Solution to Monthly Problem 11567

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How many arrangements $(a_1, \ldots, a_{2n})$ *of the multiset* $\{1, 1, 2, 2, \ldots, n, n\}$ *satisfy the following two conditions: (i) All entries between two occurrences of any given value* i *exceed* i*, and (ii) No three entries increase from left to right with the last two adjacent? (When* $n = 3$*, one such arrangement is* 122133*.) Proposed by David Callan.*

Consider an arrangement $A = (a_1, \ldots, a_{2n})$ where $a_i = 1$ and $a_j = 1$ for $i < j$. Divide the arrangement into three subarrangements, $X := (a_1, a_2 \ldots, a_{i-1})$, $Y := (a_i, a_{i+1}, a_{i+2}, \ldots, a_{j-1})$, and $Z := (a_j, a_{j+1}, a_{j+2}, \ldots, a_{2n})$. Note that if A satisfies conditions (i) and (ii), then each of X,Y,and Z satisfy them as well.

Consider Y first: any term $a_k \neq 1$ for $i < k < j$ must have its partner $a_\ell = a_k$ also enclosed by the 1's, that is, $i < \ell < j$. Otherwise A would contain either the subsequence $a_\ell < a_i < a_k$ or $a_k < a_j < a_\ell$, both of which violate condition (i). To satisfy condition (ii), the letters of Y besides a_i must be nonincreasing: otherwise if $a_k < a_{k+1}$ then $a_i a_k a_{k+1}$ would violate condition (ii). If y is the number of pairs in Y (i.e. $y = \frac{j-i-1}{2}$), we see that there are $\binom{n-1}{y}$ possible subarrangements Y since we may freely choose the y pairs of letters.

We next consider Z. Similarly to the case for Y, the letters of Z other than the initial $a_j = 1$ must be nonincreasing or else Z violates condition (ii). Also, if $a_k \neq 1$ is a letter in Z, then its partner $a_\ell = a_k$ must also appear in Z. Otherwise A would violate condition (i), as we saw in case in Y. Thus we see that Z exhibits a similar structure of nonincreasing pairs as in Y. Hence if z is the number of pairs in Z (i.e., $z = \frac{2n-j}{2}$), there are $\binom{n-1-y}{z}$ possible subarrangements Z given Y.

Last consider subarrangement X. Since Y and Z each contained both copies of each number $\{1, 1, 2, 2 \ldots, n, n\}$, we see that if a_k appears in X then so does its partner $a_\ell = a_k$. If x is the number of pairs in X, then we see that X is order-isomorphic to an arrangement of $\{1, 1, 2, 2 \ldots, x, x\}$ which satisfies (i) and (ii) by replacing the i^{th} smallest letters of X with i.

Let B_n be the number of arrangements of $\{1, 1, \ldots, n, n\}$ satisfying conditions (i) and (ii). Given the locations $a_i = a_j = 1$ and $i < j$, the arguments above imply that there are B_x ways to construct subarrangement X, $\binom{n-1}{y}$ ways to construct Y, and $\binom{n-1-y)}{z}$ ways to construct Z, where $x + y + z = n - 1$.

Therefore we get the following recurrence for B_n:

$$\begin{aligned} B_n &= \sum_{x+y+z=n-1} B_x \binom{n-1}{y} \binom{n-1-y)}{z} \\ &= \sum_{x=0}^{n-1} B_x \left(\sum_{y=0}^{n-1-x} \binom{n-1}{y} \binom{n-1-y}{n-1-x-y} \right) \\ &= \sum_{x=0}^{n-1} B_x \left(\binom{n-1}{n-1-x} 2^{n-1-x} \right). \end{aligned}$$

The first 10 terms of this sequence are:

$$1, 3, 11, 49, 257, 1539, 10299, 75905, 609441, 5284451$$

which corresponds to OEIS sequence A004211, `http://oeis.org/A004211`. There it is noted that $B_n = \sum_{k=0}^{n} 2^{n-k} S_2(n,k)$ where $S_2(n,k)$ is the Stirling number of the second kind. It is not immediately clear why these two sequences should be equal.