Combo Project Seven

Combinatorics (01:640:454)
Project Write-Up

Team 7

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Submission Date

December 14th, 2020

Section Zero: Table of Contents

Section One: Background and Approach

A. Background

1. Introduction

Over the course of this semester, we have learned about many powerful combinatorial techniques used to analyze various problems such as the number of walks in a lattice, the number of words in an alphabet S which add up to n, the number of set partitions for a set of size n, and so on. First, we formulated these problems as recurrence equations that one can solve by writing a basic recurrence program in Maple with initial conditions. We quickly saw, however, that these programs, however cleverly written, are severely limited in computational power when it comes to large values of n.

Fortunately, through the use of generating functions, one is able to calculate a particular value by extracting the coefficient in the taylor series expansion of the function - this greatly reduces the time complexity and allows us to find solutions for very large n in the thousands and higher. This enables us to see trends in sequences. For example, we can see if a sequence converges, slows down, speeds up, decreases or increases, and much more useful information which would enhance our understanding of the sequence of study.

One of the main goals of this class has been to learn how to use programs in conjunction with theory. In particular, we have written programs to find the coefficients for complicated generating functions, we studied structures such as labelled trees by computing averages, standard deviation, and moments of the number of leaves in the tree over many iterations, and we often see the law of large numbers in action.

As a result, the goal of this project will be to use various techniques learned in homeworks, lectures, and maple programs to learn about some interesting sequences generated from a generating function of the form: 1/(1-a*x-b*y-c*x*y-d*x*z-e*y*z-f*x*y*z), where a,b,c,d,e,f are nonnegative, small integers. We will divide functions of this form into two special cases - the two variable, x and y, case and the three variable, x,y, and z case. In two variables, we will be particularly interested in generating functions which fit the form 1/(1-a*x-b*y) or 1/(1-a*x-b*y-c*x*y). In three variables, we will be particularly interested in generating functions which fit the form 1/(1-a*x-b*y-c*z).

B. Approach/Purpose

In mathematics, one can often learn a lot about an object of study by performing actions on it. For example, in group theory, one can analyze the action of a group G on a subgroup H, and this can illuminate many interesting properties of H. Following this reasoning, we will perform many different operations on a function of interest and see what kind of interesting integer sequences we may get out of it.

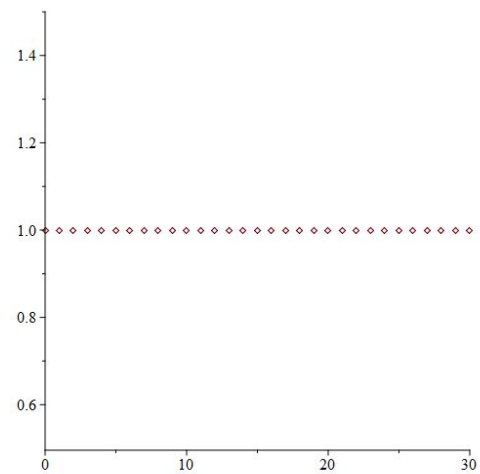
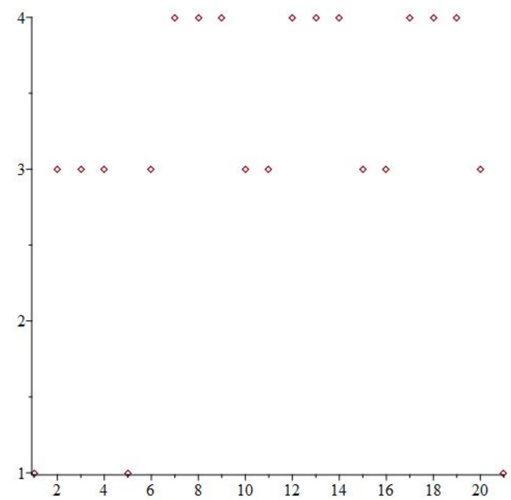
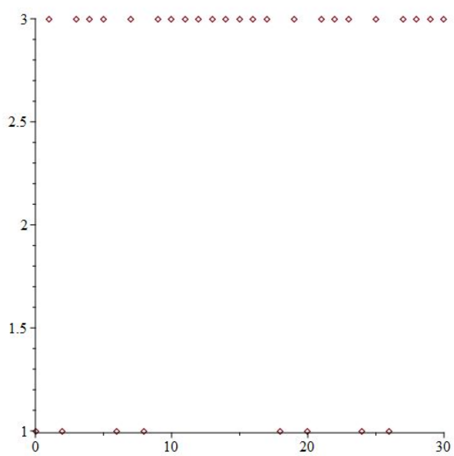
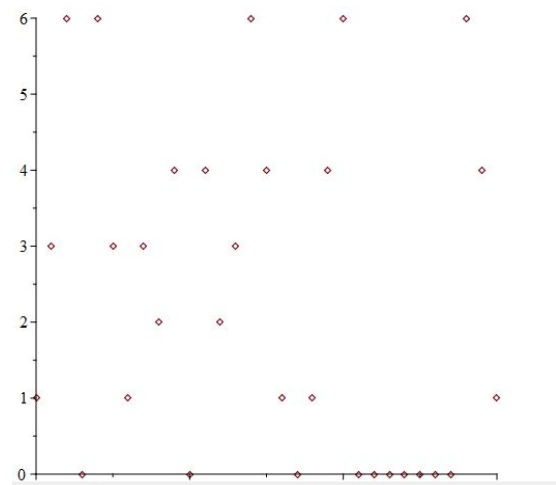
Therefore, many of the modules in this project will perform mathematical operations on one or more generating functions. One of the popular techniques that we have seen throughout this semester is taking the modulus of elements in a sequence, and in some cases there exists an integer m>1 such that a(n) mod m is equal to 0 for all n. So, we will have a function which computes the modulus of elements in a sequence. Also, when we were studying graphs we saw that computing f^k/k!, where k > 1 is an integer, gives us exactly k components in the graph. We will figure out whether this operation reveals any interesting properties for some of the fundamental functions. On top of that, we will have modules which look at the element-wise ratios of sequences corresponding to similar generating functions from the three main classes: 1/(1-a*x-b*y), 1/(1-a*x-b*y-c*x*y), or 1/(1-a*x-b*y-c*z). This may expose some kind of interesting relationship between functions. Furthermore, we will have modules which multiply, add, and subtract sequences corresponding to generating functions. Ultimately, we will compile the interesting integer sequences found in our output file database.

Furthermore, the existing Maple package has come with a few modules we can use for analytics. For example, we have modules to find the growth constant, critical exponent, and estimated asymptotics based off of the operator ope. We will utilize these modules to find any patterns in functions in the three main classes of interest: 1/(1-a*x-b*y), 1/(1-a*x-b*y-c*x*y), or 1/(1-a*x-b*y-c*z).

Section Two: Trends and Analytics

1. Modulo Division

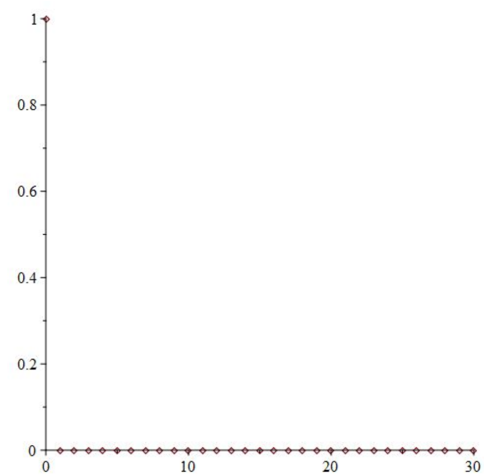
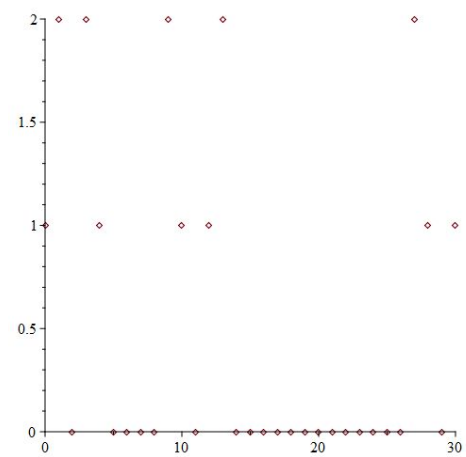
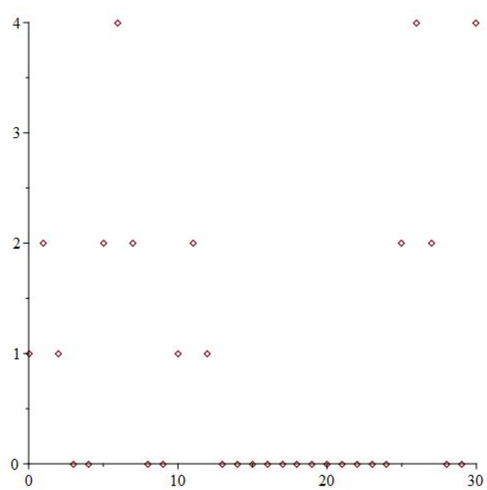
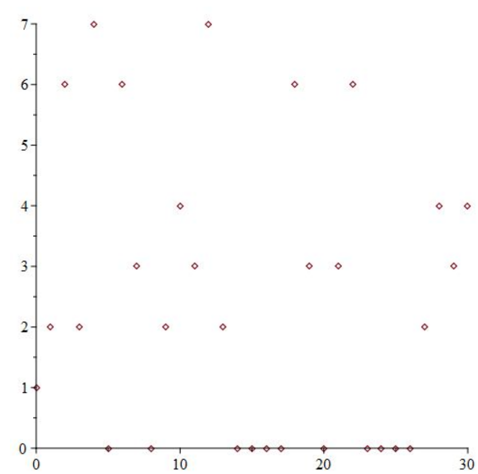
A. Diagrams for f:=1/(1-x-y-x*y)

a(n,n) mod 2; f:=1/(1-x-y-x*y); n=1..30	**a(n,n) mod 5; f:=1/(1-x-y-x*y); n=1..30**
	
a(n,n) mod 6; f:=1/(1-x-y-x*y); n=1..30	**a(n,n) mod 7; f:=1/(1-x-y-x*y); n=1..30**
	

B. Analysis for f:=1/(1-x-y-x*y)

Conjecture One: All elements in the diagonal sequence a(n,n) are odd. We see that in the top left diagram, when we take an element modulus 2 we get 1, which implies that the value is odd for n=0..30.
Furthermore, for many values of m, the elements modulus m are divided into two groups as seen for m=5,6. However, this is not always the case, as for m=7, the values for each element are varying.

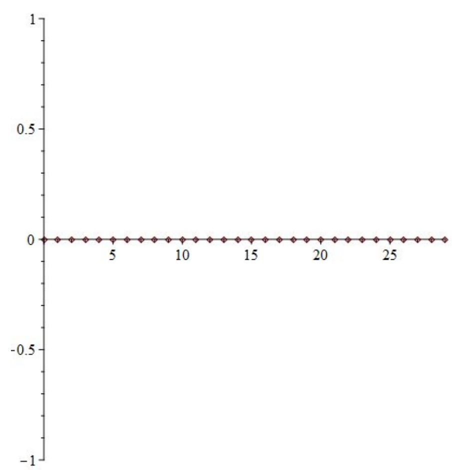
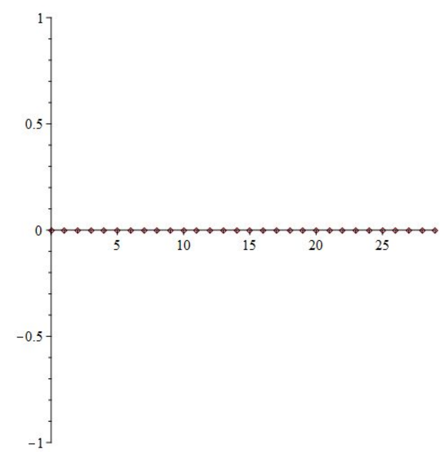
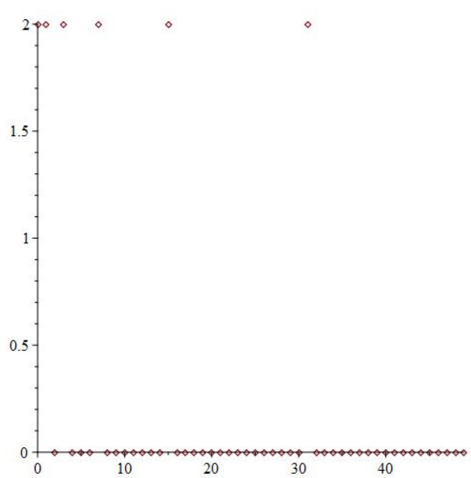
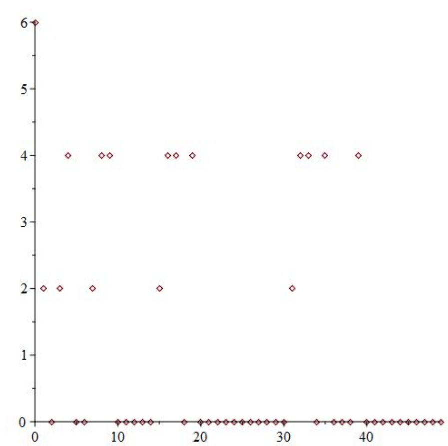
C. Diagrams for f:=1/(1-x-y)

a(n,n) mod 2; f:=1/(1-x-y); n=1..30	**a(n,n) mod 3; f:=1/(1-x-y); n=1..30**
	
a(n,n) mod 5; f:=1/(1-x-y); n=1..30	**a(n,n) mod 9; f:=1/(1-x-y); n=1..30**
	

D. Analysis for f:=1/(1-x-y)

Conjecture Two: All elements in the diagonal sequence a(n,n) are even. We see that in the top left diagram, when we take an element modulus 2 we get 0, which implies that the values are even for n=0..30.
Also, for many values of m, if m is even then a(n,n) mod m is nicely divided into specific groups for all n. However, when m is odd, a(n,n) mod m is sporadic and takes on all values for all n. This likely has to do with a(n,n) being even for all n.

E. Diagrams for f:=1/(1-x-y-z)

<table>
<tr><th>a(n,n) mod 2; f:=1/(1-x-y-z); n=1..30</th><th>a(n,n) mod 3; f:=1/(1-x-y-z); n=1..30</th></tr>
<tr><td></td><td></td></tr>
<tr><th>a(n,n) mod 4; f:=1/(1-x-y-z); n=1..50</th><th>a(n,n) mod 8; f:=1/(1-x-y-z); n=1..50</th></tr>
<tr><td></td><td></td></tr>
</table>

F. Analysis for f:=1/(1-x-y-z)

Conjecture Three: All elements in the diagonal sequence a(n,n) are even. We see that in the top left diagram, when we take an element modulus 2 we get 0, which implies that the values are even for n=0..30.

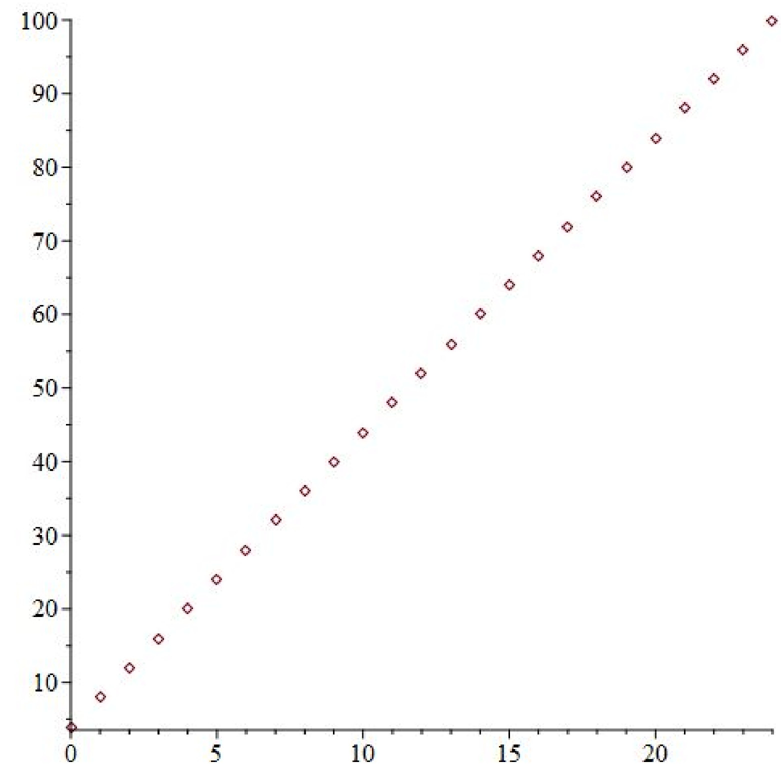
Conjecture Four: As n increases, a(n,n) modulus becomes more sparse. This means that as n grows the number of 0s before the next nonzero term becomes greater.

Conjecture Five: As m increases, a(n,n) mod m becomes less sparse. This means that as m grows, the number of zeros between nonzeros becomes smaller. As evidenced by the differences for m=4 and m=8.

Also, for many values of m, if m is even then a(n,n) mod m is nicely divided into specific groups for all n. However, when m is odd, a(n,n) mod m is sporadic and takes on all values for all n. This likely has to do with a(n,n) being even for all n.

2. Growth Constant Analysis

A. Diagrams for Growth Constant of f:=1/(1-x-y-n*x*y)

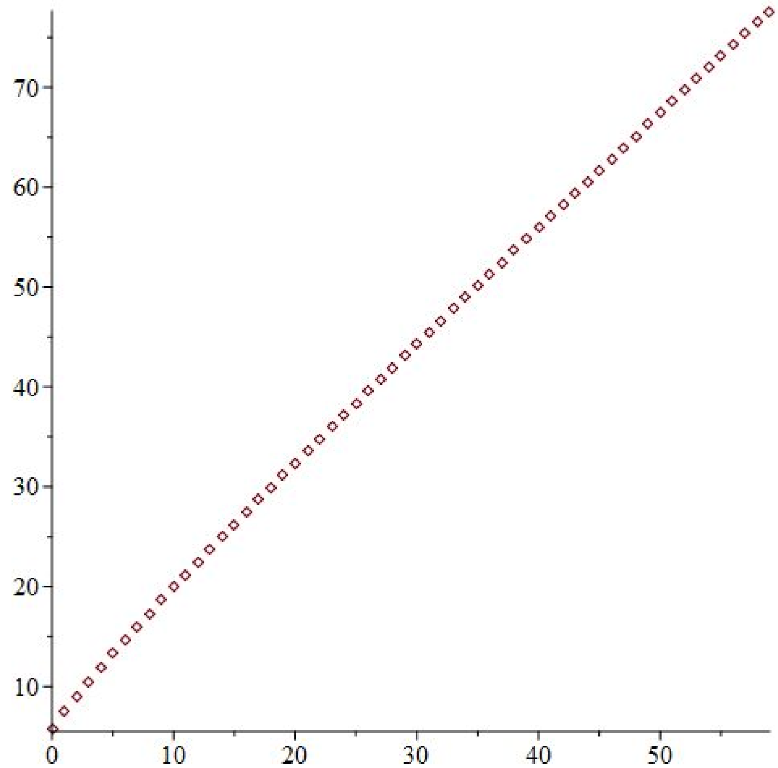
GrowthConstant(f): f:=1/(1-n*x-y); n=0..24


B. Analysis for Growth Constant of f:=1/(1-n*x-y)

Conjecture Six: The growth constant of the generating function f:=1/(1-n*x-y) follows a linear function, g(n)=4*n for all n.

C. Diagrams for Growth Constant of f:=1/(1-x-y-n*x*y)

GrowthConstant(f): f:=1/(1-x-y-n*x*y); n=0..59



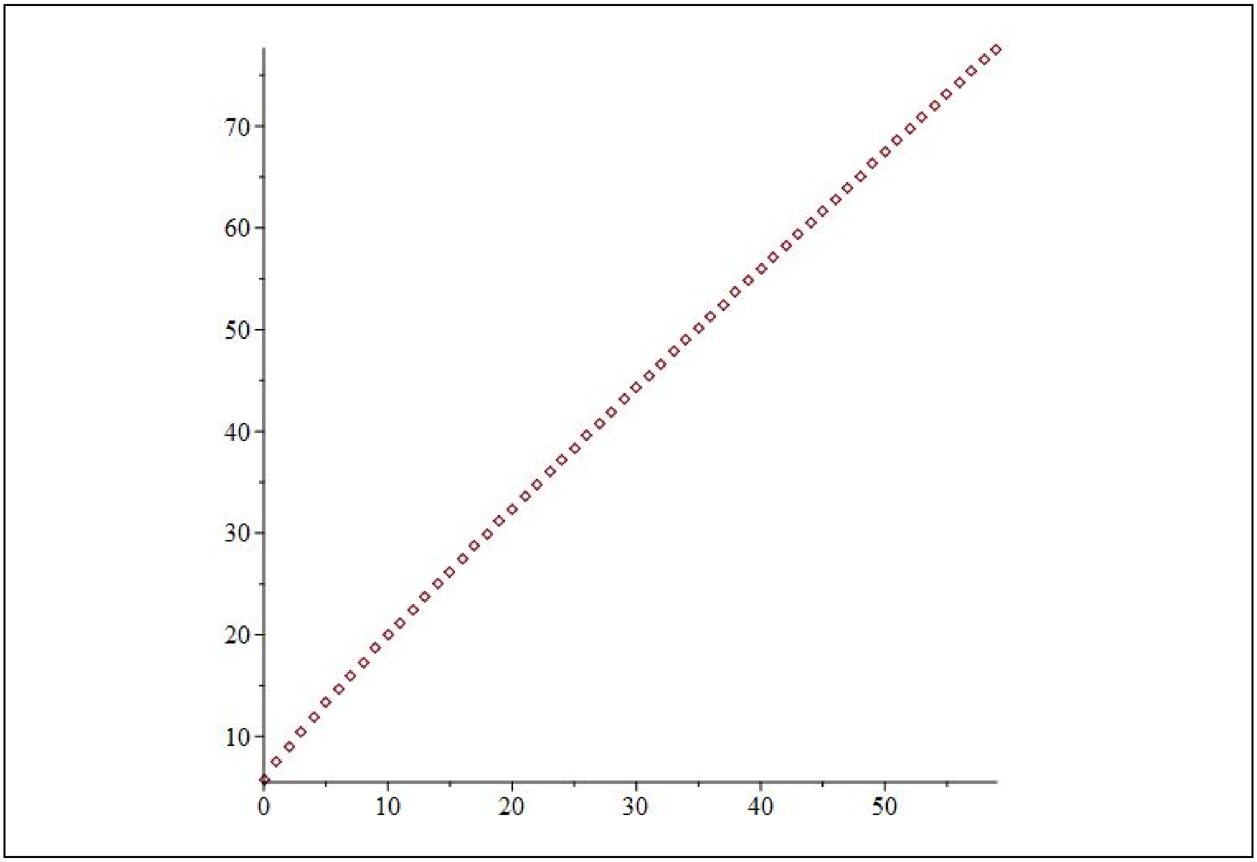
D. Analysis for Growth Constant of f:=1/(1-x-y-n*x*y)

Data: L:=[3 + 2*sqrt(2), 4 + 2*sqrt(3), 9, 6 + 2*sqrt(5), 7 + 2*sqrt(6), 8 + 2*sqrt(7), 9 + 4*sqrt(2), 16, 11 + 2*sqrt(10), 12 + 2*sqrt(11), 13 + 4*sqrt(3), 14 + 2*sqrt(13), 15 + 2*sqrt(14), 16 + 2*sqrt(15), 25, 18 + 2*sqrt(17), 19 + 6*sqrt(2), 20 + 2*sqrt(19), 21 + 4*sqrt(5), 22 + 2*sqrt(21), 23 + 2*sqrt(22), 24 + 2*sqrt(23), 25 + 4*sqrt(6), 36, 27 + 2*sqrt(26), 28 + 6*sqrt(3), 29 + 4*sqrt(7), 30 + 2*sqrt(29), 31 + 2*sqrt(30), ...]

Conjecture Seven: The growth constant of the generating function f:=1/(1-x-y-n*x*y) follows a concave down shape for all n. The rate of increase of the growth constant is decreasing as n gets larger.

E. Diagram for Growth Constant of f:=1/(1-x-y-n*z)

GrowthConstant(f): f:=1/(1-x-y-n*z); n=0..59



F. Analysis for Growth Constant of f:=1/(1-x-y-n*z)

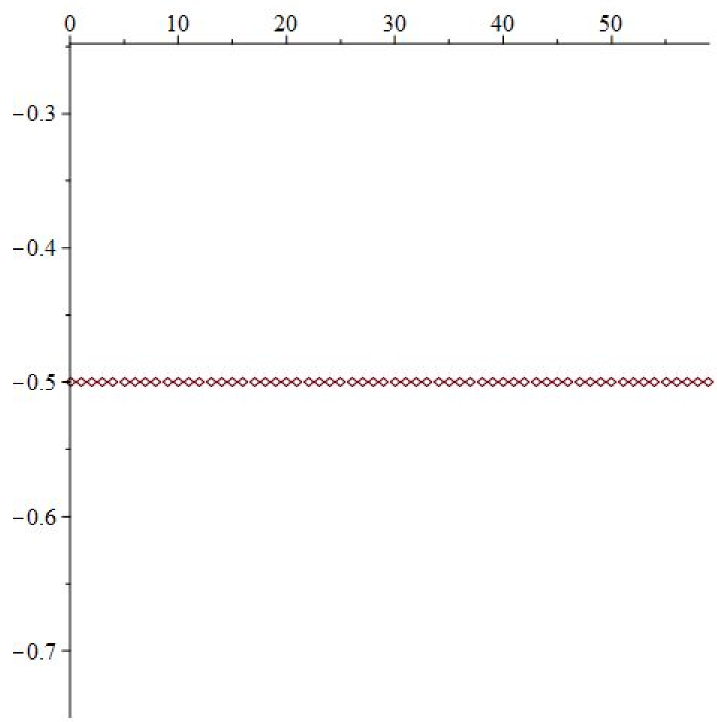
Data: L:=[27, 54, 81, 108, 135, 162, 189, 216, 243, 270, 297, 324, 351, 378, 405, 432, 459, 486, 513, 540, 567, 594, 621, 648, 675, 702, 729, 756, 783, 810, 837, 864, 891, 918, 945, 972, 999, ...]

Conjecture Eight: The growth constant of the generating function f:=1/(1-x-y-n*z) follows a linear function g(n)=27*n for all n. The rate of increase of the growth constant is constant.

3. Critical Exponent Analysis

A. Diagram for Critical Exponent of f:=1/(1-n*x-y)

CriticalExponent(f): f:=1/(1-n*x-y); n=0..59



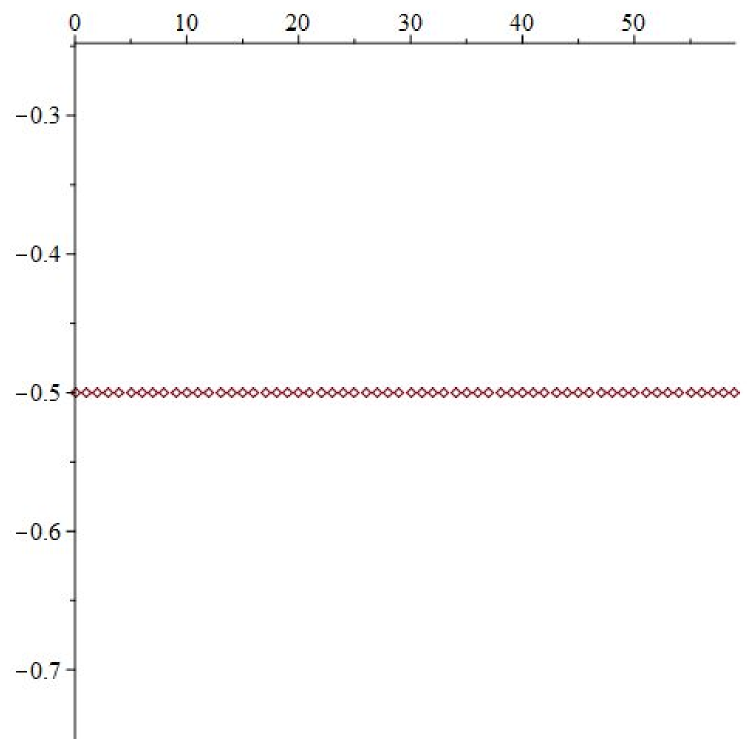
B. Analysis for Critical Exponent of f:=1/(1-n*x-y)

Data: L:=[-1/2, ...]

Conjecture Eight: The critical exponent of the generating function f:=1/(1-n*x-y) is -½ for all n. The rate of increase of the growth constant is constant.

C. Diagram for Critical Exponent of f:=1/(1-x-y-n*x*y)

CriticalExponent(f): f:=1/(1-x-y-n*x*y); n=0..59



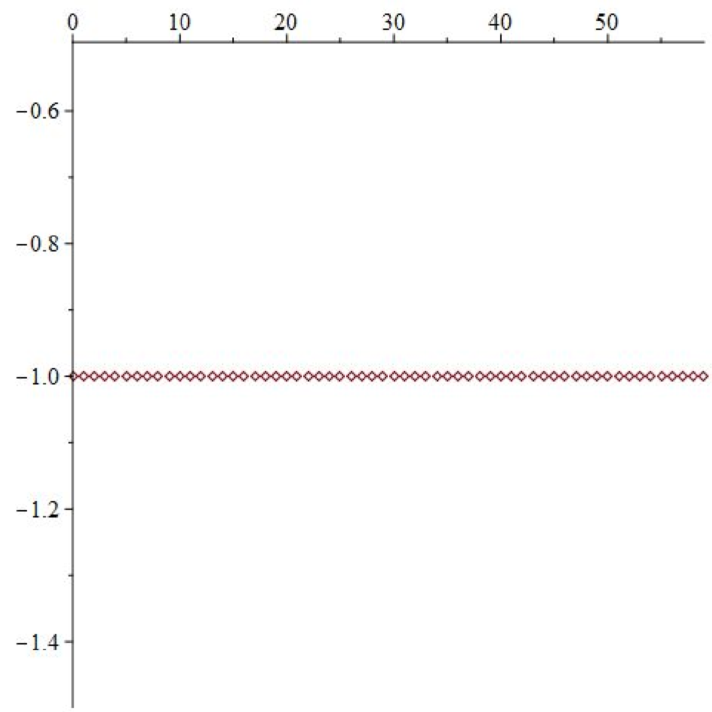
D. Analysis for Critical Exponent of f:=1/(1-x-y-n*x*y)

Data: L:=[-1/2, ...]

Conjecture Nine: The critical exponent of the generating function f:=1/(1-x-y-n*x*y) is -½ for all n. The rate of increase of the growth constant is constant.
Conjecture Ten: The critical exponent of f:=1/(1-x-y-n*x*y) is equal to the critical exponent of f:=1/(1-n*x-y) for all n.

E. Diagram for Critical Exponent of f:=1/(1-x-y-n*z)

CriticalExponent(f): f:=1/(1-x-y-n*z); n=0..59



F. Analysis for Critical Exponent of f:=1/(1-x-y-n*z)

Data: L:=-1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, ...]

Conjecture Eleven: The critical exponent of f:=1/(1-x-y-n*z) is -1 for all n. The rate of change is 0.

4. Asymptotics Analysis

A. Asymptotic for f:=1/(1-n*x-y)

Function	Asymptote
1/(1-x-y)	4^n*(1 - 10/(9*n) + 14/(9*n^2) - 8/(7*n^3) + 85/(189*n^4) - 142/(189*n^5))/sqrt(n)
1/(1-2x-y)	8^n*(1 - 10/(9*n) + 14/(9*n^2) - 8/(7*n^3) + 85/(189*n^4) - 142/(189*n^5))/sqrt(n)
1/(1-3x-y)	12^n*(1 - 10/(9*n) + 14/(9*n^2) - 8/(7*n^3) + 85/(189*n^4) - 142/(189*n^5))/sqrt(n)

B. Analysis of Asymptotes for f:=1/(1-n*x-y)

We see that as n increases by 1, the base of the leading term increases by 4. Note that the growth constant for this generating function also increased by 4 for every increase of 1 in n. This leads to following conjectures:
Conjecture Twelve: The asymptote for f:=1/(1-i*x-y), for any n, is (4*i)^n*1 - 10/(9*n) + 14/(9*n^2) - 8/(7*n^3) + 85/(189*n^4) - 142/(189*n^5))/sqrt(n)
Conjecture Thirteen: The increase by 4 in the base of the leading term represents the increase by 4 in the growth constant for this generating function.

C. Asymptotic for f:=1/(1-x-y-n*z)

Function	Asymptote
1/(1-x-y-z)	27^n*(1 - 28/(17*n) + 40/(17*n^2) - 30/(17*n^3) + 12/(17*n^4) - 11/(17*n^5))/n
1/(1-x-y-2z)	54^n*(1 - 28/(17*n) + 40/(17*n^2) - 30/(17*n^3) + 12/(17*n^4) - 11/(17*n^5))/n
1/(1-x-y-3z)	81^n*(1 - 28/(17*n) + 40/(17*n^2) - 30/(17*n^3) + 12/(17*n^4) - 11/(17*n^5))/n

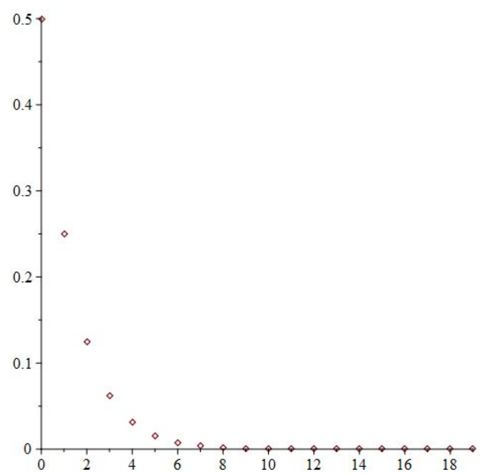
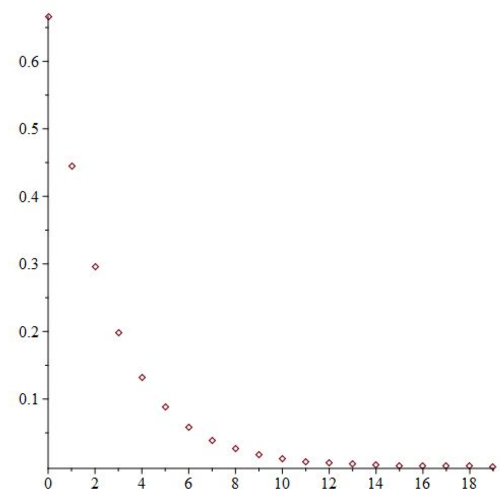
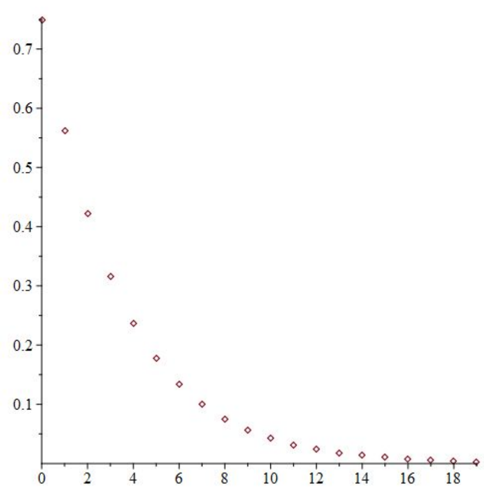
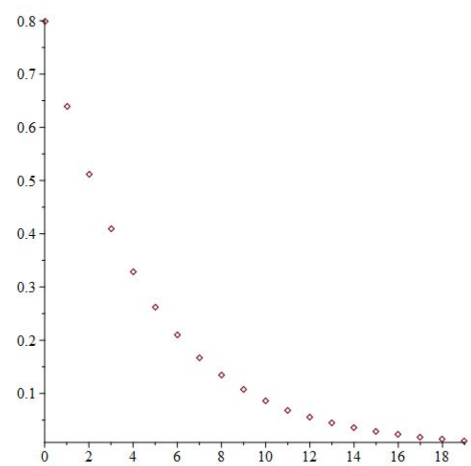
D. Analysis of Asymptotes for f:=1/(1-n*x-y)

We see that as n increases by 1, the base of the leading term in the asymptote increases by 27. Note that the growth constant for this generating function also increased by 27 for every increase of 1 in n. This leads to the following conjectures:

Conjecture Fourteen: The asymptote for f:=1/(1-x-y-i*z), for any n, is (27*i)^n*(1 - 28/(17*n) + 40/(17*n^2) - 30/(17*n^3) + 12/(17*n^4) - 11/(17*n^5))/n
Conjecture Fifteen: The increase in the base of the leading term in the asymptote represents the increase by 4 in the growth constant for this generating function.
Possible Theorem: If the growth constant of any generating function increases linearly, then the base of the leading term in the asymptote increases by that exact same linear rate.

5. Ratios

A. Diagrams for ratio of 1/(1-nx-y),1/(1-(n+1)x-y)

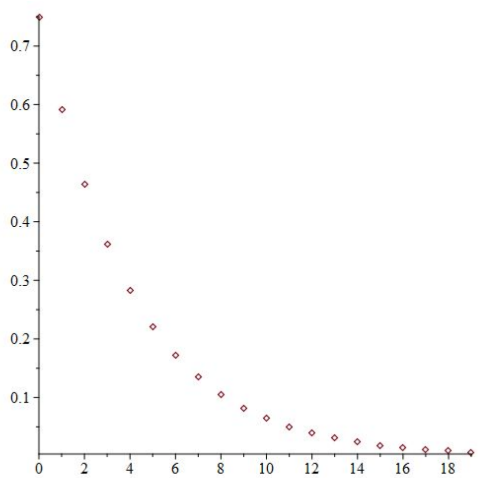
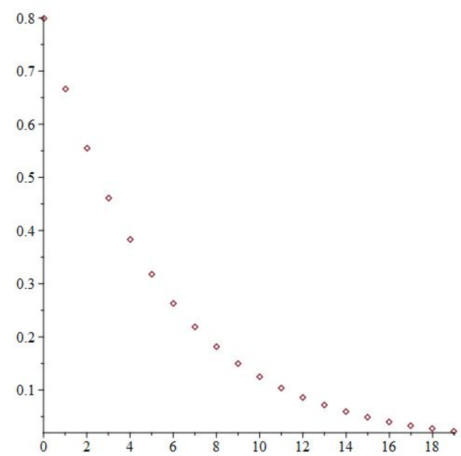
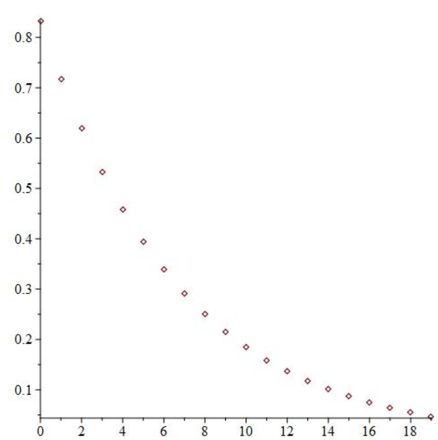
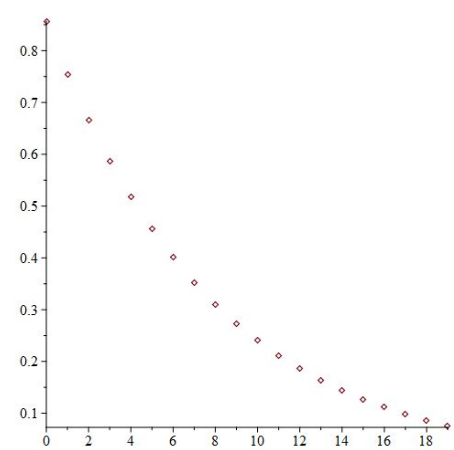
Ratio of (1/(1-x-y), 1/(1-2x-y))	**Ratio of (1/(1-2x-y), 1/(1-3x-y))**
	
Ratio of (1/(1-3x-y), 1/(1-4x-y))	**Ratio of (1/(1-4x-y), 1/(1-5x-y))**
	

B. Analysis for the ratio of 1/(1-nx-y),1/(1-(n+1)x-y)

As n increases, the ratio starts to decrease more slowly. In the above diagrams, when n=1, the ratio decreases rapidly to 0. This implies that the sequence for 1/(1-2x-y) grows much more rapidly than that for 1/(1-x-y). This can be applied to all of our cases.

Conjecture Sixteen: The ratio between 1/(1-nx-y) and 1/(1-(n+1)x-y) approaches zero for all n
Conjecture Seventeen: As n increases, the ratio decreases more slowly.

C. Diagrams for ratio of 1/(1-x-y-n*x*y),1/(1-x-y-(n+1)*x*y)

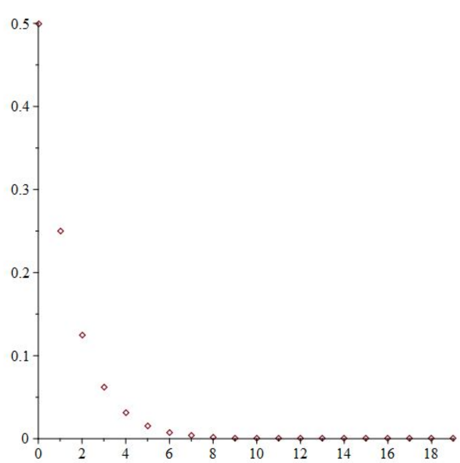
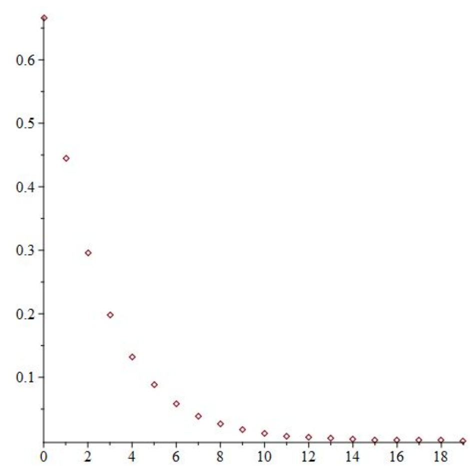
<table>
<tr><th>Ratio of (1/(1-x-y-xy), 1/(1-x-y-2xy))</th><th>Ratio of (1/(1-x-y-2xy), 1/(1-x-y-3xy))</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><th>Ratio of (1/(1-x-y-3xy), 1/(1-x-y-4xy))</th><th>Ratio of (1/(1-x-y-4xy), 1/(1-x-y-5xy))</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

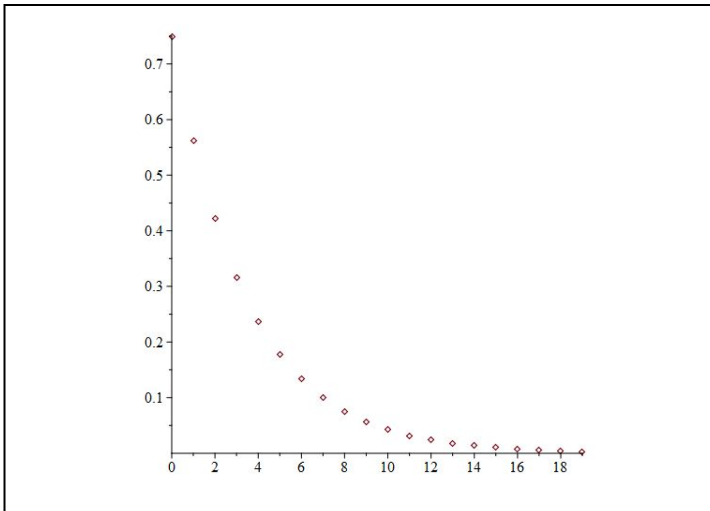
D. Analysis for the ratio of 1/(1-x-y-n*x*y),1/(1-x-y-(n+1)*x*)

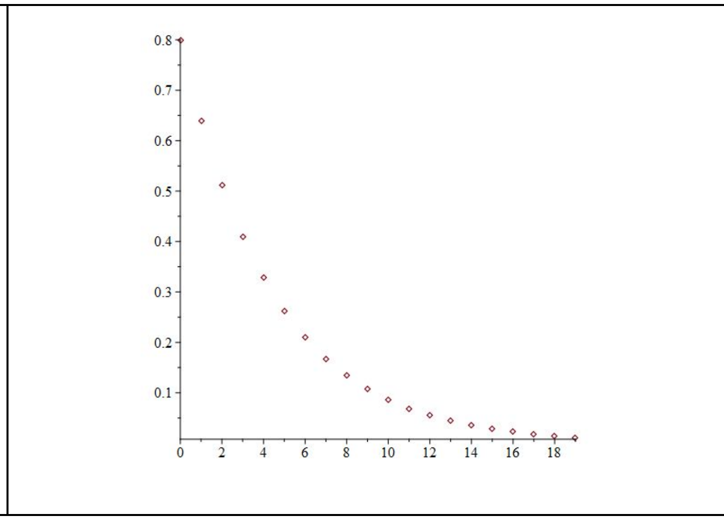
As n increases, the ratio starts to decrease more slowly. In the above diagrams, when n=1, the ratio decreases rapidly to 0. This implies that the sequence for 1/(1-2x-y) grows much more rapidly than that for 1/(1-x-y). However, we see that this applies to all of our cases. Furthermore, for these functions we see that while the ratio does decrease more slowly as n increases, the rate of change is fairly small.

Conjecture Eighteen: The ratio between 1/(1-x-y-nxy) and 1/(1-x-y-(n+1)xy) approaches zero for all n
Conjecture Nineteen: As n increases, the ratio decreases more slowly. Although, the rate of change is smaller than that for 1/(1-nx-y), 1/(1-(n+1)x-y).

E. Diagrams for ratio of 1/(1-x-y-nz),1/(1-x-y-(n+1)z)

Ratio of (1/(1-x-y-z), 1/(1-x-y-2z))	**Ratio of (1/(1-x-y-2z), 1/(1-x-y-3z))**
	
Ratio of (1/(1-x-y-3z), 1/(1-x-y-4z))	**Ratio of (1/(1-x-y-4z), 1/(1-x-y-5z))**





F. Analysis for the ratio of 1/(1-x-y-n*z),1/(1-x-y-(n+1)*z)

As n increases, the ratio starts to decrease more slowly. These graphs are very similar to those for 1/(1-nx-y), 1/(1-(n+1)x-y), implying that the ratios when one coefficient is changing are close.

Conjecture Twenty: The ratio between 1/(1-x-y-nz) and 1/(1-x-y-(n+1)z) approaches zero for all n
Conjecture Nineteen: The difference between the ratios of these functions and that for 1/(1-x-y-nz) and 1/(1-x-y-(n+1)z) gets increasingly small as n increases.