FIRST PART

. . . AND GAMES

But leave the Wise to wrangle, and with me
The Quarrel of the Universe let be:
And, in some corner of the Hubbub coucht,
Make Game of that which makes as much of Thee

The Rubaiyat of Omar Khayyam

CHAPTER 7

Playing Several Games at Once

For when the One Great Scorer comes
to write against your name,
He marks—not that you won or lost—
but how you played the game.

Grantland Rice,
Alumnus Football

The games we shall consider are in spirit closer to Chess than to Football. We imagine them played, on some kind of board perhaps, between two players whose usual names are *Left* and *Right*. [Aliases (respectively) Black and White, Vertical and Horizontal, Arthur and Bertha.] Our own sympathies are usually with Left.

The games these people play have *positions*, and in any position P, there are rules which restrict Left to move to any one of certain positions (typically P^L) called the *Left options* of P, while Right may similarly move only to certain positions (typically P^R) called the *Right options* of P. Since we are interested only in the abstract structure of games, we can regard any position P as being completely determined by its Left and Right options, and so we shall write $P = \{P^L \mid P^R\}$.

Thus if in some game there is a position P from which Left may move to any one of certain positions A, B, C (only), while Right may move only to the position D, then we write $P = \{A, B, C \mid D\}$.

A game obviously *ends* when the player who is called upon to move finds himself unable to do so. So for instance the position $\{ \mid U, V, W, X\}$, with Left about to move, obviously corresponds to an ended game. Except in Chapters 12 and 14, we adopt the *normal play* convention, according to which *a player who is unable to move when called upon to do so is the loser*. This is obviously a natural convention, for since we normally consider ourselves as losing when we cannot find any good move, we should obviously lose when we cannot find any move at all!

Our players Left and Right are usually unwilling to play games that are capable of going on forever (they are both busy men, with heavy political

responsibilities). So except for a moment in Chapter 11, we adopt the convention that in no game is there an infinite sequence of positions each of which is an option of its predecessor. [Including *in particular* the case when these options are alternately Left and Right.]

Each game G has its own proper *starting position*, the position from which we usually start to play. But for any position P of G we can obviously obtain a shortened game by starting instead at P. We find it handy to identify this game with P, so that in particular every game G will automatically be identified with its starting position.

It follows from these conventions that games can be represented by *trees*, the positions being represented by nodes (the initial position being the lowest node, or *root*), and the legal moves by branches. We shall always draw these trees so that the moves for Left are represented by leftwards slanting branches, and those for Right by rightwards slanting ones.

EXAMPLES OF SIMPLE GAMES

In Fig. 4 we draw these trees for the four simplest games (born on days 0 and 1),

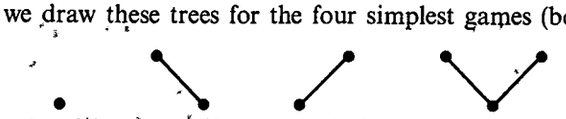


FIG. 4. The simplest games.

The simplest game of all is the *Endgame*, 0. I courteously offer you the first move in this game, and call upon you to make it. You lose, of course, because 0 is defined as the game in which it is never legal to make a move.

In the game $1 = \{0\,|\,\}$, there is a legal move for Left, which ends the game, but at no time is there any legal move for Right. If I play Left, and you Right, and you have first move again (only fair, as you lost the previous game) you will lose again; being unable to move even from the initial position. To demonstrate my skill, I shall now start from the same position, make my legal move to 0, and call upon you to make yours.

Of course you are now beginning to suspect that Left always wins, so for our next game, -1, you may play as Left and I as Right! For the last of our examples, the new game $* = \{0\,|\,0\}$, you may play whichever role you wish, provided that for this privilege you allow me to play first.

We summarise your probable conclusions:

In the game 0, there is a winning strategy for the second player

In the game 1, there is a winning strategy for Left (whoever starts)

In the game -1, there is a winning strategy for Right; and, finally,

In the game $*$, there is a winning strategy for the first player to move.

In general, we introduce corresponding notations:

$G > 0$ (G is *positive*) if there is a winning strategy for Left

$G < 0$ (G is *negative*) if there is a winning strategy for Right

$G = 0$ (G is *zero*) if there is a winning strategy for the second player,

$G \parallel 0$ (G is *fuzzy*) if there is one for the first player.

We shall also combine these symbols:

$G \geqslant 0$ means $G > 0$ *or* $G = 0$; $G \leqslant 0$ means $G < 0$ *or* $G = 0$;

$G \rhd\!\lhd 0$ means $G > 0$ *or* $G \parallel 0$; $G \lhd\!\lhd 0$ means $G < 0$ *or* $G \parallel 0$.

Thus $G \geqslant 0$ means that supposing *Right* starts, there is a winning strategy for Left, while $G \rhd\!\lhd 0$ means that there is a winning strategy for Left if *Left* starts. In slightly less formal terms, justified by Theorem 50, we can say that $G \geqslant 0$ if there is *no* winning first move for Right (the start of a winning strategy for him), while $G \rhd\!\lhd 0$ means that there *is* a winning first move for Left.

THEOREM 50. *Each game G belongs to one of the outcome classes above.*

Proof. This is equivalent to the assertion that for each game G, we have either $G \geqslant 0$ or $G \lhd\!\lhd 0$, and either $G \leqslant 0$ or $G \rhd\!\lhd 0$. Suppose that this is true of all G^L, G^R. Then if any $G^L \geqslant 0$, Left can win by first moving to this G^L, and then following with his strategy for this G^L, Right starting. If not, we have each $G^L \lhd\!\lhd 0$, and Right has a winning strategy in G, Left starting. He just sits back and waits until Left has moved to some G^L, and then applies his winning strategy (Right starting) in that G^L.

THE NEGATIVE OF A GAME

Since the legal moves for the two players are not necessarily the same, we may obtain a distinct game by reversing the roles of Left and Right throughout G. The game so obtained we call the *negative* of G. Inductively, it is the game $-G$ defined by the equation

$$-G = \{-G^R\,|\,-G^L\}.$$

Obviously, negation interchanges positive and negative games, while the negative of a zero or fuzzy game is another game of the same type.

SIMULTANEOUS DISPLAYS. SUMS OF GAMES

Left and Right are given to playing simultaneous displays of games against each other, in the following manner. Each game is placed on a table, and

when it is Left's turn to move, he selects one of the component games, and makes any move legal for Left in that game. Then Right selects some component game (possibly the same as that used by Left, possibly not), and makes a move legal for Right in that game. The game continues in this way until some player is unable to move in any of the components, when of course that player loses, according to the normal play convention.

When games G and H are played as a simultaneous display in this manner, we refer to the compound game as the *disjunctive sum* $G + H$ of the two games. Most of the rest of this book is concerned with such disjunctive sums—which we therefore simply call *sums*—but in Chapter 14 we shall consider some other kinds of simultaneous display, which will lead to other operations on games.

HOW SUMS HAPPEN—A GAME WITH DOMINOES

In fact it often happens in some real-life game that a position breaks up into a disjunctive sum, because it is obvious for some reason that moves made in one part of the position will not affect the other parts. Consider for example the following game with dominoes, suggested by Göran Andersson.

On a rectangular board ruled into squares, the players alternately place dominoes which cover two adjacent squares, Left being required to place his dominoes vertically, Right horizontally. The dominoes must not overlap, and the last player able to move is the winner.

After a time, the vacant spaces left on the board are usually in several separated regions, and the game becomes a sum of smaller games one for each region. We analyse the simplest possibilities.

A region □ contains no move for either player, and so is abstractly the game $\{\,|\,\} = 0$. Such regions can be neglected.

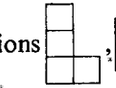
A region or has just one move for Left (to 0), but none for Right. Its value is therefore $\{0\,|\,\} = 1$, and indeed it confers an advantage of just one move upon Left. Similarly the region □□□□ is -2, since it has no move for Left, but moves for Right to 0 and -1, and we recall $\{\,|\,0, -1\} = -2$.

In general, if a position has no move for Right at any time, and at most n successive moves for Left, its value is n, and the value will be $-n$ if we reverse the roles of Left and Right here.

The region is more interesting. Left has one (stupid) move to □□ $= -1$ and another (more sensible) move to □ + □ $= 0$, whereas Right has only one move to $= 1$. So the value should be $\{0, -1\,|\,1\}$, which

the diligent reader of the zeroth part of this book will recognise as $\frac{1}{2}$. And there is indeed a definite sense in which this region represents an advantage of exactly one half of a move to Left!

Values other than numbers can occur in this domino game. The region has value $\{0\,|\,0\} = *$, since either player can move to □ $= 0$ (only), while the region has value $\{1\,|\,-1\}$ since Left moves to $= 1$, and Right by symmetry to -1.

The dominoes position with regions  , , □□□□ (only) has the value $\frac{1}{2} + 1 - 2 = -\frac{1}{2}$. Since this is *negative*, Right is half-a-move ahead, and can win the game, no matter who starts.

SUMS OF SIMPLE GAMES

Since it is never legal to move in 0, the game $G + 0$ is essentially the same as G, and we write $G + 0 = G$.

The game $1 + 1$. From the sum $1 + 1$, Left can move to $1 + 0$ or $0 + 1$, both essentially the same as 1. Since Right can never move, we have $1 + 1 = \{1, 1\,|\,\}$, and since Left's two moves are essentially the same, we can simplify this further to $1 + 1 = \{1\,|\,\}$. This game we call 2. It is a positive game, since Left has moves but Right has not.

The game $1 - 1$. We write $1 - 1$ for the sum $1 + (-1)$. In this, Left can only move to $0 + -1 = -1$ (which is a win for Right), and Right can only move to $1 + 0$, a win for Left. So neither player will really want to move, and the game is a zero game. In symbols, we have $1 - 1 \equiv \{-1\,|\,1\} = 0$.

The game $* + *$. In a similar way, $* + * \equiv \{*\,|\,*\}$, which, since $*$ is a win for the first player, is a second player win. So we have $* + * = 0$.

What do these equalities mean?

There is a famous story of the little girl who played a kind of simultaneous display against two Chess Grandmasters (surely a Big concept!). How was it that she managed to win one of the games? Anne-Louise played Black against Spassky, White against Fischer. Spassky moved first, and Anne-Louise just copied his move as the first move of her game against Fischer, then copied Fischer's reply as her own reply to Spassky's first move, and so on.

THEOREM 51. *$G - G$ is always a zero game.*

Proof. The moves legal for one player in G become legal for his opponent

in $-G$, and vice versa. So the second player can win $G - G$ by always mimicking her opponent's previous move—if Left moves to G^L in G, Right (as second player) can move to $-G^L$ in $-G$. If she plays in this way, the second player will never be lost for a move in $G - G$.

In a similar way, we can prove:

THEOREM 52. *From $G \geqslant 0$ and $H \geqslant 0$, we can deduce $G + H \geqslant 0$.*

Proof. The suppositions tell us that if Right starts, Left can win each of G and H. But he can then win $G + H$ by always replying in the component Right moves in, and making the winning reply in this component. In this way, Left cannot be lost for a move in G or H, and so will win the sum.

THEOREM 53. *If H is a zero game, then $G + H$ has the same outcome as G.*

Proof. This can be made to follow from the previous theorem, but we give it a separate proof. Play $G + H$, in exactly the same way as you would in G, never moving in the H component except to reply to an immediately previous move of your opponent in that game. This rule converts a winning strategy for you in G to one for you in $G + H$, it being understood that the same player starts in both cases.

THEOREM 54. *If $H - K$ is a zero game, then the games $G + H$ and $G + K$ have always the same outcome.*

Proof. $G + K$ has the same outcome as $(G + K) + (H - K)$, by Theorem 53. But this can be written as $(G + H) + (K - K)$, which has the same outcome as $G + H$, since $K - K$ is a zero game.

Now our aim in this book is to find out who wins sums of various games, so that if $H - K$ is a zero game, it will not matter if we replace H by K. So in this case, we shall say that H is *equal* to K, and write $H = K$. We shall not usually distinguish between equal games, and so when we speak of the game 0, we mean to refer also to the games $1 - 1$, $* + *$, and so on. On occasions when it is necessary to make these distinctions, we speak of the *form* of a game (meaning some particular game, regarded as distinct from its equals) and the *value* of a game (G and H having the same value when $G = H$).

SOME MORE GAMES

The game $\frac{1}{2}$. We define $\frac{1}{2} = \{0 \mid 1\}$, and verify the equality $\frac{1}{2} + \frac{1}{2} = 1$. In Fig. 5 we have drawn the components of the game $\frac{1}{2} + \frac{1}{2} - 1$, with letters for the names of various positions.

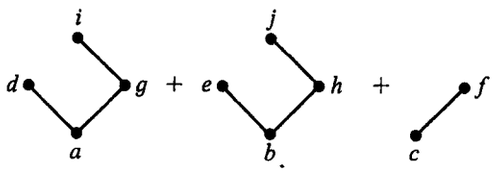
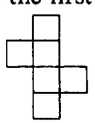


FIG. 5. Strategic proof that $\frac{1}{2} + \frac{1}{2} = 1$.

Initially, we are at the position (a, b, c). We consider first what happens if Left starts. He might as well move from a to d, to which Right replies by the move from b to h, then Left can only move from h to j, and Right makes the last move from c to f and wins.

If Right moves from b to h, Left can reply with a to d, and then wins with h to j as his reply to Right's only move c to f. If instead Right makes the move c to f, Left can reply a to d, then we have b to h for Right, followed by the winning move h to j. (Note that in all cases we have the same 4 moves $a \to d$, $b \to h$, $h \to j$, $c \to f$. This phenomenon often happens.)

Exercise. Taking $\frac{1}{4}$ as $\{0 \mid \frac{1}{2}\}$ and $\frac{3}{4}$ as $\{\frac{1}{2} \mid 1\}$, give a strategic discussion of the equality $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

The game $\uparrow$. The game $\{0 \mid *\}$ is common enough to deserve a special name, so we call it *up*, and give it the special symbol $\uparrow$. Its negative $\{* \mid 0\}$—note that $*$ is its own negative, like 0—is called *down* and given the symbol $\downarrow$. Since Left wins with the first or second move, $\uparrow$ is a positive game. It is the value of the position  in our domino game. In Fig. 6 we illustrate the remarkable equality

$$\{0 \mid \uparrow\} = \uparrow + \uparrow + *.$$

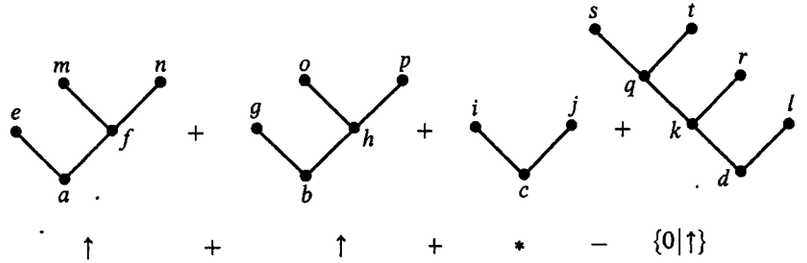


FIG. 6. The upstart equality.

In the illustrated position, the moves $a \to f$ and $d \to k$ lead collectively to the zero position $* + \uparrow + * + \downarrow$, so we can use either as a reply to the other, and then mimic our opponent's moves. So by symmetry we need only consider the moves $c \to j$, $d \to l$ for Right, and $a \to e$, $c \to i$ for Left, showing that each has its counter.

Now the moves: $c \to j$, $d \to k$ lead to a position $\uparrow + \uparrow + \downarrow = \uparrow > 0$, and $d \to l$, $c \to i$ lead to $\uparrow + \uparrow > 0$, so that $c \to j$ and $d \to l$ are bad moves for Right. Similarly, after $a \to e$, $b \to h$, we have a position $* + * + d = d$, and Right wins d, the moves being $d \to k$, $k \to r$. In the final case, Right replies to $c \to i$ with $a \to f$, and then follows one of $(f \to m, b \to h)$, $(b \to g, f \to n)$, and $(d \to k, f \to n)$ and an easy win for Right in each case. So indeed we have $\uparrow + \uparrow + * = \{0 \mid \uparrow\}$.

We close this introductory chapter with the details of a more formal approach, for those who might prefer it.

Construction. If L and R are any two sets of games, there is a game $\{L \mid R\}$. All games are constructed in this way.

Convention. If $G = \{L \mid R\}$, we write G^L for the typical element of L, G^R for the typical element of R, and refer to these (respectively) as the Left and Right options of G. Then the *legal moves* in G are, for Left, from G to G^L, and for Right, from G to G^R, and we write $G = \{G^L \mid G^R\}$.

Definition of $G \geqslant H$, etc.
$G \geqslant H$ iff (no $G^R \leqslant H$ and $G \leqslant$ no H^L). $G \leqslant H$ iff $H \geqslant G$. $G \| H$ iff neither.
$G \rhd\!\lhd H$ iff $G \nleqslant H$; $G \lhd\!\rhd H$ iff $G \ngeqslant H$; $G < H$, $G > H$, $G = H$, as usual.

Definition of $G + H$.

$$G + H = \{G^L + H, G + H^L \mid G^R + H, G + H^R\}$$

Definition of $-G$.

$$-G = \{-G^R \mid -G^L\}.$$

Then we have all the statements of the following.

Summary. The Class **Pg** of all *Partizan Games* forms a partially ordered group under addition, with 0 as zero and $-G$ as negative, when considered modulo equality. This Group strictly includes the additive Group of all numbers. The order-relation is that defined by

$G > H$ iff $G - H$ is won by Left, whoever starts
$G < H$ iff $G - H$ is won by Right, whoever starts
$G = H$ iff $G - H$ is won by the second player to move, and
$G \| H$ iff $G - H$ is won by the first player to move.

The relation $G \| H$ is the relation of incomparability for this order, meaning that we have no one of $G = H$, $G > H$, $G < H$. We say then that G and H are *confused*, or that G *is fuzzy against* H.

Formal proofs of these statements from these definitions are to be found in the zeroth part of this book where in some places we were careful to word our proofs so as to include more general games, although we were then primarily interested in numbers. Informal proofs and explanations in terms of strategies have been given in this chapter.

However, there is one point that calls for special notice. The phrase "all games are constructed in this way" justifies the proving of theorems by induction over games. Thus if for all G we can deduce that P holds at G provided it holds at all options of G, then P holds for all games. The following argument shows that this is equivalent to our requirement that there be no infinite sequence of games each an option of its predecessor.

If such a property P does not hold for some game $G = G_0$, then it must also fail for some option G_1 of G_0, and then for some option G_2 of G_1, and so on. So unless P holds for all games, we obtain an infinite option-sequence. [This proof uses the axiom of choice.]

SOME INFINITE GAMES

At first sight it might be thought that the previous discussion makes all games finite. But the game $\omega = \{0, 1, 2, 3, \ldots \mid \}$ has infinitely many positions, and yet is a perfectly good game, if a little biassed in favour of Left. For since after the first move, we reach some finite game $n = \{0, 1, 2, \ldots, n - 1\}$, which lasts at most n moves, there can be no infinite option-sequence in ω. But of course we can give no fixed estimate, before choosing the first option, for the length of an option-sequence. The tree of ω is sketched in Fig. 7.

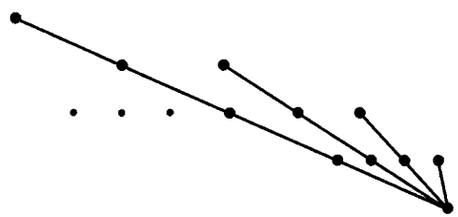


FIG. 7. The tree of ω.

MY DAD HAS MORE MONEY THAN YOURS

In this game, the players alternately name sums of money (for just two moves), and the player who names the larger amount is the winner. The game

is essentially the same as

$$\omega - \omega = \{0 - \omega, 1 - \omega, \ldots, n - \omega, \ldots \mid \omega - 0, \omega - 1, \ldots, \omega - n, \ldots\},$$

whose tree is rather complicated, though the complication is irrelevant in play. As childhood experience shows, there is not much point in starting first at this game. This observation is equivalent to the equality

$$\omega - \dot{\omega} = \dot{0}.$$

The theory of games developed in the rest of this book is a grand generalization of the earlier theory found independently by Sprague and Grundy for *impartial games*—those in which both players have the same legal moves. In the first edition of this book the term "unimpartial" was used for the wider class of games obtained by dropping this condition—we now adopt the nicer word "partizan" that was introduced in *Winning Ways*.

CHAPTER 8

Some Games are Already Numbers

"Reeling and Writhing, of course, to begin with," the Mock Turtle replied; "And then the different branches of Arithmetic— Ambition, Distraction, Uglification, and Derision."

Lewis Carroll, "Alice in Wonderland".

In this chapter we consider several games in which the values of all, or almost all, the positions are already numbers. For such a game we shall obtain a complete theory as soon as we can give some rule for calculating the number which is the value of any particular position. We shall not always be able to do this, even when we can quite easily prove that all the values are numbers.

The diligent reader of the zeroth part of this book will already know quite a lot about numbers. But for the benefit of certain other readers, we summarise some of the more basic information here.

There is a notion of *simplicity* for numbers, which we can if we like define as follows. [This is not quite the same as the notion used in the zeroth part, but the differences are inessential.]

The number 0 is the simplest possible number, followed by the numbers 1 and -1, then 2 and -2, 3 and -3, etc., and so on through all the integers. Next come all rationals with denominator 2, followed by those with denominator 4 (*not* 3), then those with denominator 8, and so on through the dyadic rationals. After these come all remaining real numbers at once, including $\frac{1}{3}$, $\sqrt{2}$, and π as examples.

For the extensions to other numbers, see the tree in Chapter 0, the discussion in Chapter 3, and some of the remarks in the appendix to the zeroth part. In this part of the book we shall mostly talk only about ordinary real numbers, and the above discussions should be enough, but for the occasional comments about other surreal numbers we shall suppose that the reader is familiar with the zeroth part.

The most important game-theoretical property of numbers is that given by the *simplicity rule*: if all the options G^L and G^R of some game G are known

to be numbers, and each G^L strictly less than each G^R, then G is itself a number, namely the simplest number x *greater than* every G^L and *less than* every G^R. (Theorem 11, Chapter 2.)

CONTORTED FRACTIONS

This game is actually played with numbers, so that it is not surprising that numbers arise in its solution. However, the complete theory is rather curious.

The typical position has a number of real numbers in boxes, and the typical legal move is to alter just one of these numbers. The number replacing a given one must have strictly smaller denominator, or, if the given number was already an integer, be an integer strictly smaller in absolute value. Irrational numbers are counted as having infinite denominator. Such a replacement will be legal for Left only if it *decreases* the number, legal for Right only if it *increases* it.

Thus from the position $\boxed{\frac{2}{5}}$ Left can move to the positions $\boxed{x}$ with $x = \frac{1}{3}, \frac{1}{4}, -\frac{1}{2}, 0, -2$, etc., since all these are less than $\frac{2}{5}$ and have denominator smaller than 5, and Right can similarly move to $\boxed{x}$ with $x = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1, 17\frac{1}{4}$, etc. But in general Left will prefer to keep the numbers as large as possible, while Right will wish to make them small, so that in fact Left will choose $x = \frac{1}{3}$ and Right $x = \frac{1}{2}$, if they play wisely. In symbolic terms, this means that we have the equation

$$\boxed{\tfrac{2}{5}} = \{\boxed{\tfrac{1}{3}} \mid \boxed{\tfrac{1}{2}}\}.$$

So it is fairly easy to see that what has happened in this game is that we have imposed a distorted notion of simplicity, under which $\frac{1}{3}$ is counted as simpler than $\frac{1}{4}$ because it has smaller denominator. Proceeding in order of this new kind of simplicity, we obtain the table

$$x = \ldots, -1, -\tfrac{1}{2}, 0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, 1, 1\tfrac{1}{2}, 2, \ldots$$

$$\boxed{x} = \ldots -1\ -\tfrac{1}{2}\ 0\ \tfrac{1}{8}\ \tfrac{1}{4}\ \tfrac{3}{8}\ \tfrac{1}{2}\ \tfrac{5}{8}\ \tfrac{3}{4}\ \tfrac{7}{8}\ 1\ 1\tfrac{1}{2}\ 2\ldots$$

in which arbitrary fractions on the top line correspond to dyadic ones on the bottom line, in the respective orders of simplicity.

The well-known rule for Farey fractions tells us how to find new entries successively—if a/b and c/d are at some time adjacent in the top line, then the next number to insert between them is $(a + c)/(b + d)$, and so this number will yield the mean of the two numbers corresponding to a/b and c/d in the bottom line. (This only happens if $bc - ad = 1$.) Thus we have the equation $\boxed{\frac{4}{7}} = \frac{9}{16}$, operating in this way on the adjacent numbers $\frac{1}{2}$ and $\frac{3}{5}$ from the top line.

The general solution requires some of the theory of continued fractions, and since this is no part of our business here, we shall simply quote the answer. The proof involves also Berlekamp's rule for interpreting sign-expansions (Chapter 3).

Each rational number x can be expanded as a simple continued fraction in two closely related ways:

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \ldots + \cfrac{1}{n+1}}} = a + \cfrac{1}{b + \cfrac{1}{c + \ldots + \cfrac{1}{n + \cfrac{1}{1}}}},$$

in view of the equation

$$\cfrac{1}{n + (1/1)} = \cfrac{1}{n+1}.$$

We obtain from this continued fraction expansion for x the dyadic rational value for $\boxed{x}$ as follows.

Write down the integer a, with its sign, as the integral part of $\boxed{x}$. For the fractional part, we have the binary expansion $\cdot 0^{b-1}1^c0^d\ldots$, where we choose the particular representation so that this ends in 1. In other words, we read the partial quotients $b, c, \ldots$ as alternate numbers of 0s and 1s, except that the first 0 is replaced by the binary point.

Thus

$$2\tfrac{13}{35} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{4}}}},$$

and so we have

$$\boxed{2\tfrac{13}{35}} = 2{\cdot}01001111 = 2\tfrac{79}{256}.$$

(The alternative form

$$2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1}}}}}$$

would yield a binary expansion ending in 0, and so is discarded.) Of course the numbers before the binary point will usually be written in decimal, so that we have a curiously mixed notation here!

For irrational x, we obtain an infinite continued fraction, and exactly the same rule works, except that we have no worries about double representation. Thus for

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}} = 1 + \frac{1}{x},$$

we have the binary expansion $1{\cdot}101010\ldots = 1\frac{2}{3}$. Since this x is the positive

root of the equation $x^2 = x + 1$, we have the mystic equation

$$\boxed{\frac{1+\sqrt{5}}{2}} = \tfrac{5}{3}.$$

The function here called $\boxed{x}$ is traditionally called "Minkowski's Question-Mark Function," and has interesting analytic properties. Its graph is shown in Fig. 8.

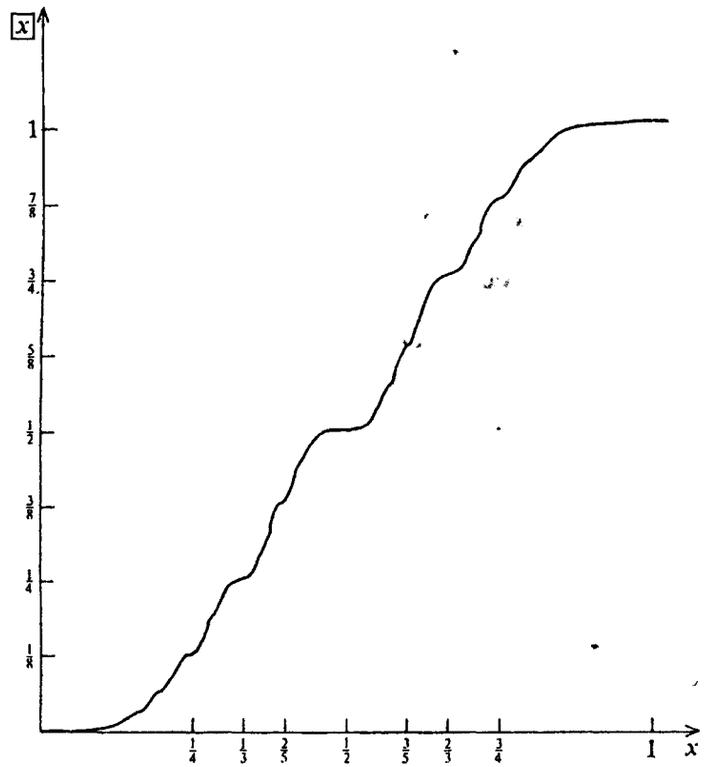


FIG. 8.

Suppose we have the position

$$\boxed{x} + \boxed{x} + \boxed{x} + \boxed{x} + \boxed{x}$$

but that Right is allowed to pass just once during the game, at any time he chooses. For what real number x is this a fair game?

The allowance for Right is equivalent to adding an extra component -1,

and so we must solve the equation $\boxed{x} = \frac{1}{5}$. Now the number $\frac{1}{5}$ has the binary expansion $0{\cdot}00110011001100\ldots$, and so the required x is the number represented by the continued fraction

$$x = \cfrac{1}{3} + \cfrac{1}{2} + \cfrac{1}{2} + \ldots.$$

Now writing t for the number

$$1 + \cfrac{1}{2} + \cfrac{1}{2} + \ldots$$

we find that

$$t = 1 + \frac{1}{1+t},$$

and so $t^2 = 2$, whence $t = \sqrt{2}$ since t is obviously positive, and this gives us the surprising answer

$$x = \frac{1}{2+\sqrt{2}}.$$

Problems. Solve the equations

$$\boxed{\sqrt{2}} + \boxed{\sqrt{3}} = \boxed{A} \qquad \left(A = \tfrac{13}{3} - \sqrt{\tfrac{26}{27}}\right)$$

$$\boxed{\sqrt{5}} - \boxed{\sqrt{2}} = \boxed{B} \qquad \left(B = 1 - \sqrt{\tfrac{2}{15}}\right)$$

$$4\boxed{e} = \boxed{C} \qquad \left(C = \frac{25e-63}{2e-5}\right)$$

$$\boxed{\pi} + \boxed{\tfrac{1}{2}} = \boxed{D} \qquad \left(D = \frac{37-10\pi}{11-3\pi}\right)$$

$$\boxed{\pi} - \boxed{\tfrac{1}{100}} = \boxed{E} \qquad \left(E = \frac{240585707\pi - 755822109}{76580827\pi - 240585706}\right)$$

We illustrate with the last equation (none of the others requires much calculation). The continued fraction for π is

$$\pi = 3 + \cfrac{1}{7} + \cfrac{1}{15} + \cfrac{1}{1} + \cfrac{1}{292} + \ldots$$

which we write as

$$3 + \cfrac{1}{7} + \cfrac{1}{15} + \cfrac{1}{1} + \cfrac{1}{77 + x}.$$

for a reason that will soon become apparent. So the expansion of $\boxed{\pi}$ will be

$$\overbrace{3}^{7}\cdot\overbrace{000000}\ \overbrace{111111111111111}^{15}\overbrace{0}^{1}\overbrace{1111111111\ldots1111}^{(76}\overbrace{1}^{+1+}\overbrace{11111\ldots1110}^{215)}\ldots,$$

and

$$0\cdot000\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 0001$$

is the corresponding expansion of $\boxed{\frac{1}{100}}$. We conclude that E must be the number

$$3+\frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{76+}\,\frac{1}{1+}\,\frac{1}{215+}\ldots,$$

or more simply

$$E=3+\frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{76+}\,\frac{1}{1+}\,\frac{1}{x}$$

Eliminating x we find the displayed answer. The calculations would have been *much* harder if we had not the good rational approximation $\pi \doteq \frac{355}{113}$!

HACKENBUSH RESTRAINED

In this game, the appearance of the numbers is less expected, but they also appear less curiously. The game has analogues and generalisations which will be considered in other chapters. This variety of Hackenbush is played on a *picture*, consisting of black edges (▮) and white edges (▯) joining nodes. It is required that each node be connected via a chain of edges to a certain dotted line called the *ground* (sometimes also called the *ceiling*, or the *walls*). Two nodes may be joined by more than one edge, and it may happen that some edge joins a node to itself. See Fig. 9.

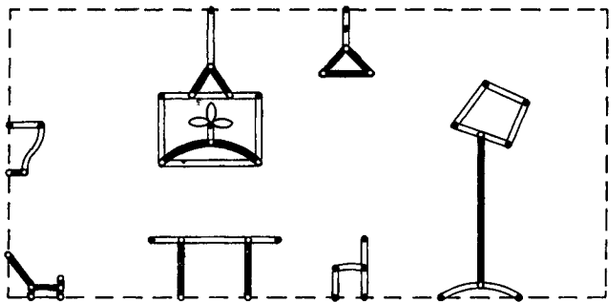


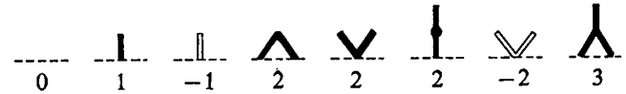
FIG. 9. A restrained Hackenbush room.

At any time when it is his turn to move, Left (Black) may *chop* through any black edge, when that edge disappears, together with any nodes and edges no longer connected to the ground. Right (White) moves in a similar way, by chopping white edges. The game ends when no edge remains to be chopped, and the player unable to move is the loser.

Thus in Fig. 9 Left might start by chopping one leg of the table, which leaves the rest of the table unaffected, but if at his next move he chops the remaining leg, the table disappears. He might alternatively chop away one petal of the flower in the picture—each of these petals is an edge whose two ends coincide. Right's first move might be to chop one of the two white edges supporting the ceiling lamp—of these the lower is the better move, since it leaves him with a further free move. Alternatively, he may chop any edge of the standard lamp except the central column, and so on.

PRELIMINARY DISCUSSION

The positions



have the indicated values. More generally, a position with just n black edges and no white ones will have value n, for Left can take the black edges in a suitable order so as to have n successive moves.

The position ▮ has value $\frac{1}{2}$, for we have the equation

$$\text{▮} = \{\cdots \mid \text{▮}\} = \{0|1\} = \tfrac{1}{2}$$

and similarly we find the equations

$$\text{▮} = \{\cdots \mid \text{▮}, \text{▯}\} = \{0|1, \tfrac{1}{2}\} = \tfrac{1}{4}, \quad \text{▮} = \{\cdots, \text{▯} \mid \text{▮}\} = \{0, \tfrac{1}{2}|1\} = \tfrac{3}{4}.$$

It appears that black edges favour Left, but less so as they get further from the ground, while white edges favour Right in a similar way.

It is not hard to give an inductive proof of the following two propositions. (They must be proved together.)

(i) On chopping a black edge, the value strictly decreases—on chopping a white one it strictly increases.
(ii) The value of every position is a number.

On the other hand, we know no simple rule which enables us to compute

this number for an *arbitrary* graph without to some extent playing the game. However, there is a complete theory for *trees*. It turns out that if Ⓟ is some position P, then the *value* of the position Ⓟ depends only on the *value* of P. If the value of P is a *real* number x, then the value of Ⓟ turns out to be the number $1{:}x$ defined by the conditions:

For real x, the number $1{:}x$ (the *ordinal sum* of 1 and x) has the first value from the series

$$\frac{x+1}{1}, \quad \frac{x+2}{2}, \quad \frac{x+3}{4}, \quad \frac{x+4}{8}, \quad \frac{x+5}{16}, \ldots$$

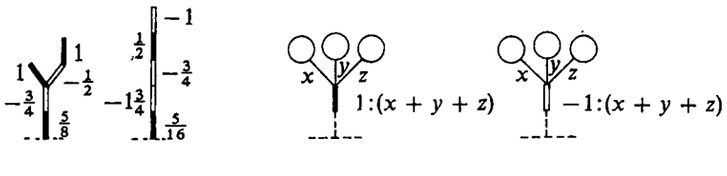
for which the numerator of the given expression exceeds 1. (We mean the numerator $x + n$ as written, not the numerator of the number $(x + n)/2^{n-1}$ when written as a rational fraction in least terms.)

In a similar way, the number $(-1){:}x$ (always negative) will have the first value from the series

$$\frac{x-1}{1}, \quad \frac{x-2}{2}, \quad \frac{x-3}{4}, \quad \frac{x-4}{8}, \quad \frac{x-5}{16}, \ldots$$

in which the numerator is exceeded by -1, This is the value of the position Ⓟ, when P has value x.

Taken together with the obvious result that the value of a position like ⓅⓆ is $x + y$, when P has value x and Q value y, these results enable us to evaluate all *trees* in Hackenbush restrained. It is customary to write the values against the edges, in the following way:



We explain the occurrence of the functions $1{:}x$ and $-1{:}x$ as follows. The moves from the position

Ⓟ are to ⎯⎯ and Ⓟᴸ for Left, Ⓟᴿ for Right.

So inductively, the appropriate function is the function $1{:}x$ defined by

$1{:}x = \{0, 1{:}x^L \mid 1{:}x^R\}$. Now this is a function which maps all numbers onto positive numbers, in order of simplicity. Thus 0, the simplest number, maps to 1, the simplest positive number. Then -1 and 1 map to the simplest positive numbers to the left and right of 1, namely $\frac{1}{2}$ and 2 respectively, and so on. We find under this map that the integers have images as follows

$x =$	-5	-4	-3	-2	-1	0	1	2	3	4	...
$1{:}x =$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4	5	...

and then that other real numbers fill in linearly, which explains the above rule.

Note that the rule does not work for *all* numbers. For instance $1{:}(-1/\omega) = 1 - (1/\omega)$ (not $1 - 1/(2\omega)$), and $1{:}\omega = \omega$ (not $\omega + 1$). But the definition in terms of simplicity works for all numbers x, and the inductive definition $1{:}G = \{0, 1{:}G^L \mid 1{:}G^R\}$ works for all games G.

We postpone further discussion of the properties of this function until Chapter 15, which is its proper home.

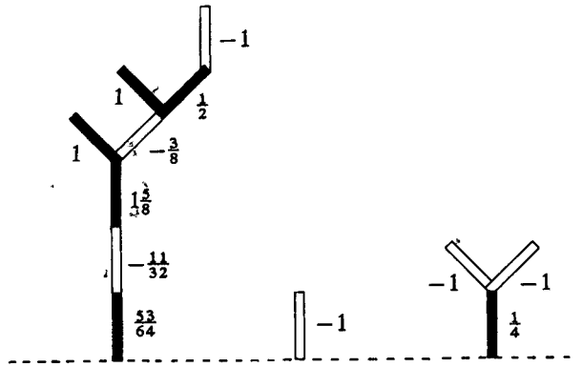


FIG. 10. A restrained Hackenbush forest.

The reader should now be able to see who wins in the position of Fig. 10. Plainly Black—he is exactly five sixtyfourths of a move ahead! (It never ceases to amaze and amuse me that such statements have a precise meaning!)

CHAINS, LOOPS AND INFINITE BEANSTALKS

It follows from the rules for trees that the sign-expansion (Chapter 3) of a *chain* can be read directly from the picture, reading + for black edges, − for white ones, from the ground upwards. So the values of the four chains

in Fig. 11 have the sign-expansions

$$+++(+-)+ = 3{\cdot}11 = 3\tfrac{3}{4}, \qquad -(-+)- = -1{\cdot}11 = -1\tfrac{3}{4}$$
$$+++(+-)++ = 3{\cdot}111 = 3\tfrac{7}{8}, \qquad -(-+)-+ = -1{\cdot}101 = -1\tfrac{5}{8}$$

where we have bracketed the first sign-change to help the reader apply Berlekamp's rule. Recall that to obtain the binary expansion of the fractional part, for positive numbers we read 0 for $-$, 1 for $+$, and the converse for negative numbers, in either case adding a final 1.

Berlekamp has given a similar rule for the value of a circuit joining the ground to itself (Fig. 11). We *break* the circuit at the node or mid-point of

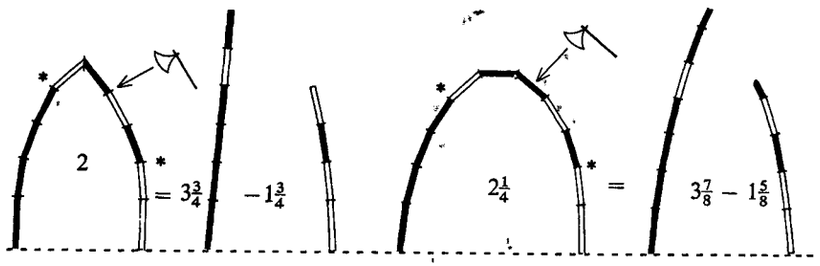


FIG. 11. Berlekamp's rule for loops.

an edge which is midway between the two sign-changes nearest the ground on each side (*s in the diagram), halves of edges appearing (as whole edges) on *both* sides of the fracture when they arise. The value of the circuit is then the sum of the values of its two component parts. The rule can also be applied to a single circuit at some distance from the ground—thus since the value of the left circuit in Fig. 11 is 2, we have the equality illustrated in Fig. 12. But we

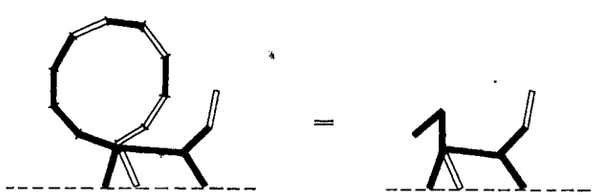


FIG. 12. A head-shrinking equality.

have no general rule for computing values of arbitrary graphs in Hackenbush restrained. Some more information will be given in Chapter 15.

It is perfectly possible to play Hackenbush on infinite trees and certain

other infinite graphs, the rules extending naturally. When we do this, arbitrary numbers can arise as values. So for instance the various beanstalks of Fig. 13 have the indicated values.

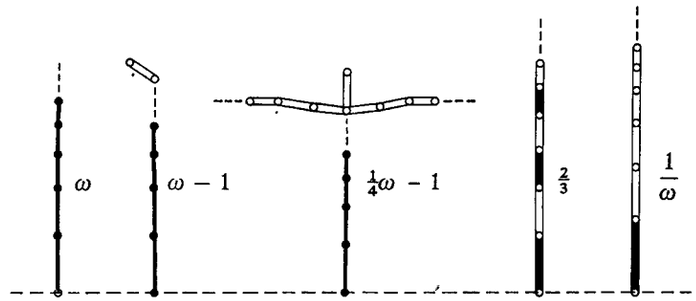


FIG. 13.

THE GAMES OF COL AND SNORT

COL is a map-colouring game introduced by Colin Vout. It is played with a map drawn on a piece of brown paper, a pot of black paint, and a pot of white paint. The players alternately colour countries of the map, subject to the conditions that no country may be coloured twice, and no two countries with a common frontier may be coloured the same colour. Of course, Left uses only black paint, and Right only white.

SNORT is a game introduced by Simon Norton. It is played between two farmers who jointly rent a certain farm, divided into fields. Mr Black buys (black) Bulls, and Mr White (white) coWs, on alternate market days. The animals bought on any one day are to be placed in a field which was previously empty, subject to the condition that no field containing cows may be adjacent to one containing bulls.

If we colour a field black or white according as it contains bulls or cows, we see that both games are played on a map (in the same sense as in the famous 4-colour map problem), the restriction in COL being that adjacent regions may not be similarly coloured, while in SNORT they may not be dissimilarly coloured. This makes it natural to discuss them in similar terms, although as we shall see later, their theories are entirely different.

It is tedious to have to draw complicated maps to specify positions, so we shall simplify the presentation as follows. We discuss COL first. The only effect of a country which has already been painted *black* in COL is to *tint* the neighbouring countries *white*, for these regions may only be painted white in future. Similarly, a *white* painted country causes its neighbours to be tinted *black*. A country that acquires tints of both colours black and white

in this way might just as well be erased from the map, since neither player will be allowed to paint it in the future.

In SNORT, these conventions are reversed—any field already coloured causes its neighbours to acquire tints of the *same* colour. But it is still true that a region tinted in both ways can be ignored. Once we have tinted regions according to these conventions, we can ignore all the regions that have actually been *painted*, for they have no further effect on the game.

So we shall represent positions in either of these games by *graphs*, as follows. The graph representing a given position will have a node for each region of that position *which has not already been coloured*, and two nodes corresponding to adjacent regions will be joined by an edge in the graph. The nodes are tinted *black* (●) or *white* (○) or *both* (⦿) or neither (·), and if we like we can omit nodes tinted both black and white. (But the ⦿ notation is still handy.) In Fig. 14 we show the graphs derived in this way from a certain partly coloured map in both COL and SNORT.

There are some further simplifications we can make. An edge joining two oppositely tinted nodes in COL may be omitted, for it has no force (the only effect of any edge is to prevent the nodes at its ends from being similarly coloured). For similar reasons edges joining similarly tinted nodes in SNORT may be deleted. We have also indicated these simplifications in Fig. 14.

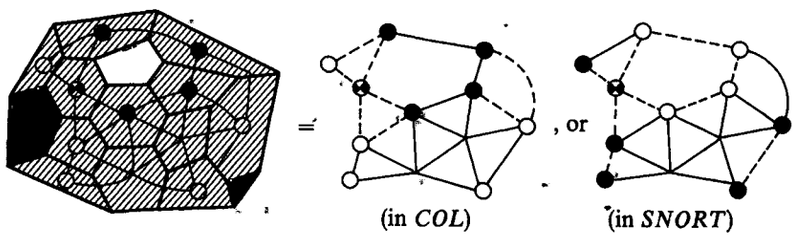


FIG. 14. How maps give graphs.

Simple graphs are now analysed in a manner which should by now be familiar. In the last pages of this chapter we give "dictionaries" for these two games. As well as evaluating simple positions, these dictionaries contain certain general statements which often enable us to simplify very complicated positions not themselves in the dictionary. The methods by which these results are proved will only appear later.

[We might remark at this point that we have found this sort of approach very useful in analysing games in general. One first analyses simple positions, building some kind of dictionary, often in a very unsystematic way. When patterns emerge, if ever, one can often prove general theorems, and then

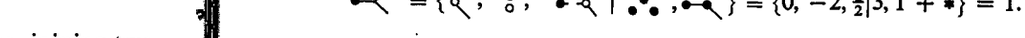
these theorems enable us to 'condense' the dictionary, and on some fortunate occasions, to give a complete theory. Almost all the games used as examples here were first discussed in this way.]

It appears that in COL the values that arise are very restricted in kind. Richard Guy and I have shown that they are all of the form x or $x + *$ for various numbers x. For the inequalities below imply trivially that

$$G^L + * \leqslant G \leqslant G^R + *$$

for any COL position G, and from this the desired result follows by induction. We do not know if denominators of 16 or more can appear in x.

All the values in the COL table can be found by the following sort of analysis. We have the equation



(found by examining the effects of the possible moves), which determines the value of the game on the left hand side in terms of simpler cases.

It is convenient to remember that the simplest number rule in its general form reads:

If there is some number x with $G^L \lhd\!| x \lhd\!| G^R$ for all G^L, G^R, then G is equal to the simplest such x.

It is also convenient to note the equality $\{x|x\} = x + *$ for all numbers x, which follows from a far more general identity later, and to note that $x + *$ is *greater* than all numbers less than x, *less* than all numbers greater than x, but *incomparable* with x. This also will be generalised later.

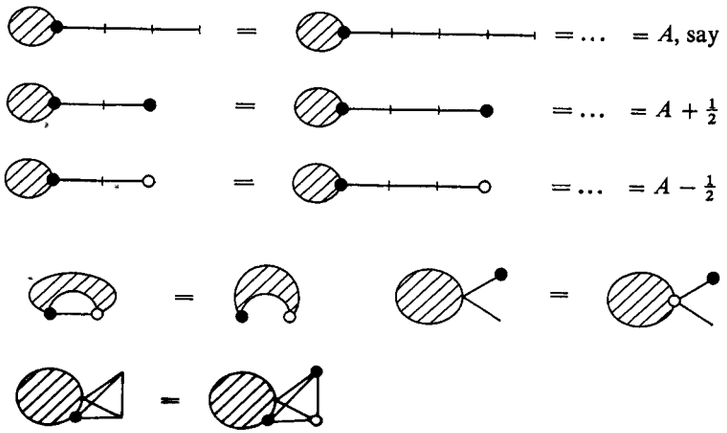
Since SNORT values are usually *not* numbers, the SNORT dictionary requires techniques which will be explained later. The abbreviations will also be generalised in Chapters 10 and 15.

A DICTIONARY OF FACTS ABOUT COL

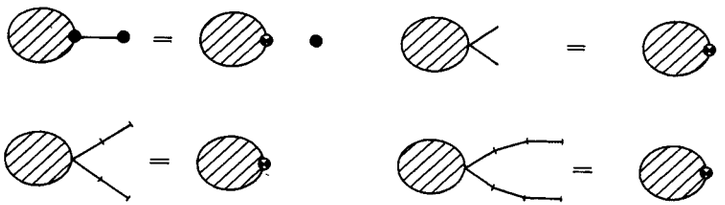
(In general each statement given here has a dual statement in which black and white are interchanged and the inequalities are reversed.)

(1) *Inequalities:* the value of a position is unaltered or increased by *either* tinting a node black (mnemonic: hindering one's opponent is no harm) *or* deleting any edge one end of which has a black tint (mnemonic: let my people go).

(2) *Equalities:* there are many circumstances in which we can say that replacing one configuration by another does not affect the value.

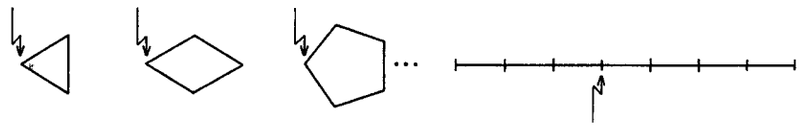


(In general, if two untinted nodes are joined to each other, and to the same set of the remaining nodes, we may tint one black and the other white.)

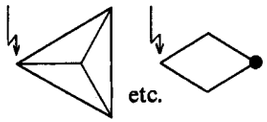


In general if in some configuration the value is unaltered both when we tint a certain node black and when we tint it white, then that node is "explosive" and may be deleted even when used to join the given configuration to another. So the above equalities are consequences of the following:

Other explosive nodes are indicated by the lightning bolts:

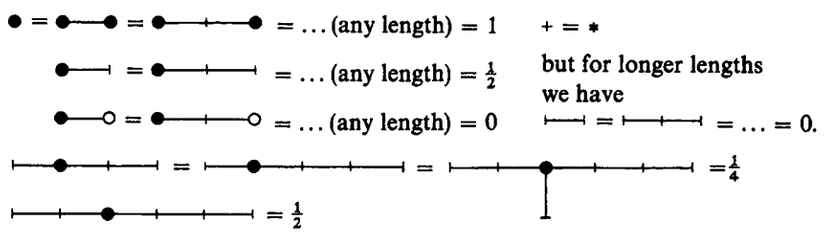


(Any node in an untinted chain with at least three others on each side.)



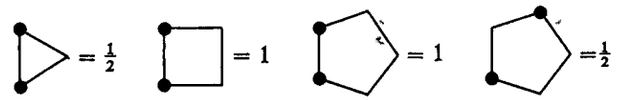
(In each case the explosive node may be tinted without affecting its explosive character.)

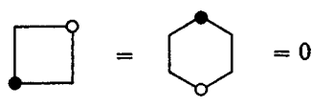
Now we list the values of some simple positions (many others can be deduced from these using the above principles and identities):



From these we can deduce the value of any tree with just one tinted node from which lead only a number of chains of untinted nodes.

We can also deduce the corresponding values if the extreme nodes are tinted. (If such a tree is completely untinted, then *either* its central node explodes by one of the above rules, *or* the value is zero.)

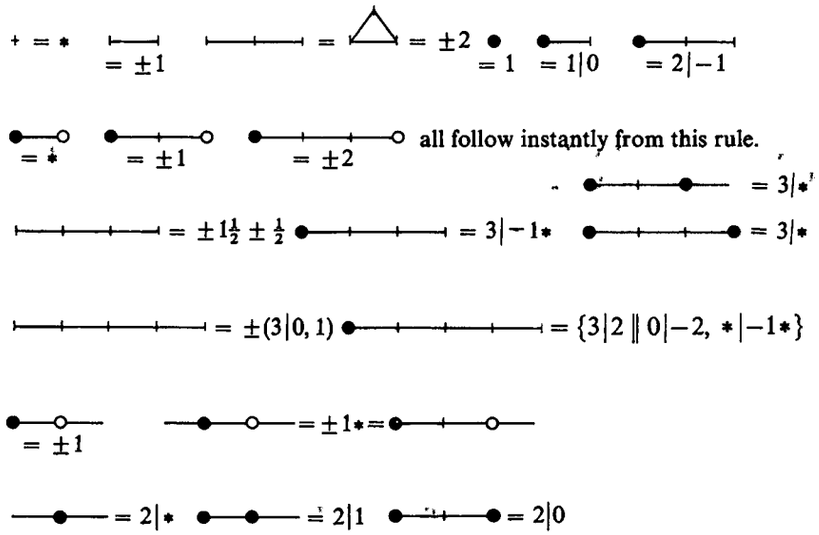




(In general a diagram which has a symmetry moving every node and reversing any tints will always have value 0.)

A SHORT SNORT DICTIONARY

It is much harder to do justice to SNORT positions, although I feel that in fact SNORT has a much richer theory than COL. There are some inequality and equality rules like those for COL, but since they are less frequently applicable we do not give many. Perhaps the most valuable rule is that if you can move in a node that is adjacent to every node not your own colour, you should do so. Our abbreviated notation is explained in Chapters 10 and 15.



Perhaps it is fortunate that positions in SNORT games tend to break up rapidly, and that we can delete edges joining two nodes with the same tint, so that in practice we need only tabulate the values of small positions. Highly connected positions succumb easily to the above rule, so that in fact it is long chains that are hardest to analyse. The reader should have little difficulty in finding the best move in actual play, even for quite large positions.

Larger COL and SNORT dictionaries will be found in *Winning Ways*.

CHAPTER 9

On Games and Numbers

And now there came both mist and snow,
And it grew wondrous cold:
And ice, mast-high, came floating by,
As green as emerald.

Samuel Taylor Coleridge,
The Ancient Mariner

We know that not all games are numbers, and that for example the game $* = \{0 \mid 0\}$ is not a number, since it is confused with 0. But since for every positive number x, we have $-x < * < x$, and since we have the equality $* + * = 0$, we can confidently handle all games whose values can be expressed as sums of numbers and $*$.

But the position in dominoes, which is equivalent to the position in SNORT, has the rather worse value $\{1 \mid -1\}$. This game G is strictly less than all numbers greater than 1, strictly greater than all numbers less than -1, and confused with all numbers between -1 and 1 inclusive. But fortunately once again, we have $G + G = 0$, so that at least the situation does not get more complicated when we consider multiples of G.

Now in general we can get a lot of information about an arbitrary game G by comparing it with all numbers. The game G will define two "Dedekind sections" in the Class of all numbers (the Left and Right values), and any number between these two sections will be confused with G, while numbers above the greatest or below the least will be comparable with G in the appropriate sense.

This information tells us between which limits G lies, but there is also a *mean value* of G, which tells us where its centre of mass lies. We shall give algorithms for computing the Left, Right, and mean values in this Chapter.

Unfortunately, there is a large part of the argument that is inapplicable to the general infinite game. We adopt the convention of considering only *short* games in detail from now on, until Chapter 16, when we consider the differences between short games and long ones. A *short* game is one which has