Average number of i-dimensional implicants

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February 14, 2011

Problem statement. `choose(nCube(n),a)` will give you all the Boolean functions of n variables with a truth vectors. Write a program, `ImplicantAve(n,a,i)` that inputs a positive integer n, a positive integer a between 1 and 2^n, and a positive integer i between 0 and n, and finds the average number of i-dimensional implicants (NOT prime implicants) amongst all such Boolean functions. Of course `Implicants(n,a,0)` should always be a. Using the numerical output of `ImplicantAve(n,a,i)` for various (small, and not small, n, a, and $i = 1$, $i = 2$, $i = 3, \dots$), conjecture an explicit (closed form) expression for `ImplicantAve(n,a,i)`. Use human ingenuity to prove your conjecture rigorously.

Theorem 1. *Abbreviating* `ImplicantAve(n,a,i)` *as* $IA(n, a, i)$*, we have*

$$IA(n, a, i) = \frac{2^{n-i}\binom{n}{i}\binom{a}{2^i}}{\binom{2^n}{2^i}}.$$

Proof. For a randomly-selected (using the uniform distribution) Boolean function G with a truth vectors, let X be the random variable that represents the number of i-dimensional implicants. For each i-dimensional implicant j, we can set the random variable X_j to be

$$X_j = \begin{cases} 1 & \text{if } j \text{ is an implicant for } G \\ 0 & \text{if } j \text{ is not an implicant for } G. \end{cases}$$

Then, $X = \sum_j X_j$.

We are trying to find $\mathbb{E}[X] = IA(n, a, i)$. By linearity of expectations, we have $\mathbb{E}[X] = \mathbb{E}\left[\sum_j X_j\right] = \sum_j \mathbb{E}[X_j]$. So, we now calculate $\mathbb{E}[X_j]$.

How many Boolean functions G have j as an implicant? As j is i-dimensional, there is a set of 2^i truth vectors that must be in G. Then, the other $a - 2^i$ vectors in G can be anything we want them to be. Thus, there are $\binom{2^n-2^i}{a-2^i}$ Boolean functions with j as an implicant.

Furthermore, there are $\binom{2^n}{a}$ Boolean functions with a truth vectors, in total. Hence,

$$\mathbb{E}[X_j] = \frac{\binom{2^n-2^i}{a-2^i}}{\binom{2^n}{a}}.$$

By use of the identity $\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$ with $r = 2^n$, $m = a$, and $k = 2^i$, we find

$$\mathbb{E}[X_j] = \frac{\binom{a}{2^i}}{\binom{2^n}{2^i}}.$$

Thus,

$$\mathbb{E}[X] = \sum_j \frac{\binom{a}{2^i}}{\binom{2^n}{2^i}}.$$

How many i-dimensional implicants j are there? There are $\binom{n}{i}$ ways to choose the i dimensions, and 2^{n-i} ways to "fill out" the other $n-i$ dimensions. (For example, with $n=5$ and $i=2$, there are $\binom{5}{2}=10$ ways to choose the two dimensions, say, 1 and 4, and 2^3 ways to fill out the other three dimensions (2,3, and 5). Thus, there are $2^{n-i}\binom{n}{i}$ implicants.

Hence,

$$IA\left(n,a,i\right)=\mathbb{E}\left[X\right]=\frac{2^{n-i}\binom{n}{i}\binom{a}{2^i}}{\binom{2^n}{2^i}}.$$

□

As a quick check:

$$\begin{aligned}IA\left(n,a,0\right)&=\frac{2^n\binom{n}{0}\binom{a}{1}}{\binom{2^n}{1}}\\&=\frac{2^n\cdot 1\cdot a}{2^n}\\&=a,\end{aligned}$$

as it should.