ıction and X a real-valued random

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ıple 2.4.7) of the discounted price -paying stock as the smallest $\mathbb{Q}$- $)_+\}_{n\geq 0}$ to prove that the price of an the same as that of a European call

3 Brownian motion

Summary

Our discrete models are only a crude approximation to the way in which stock markets actually move. A better model would be one in which stock prices can change at any instant. As early as 1900 Bachelier, in his thesis 'La théorie de la spéculation', proposed Brownian motion as a model of the fluctuations of stock prices. Even today it is the building block from which we construct the basic reference model for a continuous time market. Before we can proceed further we must leave finance to define and construct Brownian motion.

Our first approach will be to continue the heuristic of §2.6 by considering Brownian motion as an 'infinitesimal' random walk in which smaller and smaller steps are taken at ever more frequent time intervals. This will lead us to a natural definition of the process. A formal construction, due to Lévy, will be given in §3.2, but this can safely be omitted. Next, §3.3 establishes some facts about the process that we shall require in later chapters. This material too can be skipped over and referred back to when it is used.

Just as discrete parameter martingales play a key rôle in the study of random walks, so for Brownian motion we shall use continuous time martingale theory to simplify a number of calculations; §3.4 extends our definitions and basic results on discrete parameter martingales to the continuous time setting.

3.1 Definition of the process

The easiest way to think about Brownian motion is as an 'infinitesimal random walk' and that is often how it arises in applications, so to motivate the formal definition we first study simple random walks.

A characterisation of simple random walks

We declared in Example 2.3.7 that the stochastic process $\{S_n\}_{n\geq 0}$ is a simple random walk under the measure $\mathbb{P}$ if $S_n = \sum_{i=1}^{n} \xi_i$ where the ξ_i can take only the values $\{-1, +1\}$ and are independent and identically distributed under $\mathbb{P}$. We concentrate

on the *symmetric* case when

$$\mathbb{P}[\xi_i = -1] = \frac{1}{2} = \mathbb{P}[\xi_i = +1].$$

This process is often motivated as a model of the gains from repeated plays fair game. For example, suppose I play a game with a friend in which each pla equivalent to flipping a fair coin. If it comes up heads I pay her a dollar, otherw she pays me a dollar. For each n, S_n models my net gain after n plays.

Recall from Exercise 13 of Chapter 2 that $\mathbb{E}[S_n] = 0$ and $var(S_n) = n$.

Lemma 3.1.1 *$\{S_n\}_{n\geq 0}$ is a $\mathbb{P}$-martingale (with respect to the natural filtrat and*

$$cov(S_n, S_m) = n \wedge m.$$

Proof: We checked in Example 2.3.7 that $\{S_n\}_{n\geq 0}$ is a $\mathbb{P}$-martingale. It remain calculate the covariance.

$$\begin{aligned} cov(S_n, S_m) &= \mathbb{E}[S_n S_m] - \mathbb{E}[S_n]\mathbb{E}[S_m] \\ &= \mathbb{E}\left[\mathbb{E}[S_n S_m | \mathcal{F}_{m\wedge n}]\right] \qquad \text{(tower property)} \\ &= \mathbb{E}\left[S_{m\wedge n}\mathbb{E}\left[S_{m\vee n} | \mathcal{F}_{m\wedge n}\right]\right] \\ &= \mathbb{E}[S^2_{m\wedge n}] \qquad \text{(martingale property)} \\ &= var(S_{m\wedge n}) = m \wedge n. \end{aligned}$$

As a result of the independence of the random variables $\{\xi_i\}_{i\geq 1}$, if $0 \leq i \leq j \leq$ l, then $S_j - S_i$ is independent of $S_l - S_k$. More generally, if $0 \leq i_1 \leq i_2 \leq \cdots \leq$ then $\{S_{i_r} - S_{i_{r-1}} : 1 \leq r \leq n\}$ are independent. Moreover, if $j - i = l - k = m$, then $S_j - S_i$ and $S_l - S_k$ both have the same distribution as S_m.

Notation: For two random variables X and Y we write

$$X \overset{\mathcal{D}}{=} Y$$

to mean that X and Y have the same distribution.

We also write $X \sim N(\mu, \sigma^2)$ to mean that X is normally distributed witl mean μ and variance σ^2.

Combining the observations above we have

Lemma 3.1.2 *Under the measure $\mathbb{P}$ the process $\{S_n\}_{n\geq 0}$ has stationary, indep dent increments.*

Lemmas 3.1.1 and 3.1.2 are actually enough to *characterise* symmetric simple random walks.

Rescaling random walks

Recall that we want to think of Brownian motion as an infinitesimal random walk. In terms of our gambling game, the time interval between plays is δt and the stake is δx say, and we are thinking of both of these as 'tending to zero'. In order to obtain a non-trivial limit, there has to be a relationship between δt and δx. To see what this must be, we use the Central Limit Theorem (stated in §2.6). In our setting, $\mu = \mathbb{E}[\xi_i] = 0$ and $\sigma^2 = var(\xi_i) = 1$. Thus, taking $\delta t = 1/n$ and $\delta x = 1/\sqrt{n}$,

$$\mathbb{P}\left[\frac{S_n}{\sqrt{n}} \leq x\right] \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \text{ as } n \to \infty.$$

More generally,

$$\mathbb{P}\left[\frac{S_{[nt]}}{\sqrt{n}} \leq x\right] \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dy \text{ as } n \to \infty,$$

where $[nt]$ denotes the integer part of nt (Exercise 1). For the limiting process, at time t our net gain since time zero will be normally distributed with mean zero and variance t.

Definition of Brownian motion

Just as in our definition of a discrete time stochastic process, to define a continuous time stochastic process $\{X_t\}_{t\geq 0}$ (formally) requires a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ such that X_t is $\mathcal{F}$-measurable for all t. However, as in the discrete case, we shall rarely specify Ω explicitly.

Heuristically, passage to the limit in the random walk suggests that the following is a reasonable definition of Brownian motion.

Definition 3.1.3 (Brownian motion) *A real-valued stochastic process* $\{W_t\}_{t\geq 0}$ *is a* $\mathbb{P}$-Brownian motion *(or a* $\mathbb{P}$-Wiener process*) if for some real constant* σ*, under* $\mathbb{P}$*,*

1. *for each* $s \geq 0$ *and* $t > 0$ *the random variable* $W_{t+s} - W_s$ *has the normal distribution with mean zero and variance* $\sigma^2 t$*,*
2. *for each* $n \geq 1$ *and any times* $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$*, the random variables* $\{W_{t_r} - W_{t_{r-1}}\}$ *are independent,*
3. $W_0 = 0$*,*
4. W_t *is continuous in* $t \geq 0$*.*

Remarks: Conditions 1 and 2 ensure that, like its discrete counterpart, Brownian motion has stationary independent increments.

Condition 3 is a convention. Brownian motion started from x can be obtained as $\{x + W_t\}_{t\geq 0}$.

In a certain sense condition 4 is a consequence of the first three, but we should like to insist once and for all that all paths that our Brownian motion can follow are continuous. □

The parameter σ^2 is known as the *variance* parameter. By scaling of the n distribution it is immediate that $\{W_{t/\sigma}\}_{t\geq 0}$ is a Brownian motion with va parameter one.

Definition 3.1.4 *The process with $\sigma^2 = 1$ is called* standard *Brownian moti*

Assumption: Unless otherwise stated we shall always assume that $\sigma^2 = 1$

Combining conditions 1 and 2 of Definition 3.1.3, we can write down the *trar probabilities* of standard Brownian motion.

$$\mathbb{P}\left[W_{t_n} \leq x_n | W_{t_i} = x_i, 0 \leq i \leq n-1\right] = \mathbb{P}\left[W_{t_n} - W_{t_{n-1}} \leq x_n - x_{n-1}\right]$$
$$= \int_{-\infty}^{x_n - x_{n-1}} \frac{1}{\sqrt{2\pi(t_n - t_{n-1})}} \exp\left(-\frac{u^2}{2(t_n - t_{n-1})}\right)$$

Notation: We write $p(t, x, y)$ for the *transition density*

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right).$$

This is the probability density function of the random variable W_{t+s} conditio $W_s = x$.

For $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, writing $x_0 = 0$, the joint probability d function of $W_{t_1}, \ldots, W_{t_n}$ can also be written down explicitly as

$$f(x_1, \ldots, x_n) = \prod_1^n p(t_j - t_{j-1}, x_{j-1}, x_j).$$

The joint distributions of $W_{t_1}, \ldots, W_{t_n}$ for each $n \geq 1$ and all $t_1, \ldots, t_n$ are the *finite dimensional distributions* of the process.

The following analogue of Lemma 3.1.1 is immediate.

Lemma 3.1.5 *For any $s, t > 0$,*

1 $\mathbb{E}\left[W_{t+s} - W_s | \{W_r\}_{0\leq r\leq s}\right] = 0$,
2 $cov(W_s, W_t) = s \wedge t$.

In fact since the multivariate normal distribution is determined by its mea covariances and normally distributed random variables are independent if an

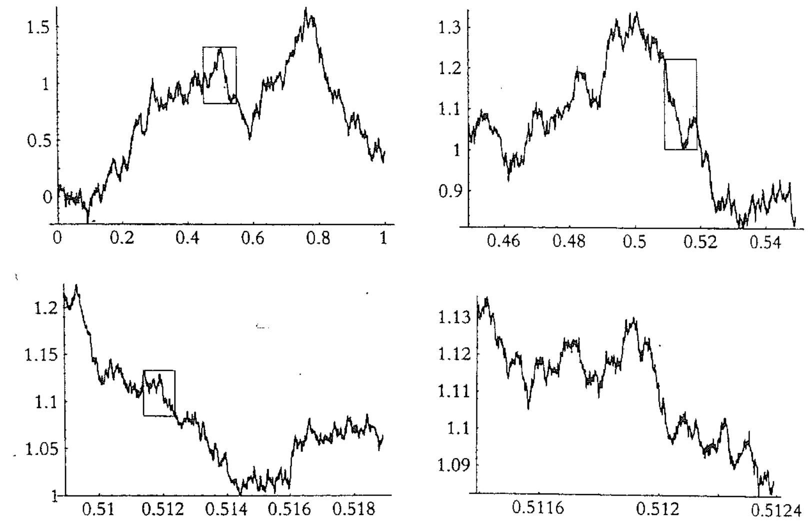


Figure 3.1 Zooming in on Brownian motion.

if their covariances are zero, this, combined with continuity of paths, characterises standard Brownian motion.

Behaviour of Brownian motion

Just because the sample paths of Brownian motion are continuous, it does not mean that they are nice in any other sense. In fact the behaviour of Brownian motion is distinctly odd. Here are just a few of its strange behavioural traits.

1. Although $\{W_t\}_{t\geq 0}$ is continuous everywhere, it is (with probability one) differentiable nowhere.
2. Brownian motion will eventually hit any and every real value no matter how large, or how negative. No matter how far above the axis, it will (with probability one) be back down to zero at some later time.
3. Once Brownian motion hits a value, it immediately hits it again *infinitely* often (and will continue to return after arbitrarily large times).
4. It doesn't matter what scale you examine Brownian motion on, it looks just the same. Brownian motion is a fractal.

Exercise 9 shows that the process cannot be differentiable at $t = 0$. We shall discuss some properties related to the hitting probabilities in §3.3 and in Exercise 8. The scaling alluded to in our last comment is formally proved in Proposition 3.3.7. It is really a consequence of the construction of the process. Figure 3.1 illustrates the result for a particular realisation of a Brownian path.

That such a bizarre process actually *exists* is far from obvious and so it is to this that we turn our attention in the next section.

3.2 Lévy's construction of Brownian motion

We have hinted that Brownian motion can be obtained as a limit of random w
However, rather than chasing the technical details of the random walk construc
in this section we present an alternative construction due to Lévy. This can be om
by readers willing to take existence of the process on trust.

A polygonal approximation.

The idea is that we can simply produce a path of Brownian motion by c
polygonal interpolation. We require just one calculation.

Lemma 3.2.1 *Suppose that $\{W_t\}_{t\geq 0}$ is standard Brownian motion. Conditi
on $W_{t_1} = x_1$, the probability density function of $W_{t_1/2}$ is*

$$p_{t_1/2}(x) \triangleq \sqrt{\frac{2}{\pi t_1}} \exp\left(-\frac{1}{2}\left(\frac{\left(x - \frac{1}{2}x_1\right)^2}{t_1/4}\right)\right).$$

In other words, the conditional distribution is a normally distributed random var
with mean $x_1/2$ and variance $t_1/4$. The proof is Exercise 11.

The construction: Without loss of generality we take the range of t to be $[0, 1]$. Lé
construction builds (inductively) a polygonal approximation to the Brownian mo
from a countable collection of *independent* normally distributed random vari
with mean zero and variance one. We index them by the dyadic points of [0,
generic variable being denoted by $\xi\left(k2^{-n}\right)$ where $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^n$

The induction begins with

$$X_1(t) = t\xi(1).$$

Thus X_1 is a linear function on $[0, 1]$.

The nth process, X_n, is linear in each interval $[(k-1)2^{-n}, k2^{-n}]$, is contin
in t and satisfies $X_n(0) = 0$. It is thus determined by the values $\{X_n(k2^{-n}),$
$1, \ldots, 2^n\}$.

The inductive step: We take

$$X_{n+1}\left(2k2^{-(n+1)}\right) = X_n\left(2k2^{-(n+1)}\right) = X_n\left(k2^{-n}\right).$$

We now determine the appropriate value for $X_{n+1}\left((2k-1)2^{-(n+1)}\right)$. Conditi
on $X_{n+1}\left(2k2^{-(n+1)}\right) - X_{n+1}\left(2(k-1)2^{-(n+1)}\right)$, Lemma 3.2.1 tells us that

$$X_{n+1}\left((2k-1)2^{-(n+1)}\right) - X_{n+1}\left(2(k-1)2^{-(n+1)}\right)$$

should be normally distributed with mean

$$\frac{1}{2}\left(X_{n+1}\left(2k2^{-(n+1)}\right) - X_{n+1}\left(2(k-1)2^{-(n+1)}\right)\right)$$

and variance $2^{-(n+2)}$.

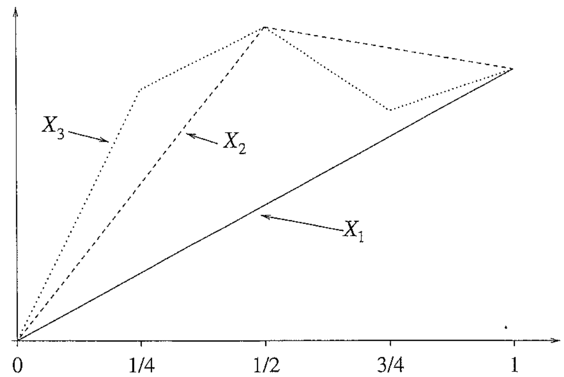


Figure 3.2 Lévy's sequence of polygonal approximations to Brownian motion.

Now if $X \sim N(0,1)$, then $aX + b \sim N(b, a^2)$ and so we take

$$\begin{aligned} X_{n+1}\left((2k-1)2^{-(n+1)}\right) - X_{n+1}\left(2(k-1)2^{-(n+1)}\right) \\ = 2^{-(n/2+1)}\xi\left((2k-1)2^{-(n+1)}\right) \\ + \frac{1}{2}\left(X_{n+1}\left(2k2^{-(n+1)}\right) - X_{n+1}\left(2(k-1)2^{-(n+1)}\right)\right). \end{aligned}$$

In other words

$$\begin{aligned} X_{n+1}\left((2k-1)2^{-(n+1)}\right) &= \frac{1}{2}X_n\left((k-1)2^{-n}\right) \\ &\quad + \frac{1}{2}X_n\left(k2^{-n}\right) + 2^{-(n/2+1)}\xi\left((2k-1)2^{-(n+1)}\right) \\ &= X_n\left((2k-1)2^{-(n+1)}\right) \\ &\quad + 2^{-(n/2+1)}\xi\left((2k-1)2^{-(n+1)}\right), \end{aligned} \tag{3.1}$$

where the last equality follows by linearity of X_n on $[(k-1)2^{-n}, k2^{-n}]$.

The construction is illustrated in Figure 3.2.

Convergence to Brownian motion

Brownian motion will be the process constructed by letting n increase to infinity. To check that it exists we need some technical lemmas. The proofs are adapted from Knight (1981).

Lemma 3.2.2

$$\mathbb{P}\left[\lim_{n\to\infty} X_n(t) \text{ exists for } 0 \le t \le 1 \textit{ uniformly} \text{ in } t\right] = 1.$$

Proof: Notice that $\max_t |X_{n+1}(t) - X_n(t)|$ will be attained at a vertex, that is for

$t \in \{(2k-1)2^{-(n+1)} : k = 1, 2, \ldots, 2^n\}$ and using (3.1)

$$\begin{aligned}\mathbb{P}\left[\max_t |X_{n+1}(t) - X_n(t)| \geq 2^{-n/4}\right] &\\ &= \mathbb{P}\left[\max_{1 \leq k \leq 2^n} \xi\left((2k-1)2^{-(n+1)}\right) \geq 2^{n/4+1}\right] \\ &\leq 2^n \mathbb{P}\left[\xi(1) \geq 2^{n/4+1}\right].\end{aligned}$$

Now using the result of Exercise 7 (with $t = 1$), for $x > 0$

$$\mathbb{P}\left[\xi(1) \geq x\right] \leq \frac{1}{x\sqrt{2\pi}} e^{-x^2/2},$$

and combining this with the fact that

$$\exp\left(-2^{(n/2+1)}\right) < 2^{-2n+2},$$

we obtain that for $n \geq 4$

$$2^n \mathbb{P}\left[\xi(1) \geq 2^{n/4+1}\right] \leq \frac{2^n}{2^{n/4+1}} \frac{1}{\sqrt{2\pi}} \exp\left(-2^{(n/2+1)}\right) \leq \frac{2^n}{2^{n/4+1}} 2^{-2n+2} <$$

Consider now for $k > n \geq 4$

$$\mathbb{P}\left[\max_t |X_k(t) - X_n(t)| \geq 2^{-n/4+3}\right] = 1 - \mathbb{P}\left[\max_t |X_k(t) - X_n(t)| \leq 2^{-n/}\right.$$

and

$$\begin{aligned}\mathbb{P}\left[\max_t |X_k(t) - X_n(t)| \leq 2^{-n/4+3}\right] &\\ &\geq \mathbb{P}\left[\sum_{j=n}^{k-1} \max_t \left|X_{j+1}(t) - X_j(t)\right| \leq 2^{-n/4+3}\right] \\ &\geq \mathbb{P}\left[\max_t \left|X_{j+1}(t) - X_j(t)\right| \leq 2^{-j/4}, j = n, \ldots, k-1\right] \\ &\geq 1 - \sum_{j=n}^{k-1} 2^{-j} \geq 1 - 2^{-n+1}.\end{aligned}$$

Finally we have that

$$\mathbb{P}\left[\max_t |X_k(t) - X_n(t)| \geq 2^{-n/4+3}\right] \leq 2^{-n+1},$$

for all $k \geq n$. The events on the left are increasing (since the maximum ca increase by the addition of a new vertex) so

$$\mathbb{P}\left[\max_t |X_k(t) - X_n(t)| \geq 2^{-n/4+3} \text{ for some } k > n\right] \leq 2^{-n+1}.$$

In particular, for $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[\text{For some } k > n \text{ and } t \leq 1,\ |X_k(t) - X_n(t)| \geq \epsilon\right] = 0,$$

which proves the lemma.

To complete the proof of existence of the Brownian motion, we must check the following.

Lemma 3.2.3 *Let $X(t) = \lim_{n\to\infty} X_n(t)$ if the limit exists uniformly and 0 otherwise. Then $X(t)$ satisfies the conditions of Definition 3.1.3 (for t restricted to $[0, 1]$).*

Proof: By construction, the properties 1–3 of Definition 3.1.3 hold for the approximation $X_n(t)$ restricted to $T_n = \{k2^{-n}: k = 0, 1, \ldots, 2^n\}$. Since we don't change X_k on T_n for $k > n$, the same must be true for X on $\bigcup_{n=1}^{\infty} T_n$. A uniform limit of continuous functions is continuous, so condition 4 holds and now by approximation of any $0 \le t_1 \le t_2 \le \cdots \le t_n \le 1$ from within the dense set $\bigcup_{n=1}^{\infty} T_n$ we see that in fact all four properties hold without restriction for $t \in [0, 1]$. □

3.3 The reflection principle and scaling

Having proved that Brownian motion actually exists, we now turn to some calculations. These will amount to no more than a small bag of tricks for us to call upon in later chapters. There are many texts devoted exclusively to Brownian motion where the reader can gain a more extensive repertoire.

By its very construction, Brownian motion has no memory. That is, if $\{W_t\}_{t\ge 0}$ is a Brownian motion and $s \ge 0$ is any fixed time, then $\{W_{t+s} - W_s\}_{t\ge 0}$ is also a Brownian motion, independent of $\{W_r\}_{0\le r\le s}$. What is also true is that for certain *random* times, T, the process $\{W_{T+t} - W_T\}_{t\ge 0}$ is again a standard Brownian motion and is independent of $\{W_s: 0 \le s \le T\}$. We have already encountered such random times in the context of discrete parameter martingales.

Definition 3.3.1 A stopping time *T for the process $\{W_t\}_{t\ge 0}$ is a random time such that for each t, the event $\{T \le t\}$ depends only on the history of the process up to and including time t.*

In other words, by observing the Brownian motion up until time t, we can determine whether or not $T \le t$.

We shall encounter stopping times only in the context of *hitting times*. For fixed a, the hitting time of level a is defined by

$$T_a = \inf\{t \ge 0 : W_t = a\}.$$

We take $T_a = \infty$ if a is never reached. It is easy to see that T_a is a stopping time since, by continuity of the paths,

$$\{T_a \le t\} = \{W_s = a \text{ for some } s, 0 \le s \le t\},$$

which depends only on $\{W_s, 0 \le s \le t\}$. Notice that, again by continuity, if $T_a < \infty$, then $W_{T_a} = a$.

Just as for random walks, an example of a random time that is *not* a stoppir is the *last* time that the process hits some level.

The reflection principle

Not surprisingly, there is often much to be gained from exploiting the syn inherent in Brownian motion. As a warm-up we calculate the distribution of 1

Lemma 3.3.2 *Let $\{W_t\}_{t\geq 0}$ be a $\mathbb{P}$-Brownian motion started from $W_0 = 0$ $a > 0$; then*

$$\mathbb{P}[T_a < t] = 2\mathbb{P}[W_t > a].$$

Proof: If $W_t > a$, then by continuity of the Brownian path, $T_a < t$. Moreove T_a is a stopping time, $\left\{W_{t+T_a} - W_{T_a}\right\}_{t\geq 0}$ is a Brownian motion, so, by syn $\mathbb{P}[W_t - W_{T_a} > 0 | T_a < t] = 1/2$. Thus

$$\begin{aligned}\mathbb{P}[W_t > a] &= \mathbb{P}[T_a < t, W_t - W_{T_a} > 0] \\ &= \mathbb{P}[T_a < t]\mathbb{P}[W_t - W_{T_a} > 0 | T_a < t] \\ &= \frac{1}{2}\mathbb{P}[T_a < t].\end{aligned}$$

A more refined version of this idea is the following.

Lemma 3.3.3 (The reflection principle) *Let $\{W_t\}_{t\geq 0}$ be a standard Browni tion and let T be a stopping time. Define*

$$\tilde{W}_t = \begin{cases} W_t, & t \leq T, \\ 2W_T - W_t, & t > T; \end{cases}$$

then $\{\tilde{W}_t\}_{t\geq 0}$ is also a standard Brownian motion.

Notice that if $T = T_a$, then the operation $W_t \mapsto \tilde{W}_t$ amounts to reflecting the of the path after the first hitting time on a in the line $x = a$ (see Figure 3.3). W prove the general form of the reflection principle here. Instead we put it into The following result will be the key to pricing certain *barrier options* in Chaj

Lemma 3.3.4 (Joint distribution of Brownian motion and its maximum) *Let* $\max_{0\leq s\leq t} W_s$, *the maximum level reached by Brownian motion in the time i* $[0, t]$. *Then for $a > 0$, $a \geq x$ and all $t \geq 0$,*

$$\mathbb{P}[M_t \geq a, W_t \leq x] = 1 - \Phi\left(\frac{2a - x}{\sqrt{t}}\right),$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

is the standard normal distribution function.

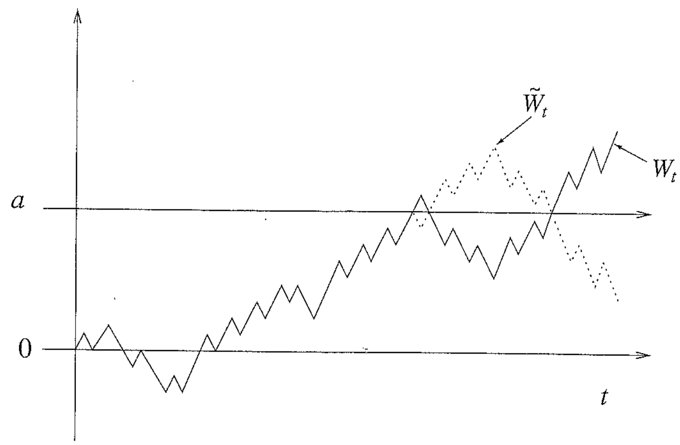


Figure 3.3 The reflection principle when $T = T_a$.

Proof: Notice that $M_t \geq 0$ and is non-decreasing in t and if, for $a > 0$, T_a is defined to be the first hitting time of level a, then $\{M_t \geq a\} = \{T_a \leq t\}$. Taking $T = T_a$ in the reflection principle, for $a \geq 0$, $a \geq x$ and $t \geq 0$,

$$\begin{aligned}
\mathbb{P}[M_t \geq a,\, W_t \leq x] &= \mathbb{P}[T_a \leq t,\, W_t \leq x] \\
&= \mathbb{P}[T_a \leq t,\, 2a - x \leq \tilde{W}_t] \\
&= \mathbb{P}[2a - x \leq \tilde{W}_t] \\
&= 1 - \Phi\left(\frac{2a - x}{\sqrt{t}}\right).
\end{aligned}$$

In the third equality we have used the fact that if $\tilde{W}_t \geq 2a - x$ then necessarily $\{\tilde{W}_s\}_{s\geq 0}$, and consequently $\{W_s\}_{s\geq 0}$, has hit level a *before* time t. □

For pricing a perpetual American put option in Chapter 6 we shall use the following result.

Proposition 3.3.5 *Set $T_{a,b} = \inf\{t \geq 0 : W_t = a + bt\}$, where $T_{a,b}$ is taken to be infinite if no such time exists. Then for $\theta > 0$, $a > 0$ and $b \geq 0$*

$$\mathbb{E}\left[\exp\left(-\theta T_{a,b}\right)\right] = \exp\left(-a\left(b + \sqrt{b^2 + 2\theta}\right)\right).$$

Proof: We defer the proof of the special case $b = 0$ until Proposition 3.4.9 when we shall have powerful martingale machinery to call upon. Here, assuming that result, we deduce the general result.

Fix $\theta > 0$, and for $a > 0$, $b \geq 0$, set

$$\psi(a, b) = \mathbb{E}\left[e^{-\theta T_{a,b}}\right].$$

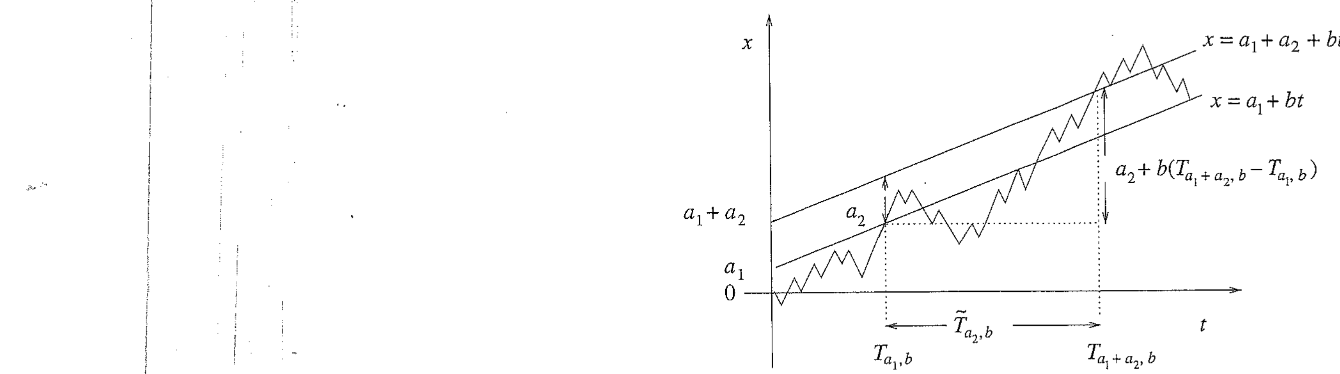


Figure 3.4 In the notation of Proposition 3.3.5, $T_{a_1+a_2,b} = T_{a_1,b} + \tilde{T}_{a_2,b}$ where $\tilde{T}_{a_2,b}$ has the sam distribution as $T_{a_2,b}$.

Now take any two values for a, a_1 and a_2 say, and notice (see Figure 3.4) that

$$T_{a_1+a_2,b} = T_{a_1,b} + \left(T_{a_1+a_2,b} - T_{a_1,b}\right) \stackrel{\mathcal{D}}{=} T_{a_1,b} + \tilde{T}_{a_2,b},$$

where $\tilde{T}_{a_2,b}$ is independent of $T_{a_1,b}$ and has the same distribution as $T_{a_2,b}$. In words,

$$\psi(a_1 + a_2, b) = \psi(a_1, b)\psi(a_2, b),$$

and this implies that

$$\psi(a, b) = e^{-k(b)a},$$

for some function $k(b)$.

Since $b \geq 0$, the process must hit level a *before* it can hit the line $a + b$ use this to break $T_{a,b}$ into two parts; see Figure 3.5. Writing f_{T_a} for the proba density function of the random variable T_a and conditioning on T_a, we obtain

$$\begin{aligned}
\psi(a, b) &= \int_0^\infty f_{T_a}(t)\mathbb{E}\left[e^{-\theta T_{a,b}}\middle| T_a = t\right] dt \\
&= \int_0^\infty f_{T_a}(t)e^{-\theta t}\mathbb{E}\left[e^{-\theta T_{bt,b}}\right] dt \\
&= \int_0^\infty f_{T_a}(t)e^{-\theta t}e^{-k(b)bt} dt \\
&= \mathbb{E}\left[e^{-(\theta+k(b)b)T_a}\right] \\
&= \exp\left(-a\sqrt{2(\theta + k(b)b)}\right).
\end{aligned}$$

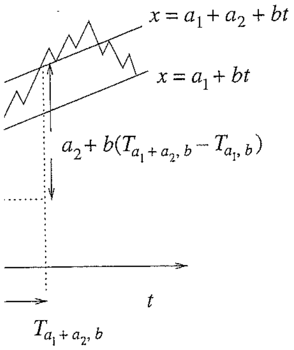
We now have two expressions for $\psi(a, b)$. Equating them gives

$$k^2(b) = 2\theta + 2k(b)b.$$

Since for $\theta > 0$ we must have $\psi(a, b) \leq 1$, we choose

$$k(b) = b + \sqrt{b^2 + 2\theta},$$

which completes the proof.



$_{1,b} + \tilde{T}_{a_2,b}$ where $\tilde{T}_{a_2,b}$ has the same

and notice (see Figure 3.4) that

$\cdot\, T_{a_1,b}) \stackrel{\mathcal{D}}{=} T_{a_1,b} + \tilde{T}_{a_2,b},$

he same distribution as $T_{a_2,b}$. In other

$_1, b)\psi(a_2, b),$

$^{-k(b)a},$

: *before* it can hit the line $a + bt$. We
ıre 3.5. Writing f_{T_a} for the probability
.d conditioning on T_a, we obtain

$\left[e^{-\theta T_{a,b}} \middle| T_a = t\right] dt$

$^{-\theta t}\mathbb{E}\left[e^{-\theta T_{bt,b}}\right] dt$

$^{-\theta t}e^{-k(b)bt}dt$

$^{b)T_a}\Big]$

$\overline{\theta + k(b)b)}\Big).$

ɔuating them gives

$2k(b)b.$

ve choose

$\overline{b^2 + 2\theta},$

□

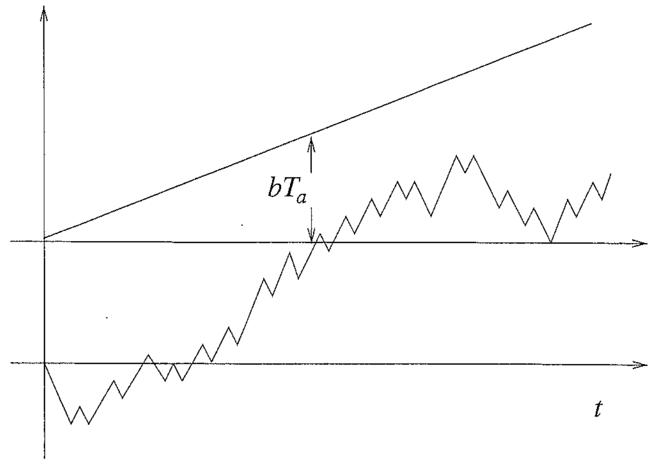


Figure 3.5 In the notation of Proposition 3.3.5, $T_{a,b} = T_a + \tilde{T}_{bT_a,b}$ where $\tilde{T}_{bT_a,b}$ has the same distribution as $T_{bT_a,b}$.

Definition 3.3.6 *For a real constant μ, we refer to the process $W_t^\mu = W_t + \mu t$ as a* Brownian motion with drift μ.

In the notation above, $T_{a,b}$ is the first hitting time of the level a by a Brownian motion with drift $-b$.

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We conclude this section with the following useful result.

Proposition 3.3.7
If $\{W_t\}_{t\geq 0}$ is a standard Brownian motion, then so are

1. $\{cW_{t/c^2}\}_{t\geq 0}$ *for any real c,*
2. $\{tW_{1/t}\}_{t\geq 0}$ *where $tW_{1/t}$ is taken to be zero when $t = 0$,*
3. $\{W_s - W_{s-t}\}_{0\leq t\leq s}$ *for any fixed $s \geq 0$.*

Proof: The proofs of 1–3 are similar. For example in the case of 2, it is clear that $tW_{1/t}$ has continuous sample paths (at least for $t > 0$) and that for any $t_1, \ldots, t_n$, the random variables $\{t_1 W_{1/t_1}, \ldots, t_n W_{1/t_n}\}$ have a multivariate normal distribution. We must just check that the covariance takes the right form, but

$$\mathbb{E}\left[sW_{1/s}tW_{1/t}\right] = st\mathbb{E}\left[W_{1/s}W_{1/t}\right] = st\left(\frac{1}{s} \wedge \frac{1}{t}\right) = s \wedge t,$$

and the proof is complete. □

3.4 Martingales in continuous time

Just as in discrete time, the notion of a martingale plays a key rôle in our continuous time models.

Recall that in discrete time, a sequence $X_0, X_1, \ldots, X_n$ for which $\mathbb{E}[|X_r|]$ for each r is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$ and a prob measure $\mathbb{P}$ if

$$\mathbb{E}\left[X_r|\mathcal{F}_{r-1}\right] = X_{r-1} \qquad \text{for all } r \geq 1.$$

We can make entirely analogous definitions in continuous time.

Filtrations

Definition 3.4.1 *Let $\mathcal{F}$ be a σ-field. We call $\{\mathcal{F}_t\}_{t\geq 0}$ a* filtration *if*

1. *$\mathcal{F}_t$ is a sub-σ-field of $\mathcal{F}$ for all t, and*
2. *$\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s < t$.*

As in the discrete setting we are primarily concerned with the *natural* filt $\{\mathcal{F}_t^X\}_{t\geq 0}$, associated with a stochastic process $\{X_t\}_{t\geq 0}$. As before, $\mathcal{F}_t^X$ encod information generated by the stochastic process X on the interval $[0, t]$. T $A \in \mathcal{F}_t^X$ if, based upon observations of the trajectory $\{X_s\}_{0\leq s\leq t}$, it is possi decide whether or not A has occurred.

Notation: If the value of a stochastic variable Z can be completely de mined given observations of the trajectory $\{X_s\}_{0\leq s\leq t}$ then we write

$$Z \in \mathcal{F}_t^X.$$

More than one process can be measurable with respect to the same filtration.

Definition 3.4.2 *If $\{Y_t\}_{t\geq 0}$ is a stochastic process such that we have $Y_t \in$ all $t \geq 0$, then we say that $\{Y_t\}_{t\geq 0}$ is* adapted *to the filtration $\{\mathcal{F}_t^X\}_{t\geq 0}$.*

Example 3.4.3

1. *The stochastic process*
$$Z_t = \int_0^t X_s ds$$
is adapted to $\{\mathcal{F}_t^X\}_{t\geq 0}$.
2. *The process $M_t = \max_{0\leq s\leq t} W_s$ is adapted to the filtration $\{\mathcal{F}_t^W\}_{t\geq 0}$.*
3. *The stochastic process $Z_t \triangleq W_{t+1}^2 - W_t^2$ is* not *adapted to the filtration gener $\{W_t\}_{t\geq 0}$.*

Notice that just as in the discrete world we have divorced the rôles of the sto process and the probability measure. Thus a process may be a Brownian under the probability measure $\mathbb{P}$, but the *same* process not be a Brownian under a different measure $\mathbb{Q}$.

Martingales

Definition 3.4.4 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t$ family $\{M_t\}_{t\geq 0}$ of random variables on this space with $\mathbb{E}[|M_t|] < \infty$ for all t*

a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale *if it is adapted to* $\{\mathcal{F}_t\}_{t\geq 0}$ *and for any* $s \leq t$,

$$\mathbb{E}^{\mathbb{P}}\left[M_t \mid \mathcal{F}_s\right] = M_s.$$

By restricting the conditions to $t \in [0, T]$, *we define martingales parametrised by* $[0, T]$.

Generally we shall be sloppy about specifying the filtration. In all of our examples there will be a Brownian motion around and it will be implicit that the filtration is that generated by the Brownian motion.

A more general notion is that of local martingale.

Definition 3.4.5 *A process* $\{X_t\}_{t\geq 0}$ *is a* local $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$-martingale *if there is a sequence of* $\{\mathcal{F}_t\}_{t\geq 0}$*-stopping times* $\{T_n\}_{n\geq 1}$ *such that* $\{X_{t\wedge T_n}\}_{t\geq 0}$ *is a* $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$*-martingale for each n and*

$$\mathbb{P}\left[\lim_{n\to\infty} T_n = \infty\right] = 1.$$

All martingales are local martingales but the converse is false. It is because of this distinction that we impose boundedness conditions in many of our results of Chapter 4.

Lemma 3.4.6 *Let* $\{W_t\}_{t\geq 0}$ *generate the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$. *If* $\{W_t\}_{t\geq 0}$ *is a standard Brownian motion under the probability measure* $\mathbb{P}$, *then*

1. W_t *is a* $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$*-martingale,*
2. $W_t^2 - t$ *is a* $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$*-martingale,*
3. $$\exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)$$

is a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$*-martingale, called an* exponential martingale.

Proof: The proofs are all rather similar. For example, consider $M_t = W_t^2 - t$. Evidently $\mathbb{E}[|M_t|] < \infty$. Now

$$\begin{aligned}
\mathbb{E}\left[W_t^2 - W_s^2 \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[(W_t - W_s)^2 + 2W_s(W_t - W_s) \middle| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[(W_t - W_s)^2 \middle| \mathcal{F}_s\right] + 2W_s\mathbb{E}\left[(W_t - W_s) \mid \mathcal{F}_s\right] \\
&= t - s.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}\left[W_t^2 - t \middle| \mathcal{F}_s\right] &= \mathbb{E}\left[W_t^2 - W_s^2 + W_s^2 - (t-s) - s \middle| \mathcal{F}_s\right] \\
&= (t-s) + W_s^2 - (t-s) - s = W_s^2 - s.
\end{aligned}$$

$\square$

Optional stopping

What we should really like is the continuous time analogue of the Optional Sto
Theorem. In general, we have to be a little careful (see Exercise 17 for wh
go wrong). Problems arise if the sample paths of our martingale are not suffic
'nice'. In all our examples the stochastic process will have càdlàg sample path

Definition 3.4.7 *The function $f : \mathbb{R} \to \mathbb{R}$ is* càdlàg *if it is right continuou
left limits.*

In particular, continuous functions are automatically càdlàg (continues à
limites à gauche).

Theorem 3.4.8 (Optional Stopping Theorem) *If $\{M_t\}_{t\geq 0}$ is a càdlàg mart
with respect to the probability measure $\mathbb{P}$ and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and if τ_1
are two stopping times such that $\tau_1 \leq \tau_2 \leq K$ where K is a finite real number,*

$$\mathbb{E}\left[\left|M_{\tau_2}\right|\right] < \infty$$

and

$$\mathbb{E}\left[M_{\tau_2}\middle|\mathcal{F}_{\tau_1}\right] = M_{\tau_1}, \qquad \mathbb{P}\text{-}a.s.$$

Remarks:

1. The term 'a.s.' (almost surely) means with ($\mathbb{P}$-) probability one.
2. Notice in particular that if τ is a bounded stopping time then $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$

Brownian hitting time distribution

Just as in the discrete case the Optional Stopping Theorem will be a po
tool. We illustrate by calculating the moment generating function for the
time T_a of level a by Brownian motion. (This result was essential to our pr
Proposition 3.3.5.)

Proposition 3.4.9 *Let $\{W_t\}_{t\geq 0}$ be a Brownian motion and let $T_a = \inf\{s$
$W_s = a\}$ (or infinity if that set is empty). Then for $\theta > 0$,*

$$\mathbb{E}\left[e^{-\theta T_a}\right] = e^{-\sqrt{2\theta}|a|}.$$

Proof: We assume that $a \geq 0$. (The case $a < 0$ follows by symmetry.) We
like to apply the Optional Stopping Theorem to the martingale

$$M_t = \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)$$

and the random time T_a, but we encounter a familiar obstacle. We *cannot*
the Theorem directly to T_a as it may not be bounded. Instead we take $\tau_1 =$
$\tau_2 = T_a \wedge n$. This gives us that

$$\mathbb{E}\left[M_{T_a\wedge n}\right] = 1.$$

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of our martingale are not sufficiently
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the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and if τ_1 and τ_2
where K is a finite real number, then

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, $\mathbb{P}$-*a.s.*

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□

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0 follows by symmetry.) We should
o the martingale

$-\frac{1}{2}\sigma^2 t\Big)$

a familiar obstacle. We *cannot* apply
ounded. Instead we take $\tau_1 = 0$ and

$= 1.$

So

$$
\begin{aligned}
1 = \mathbb{E}\left[M_{T_a \wedge n}\right] &= \mathbb{E}\left[M_{T_a\wedge n}\middle| T_a < n\right]\mathbb{P}\left[T_a < n\right] \\
&\quad + \mathbb{E}\left[M_{T_a\wedge n}\middle| T_a > n\right]\mathbb{P}\left[T_a > n\right].
\end{aligned} \tag{3.2}
$$

Now, by Lemma 3.3.2 and the result of Exercise 7,

$$\mathbb{P}\left[T_a < n\right] \to 1 \qquad \text{as } n \to \infty.$$

Also, if $T_a < \infty$, $\lim_{n\to\infty} M_{T_a\wedge n} = M_{T_a}$, whereas if $T_a = \infty$, $W_t \leq a$ for all t and so $\lim_{n\to\infty} M_{T_a\wedge n} = 0$. Letting $n \to \infty$ in equation (3.2) then yields

$$\mathbb{E}\left[M_{T_a}\right] = 1.$$

Taking $\sigma^2 = 2\theta$ completes the proof. □

ominated
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Arguments of this type are often simplified by an application of the Dominated Convergence Theorem.

Theorem 3.4.10 (Dominated Convergence Theorem) *Let $\{Z_n\}_{n\geq 1}$ be a sequence of random variables with $\lim_{n\to\infty} Z_n = Z$. If there is a random variable Y with $|Z_n| < Y$ for all n and $\mathbb{E}[Y] < \infty$, then*

$$\mathbb{E}[Z] = \lim_{n\to\infty} \mathbb{E}[Z_n].$$

In the proof of Proposition 3.4.9, since

$$0 \leq M_{T_a\wedge n} = \exp\left(\sigma W_{T_a\wedge n} - \frac{1}{2}\sigma^2\left(T_a \wedge n\right)\right) \leq \exp\left(\sigma a\right),$$

we could take the *constant* $e^{\sigma a}$ as the dominating random variable Y.

Exercises

1 Suppose that $\{S_n\}_{n\geq 0}$ is a symmetric simple random walk under $\mathbb{P}$. Show that

$$\mathbb{P}\left[\frac{S_{[nt]}}{\sqrt{n}} \leq x\right] \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right) dy$$

as $n \to \infty$ where $[nt]$ is the integer part of nt.

2 Let Z be normally distributed with mean zero and variance one under the measure $\mathbb{P}$. What is the distribution of $\sqrt{t}Z$? Is the process $X_t = \sqrt{t}Z$ a Brownian motion?

3 Suppose that W_t and $\tilde{W}_t$ are independent Brownian motions under the measure $\mathbb{P}$ and let $\rho \in [-1, 1]$ be a constant. Is the process $X_t = \rho W_t + \sqrt{1-\rho^2}\tilde{W}_t$ a Brownian motion?

4 Let $\{W_t\}_{t\geq 0}$ be standard Brownian motion under the measure $\mathbb{P}$. Which following are $\mathbb{P}$-Brownian motions?

(a) $\{-W_t\}_{t\geq 0}$,
(b) $\{cW_{t/c^2}\}_{t\geq 0}$, where c is a constant,
(c) $\{\sqrt{t}W_1\}_{t\geq 0}$,
(d) $\{W_{2t} - W_t\}_{t\geq 0}$.

Justify your answers.

5 Suppose that X is normally distributed with mean μ and variance σ^2. Calcula

$$\mathbb{E}\left[e^{\theta X}\right]$$

and hence evaluate $\mathbb{E}\left[X^4\right]$.

6 Prove Lemma 3.4.6.3.

7 Prove that if $\{W_t\}_{t\geq 0}$ is standard Brownian motion under $\mathbb{P}$ then, for $x > 0$,

$$\mathbb{P}[W_t \geq x] \equiv \int_x^\infty \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t}\,dy \leq \frac{\sqrt{t}}{x\sqrt{2\pi}} e^{-x^2/2t}.$$

[Hint: Integrate by parts.]

8 Let $\{W_t\}_{t\geq 0}$ be standard Brownian motion under $\mathbb{P}$. Let $Z = \sup_t W_t$. Eviden any $c > 0$, cZ has the same distribution as Z. Deduce that, with probabilit $Z \in \{0, \infty\}$. Let $p = \mathbb{P}[Z = 0]$. By conditioning on the event $\{W_1 \leq 0\}$, pro

$$\mathbb{P}[Z = 0] \leq \mathbb{P}[W_1 \leq 0]\,\mathbb{P}[Z = 0],$$

and hence $p = 0$. Deduce that

$$\mathbb{P}\left[\sup_t W_t = +\infty, \inf_t W_t = -\infty\right] = 1.$$

9 Deduce from the result of Exercise 8 and the result of Proposition 3.3.7.2 that

$$\mathbb{P}[\text{For each } \epsilon > 0, \exists s, t \leq \epsilon \text{ such that } W_s < 0 < W_t] = 1.$$

Deduce that if $\{W_t\}_{t\geq 0}$ is differentiable at zero, then the derivative must be ze hence $|W_t| \leq t$ for all sufficiently small t. By considering $\tilde{W}_s \triangleq sW_{1/s}$, arri contradiction and deduce that Brownian motion is *not* differentiable at zero.

10 Brownian motion is not going to be adequate as a stock market model. F has constant mean, whereas the stock of a company usually grows at som if only due to inflation. Moreover, it may be too 'noisy' (that is the variance increments may be bigger than those observed for the stock) or not noisy e We can scale to change the 'noisiness' and we can artificially introduce a dri this still won't be a good model. Here is one reason why. Suppose that $\{W_t$ standard Brownian motion under $\mathbb{P}$. Define a new process $\{S_t\}_{t\geq 0}$ by $S_t = \mu t$ - where $\sigma > 0$ and $\mu \in \mathbb{R}$ are constants. Show that for all values of $\sigma > 0$, $\mu \in$ $T > 0$ there is a positive probability that S_T is negative.

under the measure $\mathbb{P}$. Which of the

ean μ and variance σ^2. Calculate

tion under $\mathbb{P}$ then, for $x > 0$,

$$\ldots dy \leq \frac{\sqrt{t}}{x\sqrt{2\pi}} e^{-x^2/2t}.$$

ler $\mathbb{P}$. Let $Z = \sup_t W_t$. Evidently, for
Z. Deduce that, with probability one,
ing on the event $\{W_1 \leq 0\}$, prove that

$$0]\,\mathbb{P}[Z = 0],$$

$$\ldots = -\infty\Big] = 1.$$

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ew process $\{S_t\}_{t\geq 0}$ by $S_t = \mu t + \sigma W_t$
hat for all values of $\sigma > 0$, $\mu \in \mathbb{R}$ and
negative.

11 Suppose that $\{W_t\}_{t\geq 0}$ is standard Brownian motion. Prove that conditional on $W_{t_1} = x_1$, the probability density function of $W_{t_1/2}$ is

$$\sqrt{\frac{2}{\pi t_1}} \exp\left(-\frac{1}{2}\left(\frac{\left(x - \frac{1}{2}x_1\right)^2}{t_1/4}\right)\right).$$

12 Let $\{W_t\}_{t\geq 0}$ be standard Brownian motion under $\mathbb{P}$. Let T_a be the 'hitting time of level a', that is

$$T_a = \inf\{t \geq 0 : W_t = a\}.$$

Then we proved in Proposition 3.4.9 that

$$\mathbb{E}\left[\exp\left(-\theta T_a\right)\right] = \exp\left(-a\sqrt{2\theta}\right).$$

Use this result to calculate

(a) $\mathbb{E}[T_a]$,
(b) $\mathbb{P}[T_a < \infty]$.

13 Let $\{W_t\}_{t\geq 0}$ denote standard Brownian motion under $\mathbb{P}$ and define $\{M_t\}_{t\geq 0}$ by

$$M_t = \max_{0\leq s\leq t} W_s.$$

Suppose that $x \geq a$. Calculate

(a) $\mathbb{P}[M_t \geq a,\, W_t \geq x]$,
(b) $\mathbb{P}[M_t \geq a,\, W_t \leq x]$.

14 Let $\{W_t\}_{t\geq 0}$ be standard Brownian motion under $\mathbb{P}$. Let $T_{a,b}$ denote the hitting time of the sloping line $a + bt$. That is,

$$T_{a,b} = \inf\{t \geq 0 : W_t = a + bt\}.$$

We proved in Proposition 3.3.5 that for $\theta > 0$, $a > 0$ and $b \geq 0$

$$\mathbb{E}\left[\exp\left(-\theta T_{a,b}\right)\right] = \exp\left(-a\left(b + \sqrt{b^2 + 2\theta}\right)\right).$$

The aim of this question is to calculate the distribution of $T_{a,b}$, without inverting the Laplace transform. In what follows, $\phi(x) = \Phi'(x)$ and

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy.$$

(a) Find $\mathbb{P}[T_{a,b} < \infty]$.
(b) Using the fact that $s W_{1/s}$ has the same distribution as W_s, show that

$$\mathbb{P}\left[T_{a,b} \leq t\right] = \mathbb{P}\left[W_s \geq as + b \text{ for some } s \text{ with } 1/t \leq s < \infty\right].$$

(c) By conditioning on the value of $W_{1/t}$, use the previous part to show that

$$\mathbb{P}\left[T_{a,b} \le t\right] = \int_{-\infty}^{b+a/t} \mathbb{P}\left[T_{b-x+a/t,a} < \infty\right]\phi(\sqrt{t}x)\,dx + 1 - \Phi\left(\frac{a+b}{\sqrt{t}}\right)$$

(d) Substitute for the probability in the integral and deduce that

$$\mathbb{P}\left[T_{a,b} \le t\right] = e^{-2ab}\Phi\left(\frac{bt-a}{\sqrt{t}}\right) + 1 - \Phi\left(\frac{a+bt}{\sqrt{t}}\right).$$

15 Let $\{W_t\}_{t\ge 0}$ be standard Brownian motion under the measure $\mathbb{P}$ and let $\{\mathcal{F}$ denote its natural filtration. Which of the following are $(\mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0})$-martingal

(a) $\exp(\sigma W_t)$,
(b) cW_{t/c^2}, where c is a constant,
(c) $tW_t - \int_0^t W_s ds$.

16 Let $\{\mathcal{F}_t\}_{0\le t\le T}$ denote the natural filtration associated to a standard $\mathbb{P}$-Bro motion, $\{W_t\}_{0\le t\le T}$. The result of Lemma 3.4.6.3 can be rewritten as

$$\mathbb{E}\left[\exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right); A\right] = \exp\left(\sigma W_s - \frac{1}{2}\sigma^2 s\right)\mathbf{1}_A, \quad \text{for all } A \in \mathcal{F}$$

Use differentiation under the integral sign to provide another proof that $\{W_t^2 -$ is a $(\mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0})$-martingale and show that the following are also $(\mathbb{P}, \{\mathcal{F}_t$ martingales:

(a) $W_t^3 - 3tW_t$,
(b) $W_t^4 - 6tW_t^2 + 3t^2$.

17 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose that the real random variable T : $\mathbb{R}$ is uniformly distributed on $[0, 1]$ under the measure $\mathbb{P}$. Define $\{X_t\}_{t\ge 0}$ by

$$X_t(\omega) = \begin{cases} 1, & T(\omega) = t, \\ 0, & T(\omega) \ne t. \end{cases}$$

Check that $\{X_t\}_{t\ge 0}$ is a $\mathbb{P}$-martingale with respect to its own filtration. Conditional expectation is only unique to within a random variable that is surely zero.]
Show that T is a stopping time for which the Optional Stopping Theorem fail

18 As before, let T_a, T_b denote the first hitting times of levels a and b respecti a $\mathbb{P}$-Brownian motion, $\{W_t\}_{t\ge 0}$, but now W_0 is not necessarily zero (see the re after Definition 3.1.3). Prove that if $a < x < b$ then

$$\mathbb{P}\left[T_a < T_b \mid W_0 = x\right] = \frac{(b-x)}{(b-a)}.$$

[Hint: Mimic the proof of the corresponding result for random walk, cf. Pr tion 2.4.4.]

19 Using the notation of Exercise 18, let $T = T_a \wedge T_b$. Prove that if $a < 0 < b$ tl

$$\mathbb{E}\left[T \mid W_0 = 0\right] = -ab.$$