There is a series of Erdos-Renyi papers on the subject, but the most concise formulation I found was in the 1961 paper (attached, page 3 bottom). Here the authors state:

$$
\begin{aligned}
N(n) \sim cn, cn < 1/2 \quad &\Rightarrow \quad |S| \approx \frac{1}{\alpha}\left(\log(n) - \frac{5}{2}\log(\log(n))\right) && (1)\\
N(n) \sim n/2, (i.e., c = 1/2) \quad &\Rightarrow \quad |S| \approx n^{2/3} && (2)\\
N(n) \sim cn, cn > 1/2 \quad &\Rightarrow \quad |S| \approx G(c)n && (3)
\end{aligned}
$$

Where $N(n)$ is the (expected?) number of edges of a random graph with n vertices, $|S|$ is the size of the largest connected component, c is some constant, $0 \leq c \leq 1$, $alpha = 2c - 1 - \log(2c)$, and $G(c)$ is the function

$$G(c) = 1 - \frac{1}{2c}\sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}(2ce^{-2c})^k$$

To translate from c to ϵ, I think we can take $N(n) = \binom{n}{2}p$, where p is the probability of an edge between two vertices. Letting $p = p_c(1+\epsilon)$, with $p_c = 1/(n-1)$, we then get $N(n) = n\cdot(1+\epsilon)/2$. Or, in other words, $c = (1+\epsilon)/2$.

Then the above set of equations becomes

$$
\begin{aligned}
\epsilon < 0 \quad &\Rightarrow \quad |C| \approx \frac{\log(n) - \frac{5}{2}\log(\log(n))}{\epsilon - log(1+\epsilon)} && (4)\\
\epsilon = 0 \quad &\Rightarrow \quad |C| \approx n^{2/3} && (5)\\
\epsilon > 0 \quad &\Rightarrow \quad |C| \approx G(\epsilon)n && (6)
\end{aligned}
$$

Of course with $G(\epsilon)$ the appropriate substitution of c to ϵ.

I did a few trials using these approximations and the results seem much better (ratios approx 0.7 to 1.2), but I'm not sure how much better, in the long run, because I was only able to do some trials with relatively small numbers.

For calculating G, I evaluated in maple with $k = 1..1000$ since I couldn't find a better way to do it.