\mathbb{E}\left[Y_t^2\right]$ to be

$$\mathbb{E}\left[Y_t^2\right] = \mathbb{E}\left[\left(\int_0^t (t-s) dW_s\right)^2\right] = \int_0^t (t-s)^2 ds = \frac{1}{3}t^3.$$

□

4.5 The Girsanov Theorem

In order to price and hedge in the Black–Scholes framework we shall need two fundamental results. The first will allow us to change probability measure so that the discounted asset prices are martingales. Recall that in our discrete time world, once

we had such a *martingale measure*, the pricing of options was reduced to calculating expectations under that measure. In the continuous world it will no longer be possible to find the martingale measure by linear algebra. Nonetheless, before stating the continuous time result, we revert to our binomial trees for guidance.

Changing probability on a binomial tree

We use the notation of Chapter 2. Suppose that, under the probability measure $\mathbb{P}$, if the value of an asset at time $i\delta t$ is known to be S_i then its value at time $(i+1)\delta t$ is $S_i u$ with probability p and it is $S_i d$ with probability $1-p$.

As we saw in Chapter 2, if we let $\mathbb{Q}$ be the probability measure under which the probability of an up jump is $q = (1-d)/(u-d)$ and of a down jump is $(u-1)/(u-d)$, then the process $\{S_i\}_{0\le i\le N}$ is a $\mathbb{Q}$-martingale.

We can regard the measure $\mathbb{Q}$ as a *reweighting* of the measure $\mathbb{P}$. For example, consider a path $S_0, S_1, \dots, S_i$ through the tree. Its probability under $\mathbb{P}$ is $p^{N(i)}(1-p)^{i-N(i)}$, where $N(i)$ is the number of up jumps that the path makes. Under $\mathbb{Q}$ its probability is $L_i p^{N(i)}(1-p)^{i-N(i)}$ where

$$L_i = \left(\frac{q}{p}\right)^{N(i)}\left(\frac{1-q}{1-p}\right)^{i-N(i)}.$$

Evidently L_i depends on the path that the stochastic process takes through the tree and can itself be thought of as a stochastic process adapted to the filtration $\{\mathcal{F}_i\}_{1\le i\le N}$. Moreover, L_i/L_{i-1} is q/p if $S_i/S_{i-1} = u$ and is $(1-q)/(1-p)$ if $S_i/S_{i-1} = d$, so that

$$\mathbb{E}^{\mathbb{P}}\left[L_i \middle| \mathcal{F}_{i-1}\right] = L_{i-1}\left(p\frac{q}{p} + (1-p)\frac{1-q}{1-p}\right) = L_{i-1}.$$

In other words, $\{L_i\}_{0\le i\le N}$ is a $\left(\mathbb{P}, \{\mathcal{F}_i\}_{1\le i\le N}\right)$-martingale with $\mathbb{E}\left[L_i\right] = L_0 = 1$.

If we wish to calculate the expected value of a claim in the $\mathbb{Q}$-measure, it is given by

$$\mathbb{E}^{\mathbb{Q}}\left[C\right] = \mathbb{E}^{\mathbb{P}}\left[L_N C\right].$$

> **Notation:** We have obtained the *Radon–Nikodym derivative* of $\mathbb{Q}$ with respect to $\mathbb{P}$. It is customary to write
>
> $$L_i = \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_i}.$$

Change of measure in the continuous world

We have shown that the process of changing to the martingale measure can be viewed as a reweighting of the probabilities of paths under our original measure $\mathbb{P}$ according to a positive, mean one, $\mathbb{P}$-martingale. This procedure of reweighting according to a positive martingale can be extended to the continuous setting. Our aim now is to investigate the effect of such a reweighting on the distribution of the $\mathbb{P}$-Brownian

motion. Later this will enable us to choose the right reweighting so that under the new measure obtained in this way the discounted stock price is a martingale.

Theorem 4.5.1 (Girsanov's Theorem) *Suppose that $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion with the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and that $\{\theta_t\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process such that*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta_t^2 dt\right)\right] < \infty.$$

Define

$$L_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

and let $\mathbb{P}^{(L)}$ be the probability measure defined by

$$\mathbb{P}^{(L)}[A] = \int_A L_t(\omega)\mathbb{P}(d\omega).$$

Then under the probability measure $\mathbb{P}^{(L)}$, the process $\{W_t^{(L)}\}_{0\leq t\leq T}$, defined by

$$W_t^{(L)} = W_t + \int_0^t \theta_s ds,$$

is a standard Brownian motion.

Notation: We write

$$\left.\frac{d\mathbb{P}^{(L)}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t.$$

(L_t is the *Radon–Nikodym derivative* of $\mathbb{P}^{(L)}$ with respect to $\mathbb{P}$.)

Remarks:

1 The condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta_t^2 dt\right)\right] < \infty,$$

known as Novikov's condition, is enough to guarantee that $\{L_t\}_{t\geq 0}$ is a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale. Since L_t is clearly positive and has expectation one, $\mathbb{P}^{(L)}$ really does define a probability measure.

2 Just as in the discrete world, two probability measures are equivalent if they have the same sets of probability zero. Evidently $\mathbb{P}$ and $\mathbb{P}^{(L)}$ are equivalent.

3 If we wish to calculate an expectation with respect to $\mathbb{P}^{(L)}$ we have

$$\mathbb{E}^{\mathbb{P}^{(L)}}[\phi_t] = \mathbb{E}[\phi_t L_t].$$

More generally,

$$\mathbb{E}^{\mathbb{P}^{(L)}}[\phi_t | \mathcal{F}_s] = \mathbb{E}^{\mathbb{P}}\left[\phi_t \frac{L_t}{L_s}\middle| \mathcal{F}_s\right].$$

This will be fundamental in option pricing. □

Outline of proof of theorem: We have already said that $\{L_t\}_{t\geq 0}$ is a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale. We don't prove this in full, but we find supporting evidence by finding the stochastic differential equation satisfied by L_t. We do this in two stages. First, define

$$Z_t = -\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds.$$

Then

$$dZ_t = -\theta_t dW_t - \frac{1}{2}\theta_t^2 dt.$$

Now we apply Itô's formula to $L_t = \exp(Z_t)$.

$$\begin{aligned} dL_t &= \exp(Z_t)dZ_t + \frac{1}{2}\exp(Z_t)\theta_t^2 dt \\ &= -\theta_t \exp(Z_t) dW_t = -\theta_t L_t dW_t. \end{aligned}$$

Now we use our integration by parts formula of Theorem 4.4.2 to find the stochastic differential equation satisfied by $W_t^{(L)} L_t$. Since

$$\begin{aligned} dW_t^{(L)} &= dW_t + \theta_t dt, \\ d(W_t^{(L)} L_t) &= W_t^{(L)} dL_t + L_t dW_t^{(L)} + d[M^{W^{(L)}}, M^L]_t \\ &= W_t^{(L)} dL_t + L_t dW_t + L_t\theta_t dt - \theta_t L_t dt \\ &= \left(L_t - \theta_t L_t W_t^{(L)}\right) dW_t. \end{aligned}$$

Granted enough boundedness (which is guaranteed by our assumptions), $\{W_t^{(L)} L_t\}_{t\geq 0}$ is then a $\mathbb{P}$-*martingale* and has expectation zero. Thus, under the measure $\mathbb{P}^{(L)}$, $\{W_t^{(L)}\}_{t\geq 0}$ is a martingale.

We proved in Theorem 4.2.1 that with $\mathbb{P}$-probability one, the quadratic variation of $\{W_t\}_{t\geq 0}$ is given by $[W]_t = t$. The probability measures $\mathbb{P}$ and $\mathbb{P}^{(L)}$ are equivalent and so have the same sets of probability one. Therefore $\{W_t^{(L)}\}_{t\geq 0}$ also has quadratic variation given by $[W^{(L)}]_t = t$ with $\mathbb{P}^{(L)}$-probability one. Finally, by Lévy's characterisation of Brownian motion (Theorem 4.3.5) we have that $\{W_t^{(L)}\}_{t\geq 0}$ is a $\mathbb{P}^{(L)}$-Brownian motion as required. □

We now try this in practice.

Example 4.5.2 *Let $\{X_t\}_{t\geq 0}$ be the drifting Brownian motion process*

$$X_t = \sigma W_t + \mu t,$$

where $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion and σ and μ are constants. Find a measure under which $\{X_t\}_{t\geq 0}$ is a martingale.

Solution: Taking $\theta = \mu/\sigma$, under $\mathbb{P}^{(L)}$ of Theorem 4.5.1 we have that $W_t^{(L)} = W_t + \mu t/\sigma$ is a Brownian motion, and $X_t = \sigma W_t^{(L)}$ is then a scaled Brownian motion.

Notice that, for example,

$$\mathbb{E}^{\mathbb{P}}[X_t^2] = \mathbb{E}^{\mathbb{P}}\left[\sigma^2 W_t^2 + 2\sigma\mu t W_t + \mu^2 t^2\right] = \sigma^2 t + \mu^2 t^2,$$

whereas

$$\mathbb{E}^{\mathbb{P}^{(L)}}[X_t^2] = \mathbb{E}^{\mathbb{P}^{(L)}}\left[\sigma^2 (W_t^{(L)})^2\right] = \sigma^2 t.$$

□

4.6 The Brownian Martingale Representation Theorem

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space and let $\{W_t\}_{t\geq 0}$ be a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-Brownian motion. We have seen that if $f(t,\omega)$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-predictable random variable and $\mathbb{E}\left[f^2(t,\omega)\right] < \infty$ for each $t \geq 0$, then

$$M_t \triangleq \int_0^t f(s,\omega) dW_s$$

is a $\left(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}\right)$-martingale. It is natural to ask if there are any others.

Just as in the discrete world the binomial representation theorem allowed us to represent martingales as 'discrete stochastic integrals' so here the Brownian martingale representation theorem tells us that all (nice) $\left(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}\right)$-martingales can be represented as Itô integrals. This result is also sometimes called the *predictable representation property*. It will be the key to *hedging* in our continuous world.

Definition 4.6.1 *A* $\left(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}\right)$*-martingale* $\{M_t\}_{t\geq 0}$ *is said to be* square-integrable *if*

$$\mathbb{E}\left[|M_t|^2\right] < \infty \text{ for each } t > 0.$$

Theorem 4.6.2 (Brownian Martingale Representation Theorem) *Let* $\{\mathcal{F}_t\}_{t\geq 0}$ *denote the natural filtration of the* $\mathbb{P}$*-Brownian motion* $\{W_t\}_{t\geq 0}$*. Let* $\{M_t\}_{t\geq 0}$ *be a square-integrable* $\left(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}\right)$*-martingale. Then there exists an* $\{\mathcal{F}_t\}_{t\geq 0}$*-predictable process* $\{\theta_t\}_{t\geq 0}$ *such that with* $\mathbb{P}$*-probability one,*

$$M_t = M_0 + \int_0^t \theta_s dW_s.$$

Outline of proof: We restrict our attention to $t \in [0, T]$ for some fixed T. The first step is to show that any $F \in \mathcal{F}_T$ for which $\mathbb{E}\left[F^2\right] < \infty$ can be represented as

$$F = \mathbb{E}[F] + \int_0^T \theta_s dW_s \tag{4.17}$$

for some predictable process $\{\theta_s\}_{0\leq s\leq T}$. Write $\mathcal{G}$ for the linear space of such F that *can* be represented in this way. For any $F \in \mathcal{G}$,

$$\mathbb{E}[F^2] = \mathbb{E}[F]^2 + \mathbb{E}\left[\int_0^T \theta_s^2 ds\right]. \tag{4.18}$$

This guarantees that if we take a sequence of random variables $\{F_n\}_{n\geq 1}$ in $\mathcal{G}$ for which

$$\mathbb{E}\left[(F_n - F_m)^2\right] \to 0 \quad \text{as } n, m \to \infty,$$

then they converge to a limit that also lies in $\mathcal{G}$. Now by Itô's formula, for any simple function

$$f(s) = \sum_{i=1}^{n} a_i(\omega) 1_{(t_{i-1}, t_i]}(s),$$

if we define

$$\mathcal{E}_t^f \triangleq \exp\left(\int_0^t f(s) dW_s - \frac{1}{2}\int_0^t f(s)^2 ds\right),$$

then

$$\mathcal{E}_t^f = 1 + \int_0^t f(s) \mathcal{E}_s^f dW_s.$$

In other words $\mathcal{E}_T^f \in \mathcal{G}$. We now approximate any $F \in \mathcal{F}_T$ for which $\mathbb{E}\left[F^2\right] < \infty$ by linear combinations of the $\mathcal{E}_T^f$ to see that all such F are in $\mathcal{G}$ and so can be represented as in (4.17). The identity (4.18) guarantees that if the representation holds with two different predictable processes, $\{\theta_s\}_{0\leq s\leq T}$ and $\{\psi_s\}_{0\leq s\leq T}$ say, then

$$\mathbb{E}\left[\int_0^T (\theta_s - \psi_s)^2 ds\right] = 0.$$

Now we replace $F \in \mathcal{F}_T$ by the martingale $\{M_t\}_{0\leq t\leq T}$ to complete the proof. This step is elementary. Since $M_t = \mathbb{E}[M_T | \mathcal{F}_t]$, applying the representation to M_T and then taking (conditional) expectations of both sides we obtain

$$\begin{aligned} M_t = \mathbb{E}[M_T | \mathcal{F}_t] &= \mathbb{E}[M_T] + \mathbb{E}\left[\int_0^T \theta_s dW_s \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}[M_T] + \int_0^t \theta_s dW_s. \end{aligned}$$

□

For full details of this proof see Revuz & Yor (1998).

Remarks:

1 The Martingale Representation Theorem tells us that such an $\{\mathcal{F}_t\}_{t\geq 0}$-predictable process $\{\theta_t\}_{t\geq 0}$ *exists*. Unfortunately, unlike the Binomial Representation Theorem, the proof is not constructive. When we call upon it in hedging options in Chapter 5, we are going to have to work harder to actually produce an *explicit* expression for the predictable process.

2 Notice that the quadratic variation of the martingale $\{M_t\}_{t\geq 0}$ satisfies $d[M]_t = \theta_t^2 dt$. If we have two Brownian martingales, $\{M_t^{(1)}\}_{t\geq 0}$ and $\{M_t^{(2)}\}_{t\geq 0}$, then provided $d[M^{(i)}]_t/dt$ is non-vanishing for $i = 1, 2$, the Martingale Representation Theorem tells us that each is a locally scaled version of the other. □

4.7 Why geometric Brownian motion?

We now have the main results in place that will allow us to price and hedge in stock market models based on Brownian motion. Other than suggesting that the paths of stock prices in an arbitrage-free world should be rough, we have thus far provided no justification for such models. In this short section we use Lévy's characterisation of Brownian motion to motivate the basic reference model in mathematical finance: geometric Brownian motion.

We begin by sketching Bachelier's argument for the Brownian motion model. Bachelier argued that stock markets cannot have any consistent bias in favour of either buyers or sellers:

'L'espérance mathématique du spéculateur est nulle'.

This is almost the martingale property. Assuming the stock price process to have the Markov property, he introduced the transition density

$$\mathbb{P}\big[S_t \in [y, y+dy] \big| S_s = x\big] \triangleq p\,(s,t;x,y)\,dy.$$

If the dynamics are homogeneous in space and time, then $p(s,t;x,y) = q(t-s, y-x)$ for some function q. Bachelier then 'derived' what is now known as the Chapman–Kolmogorov equation for q and showed that this is solved by the probability density function of Brownian motion.

Bachelier's argument is not rigorous, but from Lévy's characterisation of Brownian motion we know that if the stock price process is a martingale under $\mathbb{P}$ whose increments have stationary conditional variance then the stock price process must be Brownian motion under $\mathbb{P}$. It is remarkable that Bachelier's argument pre-dates Einstein's famous work on Brownian motion and, of course, Wiener's rigorous construction of the process.

Although we would not take issue with the mathematical conclusions of Bachelier's analysis, we have already discarded Brownian motion as a model. A modern approach makes different assumptions, but we need not completely abandon Bachelier's argument. His key assumption was that the increments of the stock price process were stationary. Suppose that instead we assume that the *relative* increments, $(S_t - S_s)/S_s$, measuring the *returns* are stationary. Taking logarithms, the process $\{\log S_t\}_{t\geq 0}$ should have stationary increments. We don't know that $\{\log S_t\}_{t\geq 0}$ is a martingale, so this time we can only deduce that this is Brownian motion plus a constant drift. This gives

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion and μ and σ are constants. This is the geometric Brownian motion model, originally championed by Samuelson (1965).

4.8 The Feynman–Kac representation

Our probabilistic approach to pricing options will result in a price expressed as the discounted expected value of a claim with respect to a probability measure under

which the discounted stock price is a martingale. In the simple case of European calls and puts we'll be able to find an explicit expression for the price. However, for more complicated claims numerical methods have to be brought to bear. One approach is to revert to our binomial tree model. Another is to express the price as the solution to a partial differential equation and employ, for example, finite difference methods. In fact for European options the binomial method amounts to a finite difference method for solving the Black–Scholes partial differential equation. We refer to Wilmott, Howison & Dewynne (1995) for an account of the numerical methods. Here we simply make the connection between the partial differential equation approach and the probabilistic approach to pricing derivatives.

Solving pde's probabilistically

The fact that the price can be expressed as the solution to a partial differential equation is a consequence of the deep connection between stochastic differential equations and certain parabolic partial differential equations.

Theorem 4.8.1 (Feynman–Kac stochastic representation) *Assume that the function F solves the boundary value problem*

$$\frac{\partial F}{\partial t}(t,x)+\mu(t,x)\frac{\partial F}{\partial x}(t,x)+\frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x)=0, \qquad 0\le t\le T, \qquad F(T,x)=\Phi(x). \tag{4.19}$$

Define $\{X_t\}_{0\le t\le T}$ to be the solution of the stochastic differential equation

$$dX_t=\mu(t,X_t)dt+\sigma(t,X_t)dW_t, \qquad 0\le t\le T,$$

where $\{W_t\}_{t\ge 0}$ is standard Brownian motion under the measure $\mathbb{P}$. If

$$\int_0^T \mathbb{E}\left[\left(\sigma(t,X_t)\frac{\partial F}{\partial x}(t,X_t)\right)^2\right]ds<\infty, \tag{4.20}$$

then

$$F(t,x)=\mathbb{E}^{\mathbb{P}}\left[\Phi(X_T)\,|\,X_t=x\right].$$

Proof: We apply Itô's formula to $\{F(s,X_s)\}_{t\le s\le T}$.

$$\begin{aligned} F(T,X_T) &= F(t,X_t)\\ &\quad+\int_t^T\left\{\frac{\partial F}{\partial s}(s,X_s)+\mu(s,X_s)\frac{\partial F}{\partial x}(s,X_s)+\frac{1}{2}\sigma^2(s,X_s)\frac{\partial^2 F}{\partial x^2}(s,X_s)\right\}ds\\ &\quad+\int_t^T\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)dW_s. \end{aligned}\tag{4.21}$$

Now using assumption (4.20) and Theorem 4.2.7

$$\mathbb{E}\left[\int_t^T\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)dW_s\,\middle|\,X_t=x\right]=0.$$

Moreover, since F satisfies (4.19), the deterministic integral on the right hand side of (4.21) vanishes, so, taking expectations,

$$\mathbb{E}\left[F(T, X_T) \middle| X_t = x \right] = F(t, x),$$

and substituting $F(T, X_T) = \Phi(X_T)$ gives the required result. $\square$

Example 4.8.2 *Solve*

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} &= 0, \\ F(T, x) &= \Phi(x). \end{aligned} \tag{4.22}$$

Solution: The corresponding stochastic differential equation is

$$dX_t = dW_t$$

so, by the Feynman–Kac representation,

$$F(t, x) = \mathbb{E}\left[\Phi(W_T) \middle| W_t = x \right].$$

In fact we knew this already. In §3.1 we wrote down the transition density of Brownian motion as

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t} \right). \tag{4.23}$$

This gives

$$\mathbb{E}\left[\Phi(W_T) \middle| W_t = x \right] = \int p\left(T - t, x, y\right) \Phi(y) dy.$$

To check that this really is the solution, differentiate and use the fact that $p(t, x, y)$ given by (4.23) is the fundamental solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2},$$

to obtain (4.22). $\square$

Kolmogorov equations

We can use the Feynman–Kac representation to find the partial differential equation solved by the transition densities of solutions to other stochastic differential equations.

Suppose that

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{4.24}$$

For any set B let

$$p_B(t, x; T, y) \triangleq \mathbb{P}\left[X_T \in B \middle| X_t = x \right] = \mathbb{E}\left[\mathbf{1}_B(X_T) \middle| X_t = x \right].$$

By the Feynman–Kac representation (subject to the integrability condition (4.20)) this solves

$$
\begin{aligned}
\frac{\partial p_B}{\partial t}(t, x; T, y) + A p_B(t, x; T, y) &= 0, \qquad (4.25)\\
p_B(T, x) &= \mathbf{1}_B(x),
\end{aligned}
$$

where

$$
Af(t, x) = \mu(t, x)\frac{\partial f}{\partial x}(t, x) + \frac{1}{2}\sigma^2(t, x)\frac{\partial^2 f}{\partial x^2}(t, x).
$$

Writing $|B|$ for the Lebesgue measure of the set B, the transition density of the process $\{X_s\}_{s\geq 0}$ is given by

$$
p(t, x; T, y) \triangleq \lim_{|B|\to 0} \frac{1}{|B|}\mathbb{P}\left[X_T \in B \,|\, X_t = x\right].
$$

Since the equation (4.25) is linear, we have proved the following lemma.

Lemma 4.8.3 *Subject to satisfying (4.20), the transition density of the solution $\{X_s\}_{s\geq 0}$ to the stochastic differential equation (4.24) solves*

$$
\begin{aligned}
\frac{\partial p}{\partial t}(t, x; T, y) + A p(t, x; T, y) &= 0, \qquad (4.26)\\
p(t, x; T, y) &\to \delta_y(x) \quad \textit{as } t \to T.
\end{aligned}
$$

Equation (4.26) is known as the *Kolmogorov backward equation* (it operates on the 'backward in time' variables (t, x)). The operator A is called the *infinitesimal generator* of the process $\{X_s\}_{s\geq 0}$.

We can also obtain an equation acting on the *forward* variables (T, y).

Lemma 4.8.4 *In the above notation,*

$$
\frac{\partial p}{\partial T}(t, x; T, y) = A^* p(t, x; T, y) \qquad (4.27)
$$

where

$$
A^* f(T, y) = -\frac{\partial}{\partial y}\left(\mu(T, y) f(T, y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t, Y) f(T, y)\right).
$$

Heuristic explanation: We don't prove this, but we provide some justification. By the Markov property of the process $\{X_t\}_{t\geq 0}$, for any $T > r > t$

$$
p(t, x; T, y) = \int p(t, x; r, z) p(r, z; T, y) dz.
$$

Differentiating with respect to r and using (4.26),

$$
\int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial r} p(t, x; r, z) p(r, z; T, y) + p(t, x; r, z) A p(r, z; T, y) \right\} dz = 0.
$$

Now integrate the second term by parts to obtain

$$\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial r}p(t,x;r,z)-A^{*}p(t,x;r,z)\right\}p(r,z;T,y)dz=0.$$

This holds for all $T>r$, which, if $p(r,z;T,y)$ provides a sufficiently rich class of functions as we vary T, implies the result. □

Equation (4.27) is the *Kolmogorov forward equation* of the process $\{X_s\}_{s\geq 0}$.

Example 4.8.5 *Find the forward and backward Kolmogorov equations for geometric Brownian motion.*

Solution: The stochastic differential equation is

$$dS_t=\mu S_t dt+\sigma S_t dW_t.$$

Substituting in our formula for the forward equation we obtain

$$\frac{\partial p}{\partial T}(t,x;T,y)=\frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}\left(y^2p(t,x;T,y)\right)-\mu\frac{\partial}{\partial y}\left(yp(t,x;T,y)\right),$$

and the backward equation is

$$\frac{\partial p}{\partial t}(t,x;T,y)=-\frac{1}{2}\sigma^2x^2\frac{\partial^2 p}{\partial x^2}(t,x;T,y)-\mu x\frac{\partial p}{\partial x}(t,x;T,y).$$

The transition density for the process is the lognormal density given by

$$p(t,x;T,y)=\frac{1}{\sigma y\sqrt{2\pi(T-t)}}\exp\left(-\frac{\left(\log(y/x)-(\mu-\frac{1}{2}\sigma^2)(T-t)\right)^2}{2\sigma^2(T-t)}\right).$$

□

Example 4.8.6 *Suppose that $\{X_t\}_{t\geq 0}$ solves*

$$dX_t=\mu(t,X_t)dt+\sigma(t,X_t)dW_t,$$

where $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion. For $k:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ and $\Phi:\mathbb{R}\to\mathbb{R}$ given deterministic functions, find the partial differential equation satisfied by the function

$$F(t,x)\triangleq\mathbb{E}\left[\exp\left(-\int_t^T k(s,X_s)ds\right)\Phi(X_T)\,\middle|\,X_t=x\right],$$

for $0\leq t\leq T$.

Solution: Evidently $F(T, x) = \Phi(x)$. By analogy with the proof of the Feynman–Kac representation, it is tempting to examine the dynamics of

$$Z_s = \exp\left(-\int_t^s k(u, X_u)du\right) F(s, X_s).$$

Notice that for this choice of $\{Z_s\}_{t\leq s\leq T}$ we have that if $X_t = x$

$$Z_t = F(t, x) = \mathbb{E}\left[Z_T \mid X_t = x\right].$$

Thus the partial differential equation satisfied by $F(t, x)$ is that for which $\{Z_t\}_{0\leq t\leq T}$ is a martingale.

Our strategy now is to find the stochastic differential equation satisfied by $\{Z_s\}_{t\leq s\leq T}$. We proceed in two stages. Remember that t is now fixed and we vary s. First notice that

$$d\left(\exp\left(-\int_t^s k(u, X_u)du\right)\right) = -k(s, X_s)\exp\left(-\int_t^s k(u, X_u)du\right)ds$$

and by Itô's formula

$$\begin{aligned} dF(s, X_s) &= \frac{\partial F}{\partial s}(s, X_s)ds + \frac{\partial F}{\partial x}(s, X_s)dX_s + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(s, X_s)\sigma^2(s, X_s)ds \\ &= \left\{\frac{\partial F}{\partial s}(s, X_s) + \mu(s, X_s)\frac{\partial F}{\partial x}(s, X_s) + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2 F}{\partial x^2}(s, X_s)\right\}ds \\ &\quad + \sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)dW_s. \end{aligned}$$

Now using our integration by parts formula we have that

$$\begin{aligned} dZ_s &= \exp\left(-\int_t^s k(u, X_u)du\right) \\ &\times\left\{\left\{-k(s, X_s)F(s, X_s) + \frac{\partial F}{\partial s}(s, X_s) + \mu(s, X_s)\frac{\partial F}{\partial x}(s, X_s) + \frac{1}{2}\sigma^2(s, X_s)\frac{\partial^2 F}{\partial x^2}\right\}ds \right. \\ &\quad \left. + \sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)dW_s\right\}. \end{aligned}$$

We can now read off the solution: in order for $\{Z_s\}_{t\leq s\leq T}$ to be a martingale, F must satisfy

$$\frac{\partial F}{\partial s}(s, x) + \mu(s, x)\frac{\partial F}{\partial x}(s, x) + \frac{1}{2}\sigma^2(s, x)\frac{\partial^2 F}{\partial x^2}(s, x) - k(s, x)F(s, x) = 0.$$

□

Exercises

1 Let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the natural filtration associated to a standard $\mathbb{P}$-Brownian motion $\{W_t\}_{t\geq 0}$. Define the process $\{S_t\}_{t\geq 0}$ by $S_t = f(t, W_t)$. What equation must f satisfy if S_t is to be a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale? Use your answer to check that

$$S_t = \exp(\nu t + \sigma W_t)$$

is a martingale if $\nu + \frac{1}{2}\sigma^2 = 0$ (cf. Lemma 4.3.3).

2 A function, f, is said to be *Lipschitz-continuous* on $[0, T]$ if there exists a constant $C > 0$ such that for any $t, t' \in [0, T]$

$$|f(t) - f(t')| < C|t - t'|.$$

Show that a Lipschitz-continuous function has bounded variation and zero 2-variation over $[0, T]$.

3 Let $\{W_t\}_{t\geq 0}$ denote a standard Brownian motion under $\mathbb{P}$. For a partition π of $[0, T]$, write $\delta(\pi)$ for the mesh of the partition and $0 = t_0 < t_1 < t_2 < \cdots < t_{N(\pi)} = T$ for the endpoints of the intervals of the partition. Calculate

(a)

$$\lim_{\delta(\pi)\to 0} \sum_{0}^{N(\pi)-1} W_{t_{j+1}}\left(W_{t_{j+1}} - W_{t_j}\right),$$

(b)

$$\int_0^T W_s \circ dW_s \triangleq \lim_{\delta(\pi)\to 0} \sum_{0}^{N(\pi)-1} \frac{1}{2}\left(W_{t_{j+1}} + W_{t_j}\right)\left(W_{t_{j+1}} - W_{t_j}\right).$$

This is the *Stratonovich integral* of $\{W_s\}_{s\geq 0}$ with respect to itself over $[0, T]$.

4 Suppose that the martingale $\{M_t\}_{0\leq t\leq T}$ has bounded quadratic variation and $\{A_t\}_{0\leq t\leq T}$ is Lipschitz-continuous. Let $S_t = M_t + A_t$. By analogy with Definition 4.2.2, we define the quadratic variation of $\{S_t\}_{0\leq t\leq T}$ over $[0, T]$ to be the random variable $[S]_T$ such that for any sequence of partitions $\{\pi_n\}_{n\geq 1}$ of $[0, T]$ with $\delta(\pi_n) \to 0$ as $n \to \infty$,

$$\mathbb{E}\left[\left|\sum_{j=1}^{N(\pi)} |S_{t_j} - S_{t_{j-1}}|^2 - [S]_T\right|^2\right] \to 0 \quad \text{as } n \to \infty.$$

Show that $[S]_T = [M]_T$.

5 If f is a simple function and $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion, prove that the process $\{M_t\}_{t\geq 0}$ given by the Itô integral

$$M_t = \int_0^t f(s, W_s)dW_s$$

is a $(\mathbb{P}, \{\mathcal{F}_t^W\}_{t\geq 0})$-martingale.

6 Verify that if $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion

$$\mathbb{E}\left[\left(\int_0^t W_s dW_s\right)^2\right] = \int_0^t \mathbb{E}\left[W_s^2\right] ds.$$

(If you need the moment-generating function of W_t, you may assume the result of Exercise 10.)

7 As usual, $\{W_t\}_{t\geq 0}$ denotes standard Brownian motion under $\mathbb{P}$. Use Itô's formula to write down stochastic differential equations for the following quantities.

(a) $Y_t = W_t^3$,
(b) $Y_t = \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)$,
(c) $Y_t = tW_t$.

Which are $(\mathbb{P}, \{\mathcal{F}_t^W\}_{t\geq 0})$-martingales?

8 Use a heuristic argument based on a Taylor expansion to check that for Stratonovich stochastic calculus the chain rule takes the form of the classical (Newtonian) one.

9 Mimic the calculation of Example 4.3.2 to show that if $\{W_t\}_{t\geq 0}$ is standard Brownian motion under the measure $\mathbb{P}$, then $\mathbb{E}\left[W_t^4\right] = 3t^2$.

10 Let $\{W_t\}_{t\geq 0}$ denote Brownian motion under $\mathbb{P}$ and define $Z_t = \exp(\alpha W_t)$. Use Itô's formula to write down a stochastic differential equation for Z_t. Hence find an ordinary (deterministic) differential equation for $m(t) \triangleq \mathbb{E}[Z_t]$, and solve to show that

$$\mathbb{E}\left[\exp\left(\alpha W_t\right)\right] = \exp\left(\frac{\alpha^2}{2}t\right).$$

11 *The Ornstein–Uhlenbeck process* Let $\{W_t\}_{t\geq 0}$ denote standard Brownian motion under $\mathbb{P}$. The Ornstein–Uhlenbeck process, $\{X_t\}_{t\geq 0}$, is the unique solution to *Langevin's equation*,

$$dX_t = -\alpha X_t dt + dW_t, \qquad X_0 = x.$$

This equation was originally introduced as a simple idealised model for the velocity of a particle suspended in a liquid. In finance it is a special case of the *Vasicek model* of interest rates (see Exercise 19). Verify that

$$X_t = e^{-\alpha t}x + e^{-\alpha t}\int_0^t e^{\alpha s}dW_s,$$

and use this expression to calculate the mean and variance of X_t.

12 The *Cox–Ingersoll–Ross model* of interest rates assumes that the interest rate, r, is not deterministic, but satisfies the stochastic differential equation

$$dr_t = (\alpha - \beta r_t)dt + \sigma\sqrt{r_t}dW_t,$$

where $\{W_t\}_{t\geq 0}$ is standard $\mathbb{P}$-Brownian motion. This process is known as a *squared Bessel* process. Find the stochastic differential equation followed by $\{\sqrt{r_t}\}_{t\geq 0}$ in the case $\alpha = 0$.

Suppose that $\{u(t)\}_{t\geq 0}$ satisfies the ordinary differential equation

$$\frac{du}{dt}(t) = -\beta u(t) - \frac{\sigma^2}{2}u(t)^2, \quad u(0) = \theta,$$

for some constant $\theta > 0$. Fix $T > 0$. Assuming still that $\alpha = 0$, for $0 \le t \le T$ find the differential equation satisfied by

$$\mathbb{E}\left[\exp\left(-u(T-t)r_t\right)\right].$$

Hence calculate the mean and variance of r_T and $\mathbb{P}[r_T = 0]$.

13 The *Black–Karasinski model* of interest rates is

$$dr_t = \sigma_t r_t dW_t + \left(\theta_t + \frac{1}{2}\sigma_t^2 - \alpha_t \log r_t\right) r_t dt,$$

where $\{W_t\}_{t\ge 0}$ is a standard $\mathbb{P}$-Brownian motion and σ_t, θ_t and α_t are deterministic functions of time. In the special case where σ, θ and α are constants, find r_t as a function of $\int_0^t e^{\alpha s} dW_s$.

14 Suppose that, under the measure $\mathbb{P}$,

$$dS_t = \sigma S_t dW_t,$$

where $\{W_t\}_{t\ge 0}$ is a $\mathbb{P}$-Brownian motion. Find the mean and variance of

$$Y_t \triangleq \int_0^t S_u du.$$

15 Suppose that $\{M_t\}_{t\ge 0}$ is a continuous $(\mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0})$-martingale with $\mathbb{E}\left[M_t^2\right]$ finite for all $t \ge 0$. Writing $\{[M]_t\}_{t\ge 0}$ for the associated quadratic variation process, show that $M_t^2 - [M]_t$ is a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0})$-martingale.

16 Suppose that under the probability measure $\mathbb{P}$, $\{X_t\}_{t\ge 0}$ is a Brownian motion with constant drift μ. Find a measure $\mathbb{P}^*$, equivalent to $\mathbb{P}$, under which $\{X_t\}_{t\ge 0}$ is a Brownian motion with drift ν.

17 Let $\{\mathcal{F}_t\}_{t\ge 0}$ be the natural filtration associated with a $\mathbb{P}$-Brownian motion, $\{W_t\}_{t\ge 0}$. Show that if X is an $\mathcal{F}_T$-measurable random variable with $\mathbb{E}[|X|] < \infty$ and $\mathbb{P}^*$ is a probability measure equivalent to $\mathbb{P}$, then the process

$$M_t \triangleq \mathbb{E}^{\mathbb{P}^*}\left[X | \mathcal{F}_t\right]$$

is a $(\mathbb{P}^*, \{\mathcal{F}_t\}_{0\le t\le T})$-martingale.

18 Use the Feynman–Kac stochastic representation formula to solve

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}(t,x) = 0,$$

subject to the terminal value condition

$$F(T,x) = x^4.$$

19 Suppose that the interest rate, r, is not deterministic, but is itself a random process, $\{r_t\}_{t\geq 0}$. In the *Vasicek model*, $\{r_t\}_{t\geq 0}$ is assumed to be a solution of the stochastic differential equation

$$dr_t = (b - ar_t)dt + \sigma dW_t,$$

where, as usual, $\{W_t\}_{t\geq 0}$ is standard $\mathbb{P}$-Brownian motion.
Find the Kolmogorov backward and forward differential equations satisfied by the probability density function of the process. What is the distribution of r_t as $t \to \infty$?

20 Suppose that $v(t, x)$ solves

$$\frac{\partial v}{\partial t}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2}(t, x) - rv(t, x) = 0, \quad 0 \leq t \leq T.$$

Show that for any constant θ,

$$v_\theta(t, x) \triangleq \frac{x}{\theta} v\left(t, \frac{\theta^2}{x}\right)$$

is another solution. Use the Feynman–Kac stochastic representation to find a probabilistic interpretation of this result.

21 Suppose that for $0 \leq s \leq T$,

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s, \quad X_t = x,$$

where $\{W_s\}_{t\leq s\leq T}$ is a $\mathbb{P}$-Brownian motion, and let $k, \Phi : \mathbb{R} \to \mathbb{R}$ be given deterministic functions. Find the partial differential equation satisfied by

$$F(t, x) = \mathbb{E}\left[\Phi(X_T) \mid X_t = x\right] + \int_t^T \mathbb{E}\left[k(X_s) \mid X_t = x\right] ds.$$